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# PULSATILE FLOW OF BLOOD IN A CONSTRICTED ARTERY WITH A VELOCITY SLIP

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**Abstract**

In this paper, the pulsatile flow of blood through a tapered artery with a mild stenosis is investigated. The body fluid blood is assumed to behave like a Newtonian fluid. A velocity-slip condition is employed at the stenotic wall. The effects of pulsatility, stenosis, slip condition and tapering of artery are simultaneously taken care of. A perturbation method is used to analyse the flow. Considering appropriate boundary conditions, analytic expressions for axial velocity, volumetric flow-rate and wall shear stress, have been obtained and their variations with different flow parameters are represented graphically. Biological implications of the present analysis are briefly discussed.

## 1. Introduction

The laminar flow of blood in arteries with the growth of a stenosis plays an important role in the diagnosis and clinical treatment as well as in the fundamental understanding of many cardiovascular diseases [11]. Among the various arterial diseases, the development of arteriosclerosis in blood vessels is quite common which may be attributed to accumulation of lipids in the arterial wall [1]. Arteries are narrowed by the development of atherosclerotic plaques that protrude into the lumen,

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resulting in stenosed arteries [2]. As an obstruction developed in an artery, one of the most serious consequences is the increased resistance and the associated reduction of blood flow to the particular vascular bed supplied by the artery [17]. Thus, the presence of a stenosis can lead to serious circulatory disorders [14]. It may be worth mentioning that in certain flow situations, physiological fluid blood may behave as a Newtonian fluid [15]. To understand the effects of stenosis in the lumen of an artery, many researchers [3, 4, 8, 10, 17, 18] have investigated the flow of blood through stenosed arteries, by considering blood to act as a Newtonian fluid. Thus it is reasonable to consider the body fluid blood as a Newtonian fluid.

It is found that arterial blood flow is highly pulsatile, with marked effects on instantaneous velocity distributions and the flow rate varies over a wide range during a flow cycle [6, 18]. The pulsatile flow of blood in a stenosed artery is presented in [9, 12, 13, 16]. Recently a pulsatile model of blood flow in a stenosed artery has been considered in [14]. Thus it is appropriate to consider the unsteady flow of blood inside the artery.

The study of blood flow through tapered tubes is also very important not only for an understanding of the flow behaviour of the marvellous body fluid blood in arteries, but also for the design of prosthetic blood vessels [5]. There is no doubt that tapering in arteries, is a significant aspect of mammalian arterial system. The formation of stenosis along the tapered wall may alter the flow situation to a great extent. In view of the above considerations, we are interested to study the pulsatile flow of blood through a catheterized tapered vessel with a stenosis. In this case, blood is taken as a Newtonian fluid and an axial slip in velocity, is introduced at the stenotic wall of the tapered artery.

## 2. Mathematical Formulation

We consider an axially symmetric, laminar, pulsatile and fully developed flow of blood (assumed to be Newtonian) through a tapered artery with a mild stenosis as shown in Figure 1.

The wall of the stenosed artery is assumed to be rigid. The geometry of the constricted tapered artery is mathematically modeled as [7].

$$\bar{R}(\bar{z}) = \begin{cases} \bar{R}_0 - m(\bar{z} + \bar{d}) - \dfrac{\overline{\delta s} \cos\phi}{2}\left(1 + \cos\dfrac{\pi\bar{z}}{\overline{\overline{z_0}}}\right), & |\bar{z}| \leq \overline{z_0}, \\ \bar{R}_0 - m(\bar{z} + \bar{d}), & |\bar{z}| > \overline{z_0}, \end{cases} \tag{2.1}$$

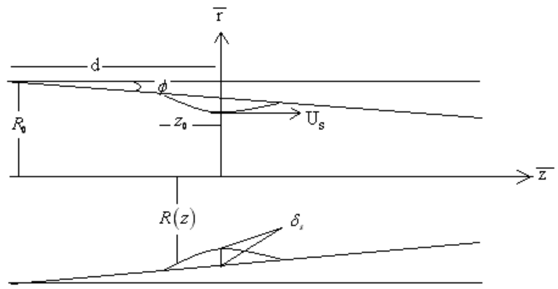


**Figure 1.** Schematic diagram of a tapered artery with stenosis.

where $\overline{R}(\bar{z})$ denotes the radius of the tapered arterial segment in the stenotic region, $\overline{R_0}$ is the constant radius of the straight artery in the non-stenotic region, $\phi$ is the angle of tapering, $\delta s \cos\phi$ is the length of the stenosis at a length $\bar{d}$ for the tapered artery, $\overline{z_0}$ is the half-length of the stenosis and $m(= \tan\phi)$ represents the slope of the tapered vessel. Let $(\bar{r}, \bar{\theta}, \bar{z})$ be the system of co-ordinates, used to analyse the flow field in the geometry as stated above, where $\bar{r}$ and $\bar{\theta}$ are along the radial and circumferential directions and, $\bar{z}$-axis is taken along the axis of the artery. It has been reported that the radial velocity is negligibly small and can be neglected for a low Reynolds number flow in a tube with mild stenosis [5, 18]. The equations of motion governing the fluid flow are given by

$$\bar{\rho}\frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}), \tag{2.2}$$

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0, \tag{2.3}$$

$$\frac{\partial \bar{p}}{\partial \bar{\theta}} = 0, \tag{2.4}$$

where $\bar{u}$ is the axial component of the velocity, $\bar{p}$ is the pressure, $\bar{\rho}$ is the density, $\bar{t}$ is the time and $\bar{\tau} = |\bar{\tau}_{\bar{r}\bar{z}}| = -\bar{\tau}_{\bar{r}\bar{z}}$ is the shear stress. In this study, blood has been considered as Newtonian fluid whose constitutive equation is given by

$$\bar{\tau} = -\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{r}}, \tag{2.5}$$

where $\bar{\mu}$ is the coefficient of viscosity for Newtonian fluid.

The boundary conditions are

(i) $\bar{\tau}$ is finite at $\bar{r} = 0,$ (2.6)

(ii) $\bar{u} = \overline{u_s}$ at $\bar{r} = \bar{R}(\bar{z}),$ (2.7)

where $\overline{u_s}$ is the slip-velocity at the stenotic wall.

Since, the pressure gradient is a function of $\bar{z}$ and $\bar{t}$, we take

$$-\frac{\partial \bar{p}}{\partial \bar{z}}(\bar{z}, \bar{t}) = A_0 + A_1 \cos(\overline{\omega}\bar{t}),$$

where $A_0$ is the steady component of the pressure gradient, $A_1$ is amplitude of the fluctuating component and $\overline{\omega} = 2\pi\bar{f}$, where $\bar{f}$ is the pulse frequency. Both $A_0$ and $A_1$ are functions of $\bar{z}$ [13]. We introduce the following non-dimensional variables

$$z = \frac{\bar{z}}{\overline{R_0}}, \quad R(z) = \frac{\bar{R}(\bar{z})}{\overline{R_0}}, \quad r = \frac{\bar{r}}{\overline{R_0}}, \quad t = \bar{t}\overline{\omega}, \quad d = \frac{\bar{d}}{\overline{R_0}}, \quad L_0 = \frac{\overline{L_0}}{\overline{R_0}},$$

$$\delta_s = \frac{\overline{\delta_s}}{\overline{R_0}}, \quad u = \frac{\bar{u}}{\frac{A_0 \overline{R_0}^2}{4\bar{\mu}}}, \quad \tau = \frac{\bar{\tau}}{\frac{A_0 \overline{R_0}}{2}}, \quad \alpha^2 = \frac{\overline{R_0}^2 \overline{\omega}\bar{\rho}}{\bar{\mu}}, \quad e = \frac{A_1}{A_0}, \tag{2.8}$$

where $\overline{R_0}$ is the radius of the normal artery and $\alpha$ is the pulsatile Reynolds number or generalized Womersley frequency parameter.

Using non-dimensional variables equations (2.2) and (2.5) reduce to

$$\alpha^2 \frac{\partial u}{\partial t} = 4(1 + e\cos t) - \frac{2}{r}\frac{\partial}{\partial r}(r\tau), \tag{2.9}$$

$$\tau = -\frac{1}{2}\frac{\partial u}{\partial r}. \tag{2.10}$$

The boundary conditions in the non-dimensional form are

(i) $\tau$ is finite at $r = 0,$ (2.11)

(ii) $u = u_s$ at $r = R(z).$ (2.12)

The geometry of stenosis in dimensionless form is given by

$$R(z) = \begin{cases} 1 - m(z + d) - \dfrac{\delta_s \cos\phi}{2}\left(1 + \cos\dfrac{\pi z}{z_0}\right), & |z| \le z_0, \\ 1 - m(z + d), & |z| > z_0. \end{cases} \tag{2.13}$$

The non-dimensional volumetric flow rate is given by

$$Q(z, t) = 4\int_0^{R(z)} ru(z, r, t)dr, \tag{2.14}$$

where $Q(z, t) = \dfrac{\overline{Q}(\bar{z}, \bar{t})}{\dfrac{\pi(\overline{R_0})^4 A_0}{8\overline{\mu}}}$, $\overline{Q}(\bar{z}, \bar{t})$ is the volumetric flow rate.

## 3. Method of Solution

Considering the Womersley parameter to be small, the velocity $u$ and shear stress $\tau$ can be expressed in the following form

$$u(z, r, t) = u_0(z, r, t) + \alpha^2 u_1(z, r, t) + \cdots, \tag{3.1}$$

$$\tau(z, r, t) = \tau_0(z, r, t) + \alpha^2 \tau_1(z, r, t) + \cdots. \tag{3.2}$$

Using (3.1) and (3.2) in (2.9), we have

$$\frac{\partial}{\partial r}(r\tau_0) = 2r(1 + e\cos t), \tag{3.3}$$

$$\frac{\partial u_0}{\partial t} = -\frac{2}{r}\frac{\partial}{\partial r}(r\tau_1). \tag{3.4}$$

Integrating (3.3) and using the boundary condition (2.11), we have

$$\tau_0 = f(t)r, \quad 0 \le r \le R(z), \tag{3.5}$$

where

$$f(t) = 1 + e\cos t. \tag{3.6}$$

Introducing (3.1) and (3.2) into equation (2.10), we obtain

$$-\frac{\partial u_0}{\partial r} = 2\tau_0, \quad 0 \le r \le R(z), \tag{3.7}$$

$$-\frac{\partial u_1}{\partial r} = 2\tau_1, \quad 0 \le r \le R(z). \tag{3.8}$$

Integrating (3.7) with the help of (3.5) and using the boundary condition (2.12), we get

$$u_0 = u_s + f(t)\left((R(z))^2 - r^2\right), \quad 0 \le r \le R(z). \tag{3.9}$$

Integrating equation (3.4) with the help of (3.9) and using the boundary condition (2.11), we get

$$\tau_1 = -\frac{1}{8} f'(t)(2(R(z))^2 r - r^3), \quad 0 \leq r \leq R(z). \tag{3.10}$$

Integrating equation (3.8) with the help of (3.10) and using boundary condition (2.12), we get

$$u_1 = u_s + \frac{1}{16} f'(t)(4(R(z))^2 r^2 - 3(R(z))^4 - r^4), \quad 0 \leq r \leq R(z). \tag{3.11}$$

The expression for axial velocity $u(r, z, t)$ can be obtained from equations (3.1), (3.9) and (3.11) and the expression for shear stress $\tau(r, z, t)$ can be found from equations (3.2), (3.5) and (3.10).

The wall shear stress is given by

$$\tau_w = (\tau_0 + \alpha^2 \tau_1)_{r=R(z)} = f(t)R(z) - \frac{\alpha^2}{8} f'(t)(R(z))^3. \tag{3.12}$$

The expression for volumetric flow rate is given by

$$Q(z, t) = (R(z))^2 \left\{ 2(1 + \alpha^2) u_s + f(t)(R(z))^2 - \frac{\alpha^2}{6} f'(t)(R(z))^4 \right\}. \tag{3.13}$$

## 4. Results and Discussions

The objective of the present investigation is to study the combined effect of stenosis, tapering of artery and slip velocity at wall on the pulsatile flow of blood through a constricted tapered artery, considering blood as to behave like a Newtonian fluid. The governing equations of flow are solved using perturbation analysis assuming that the Womersley frequency parameter is very small which is valid for physiological situations in small blood vessels. Analytic expressions for axial velocity, flow rate and wall shear stress are found out and their variations with different flow parameters are represented graphically.

### 4.1. Axial velocity profile

Variation of axial velocity with the radial distance for different values of $z$ and $t$ are given by Figure 1 and Figure 2, respectively.

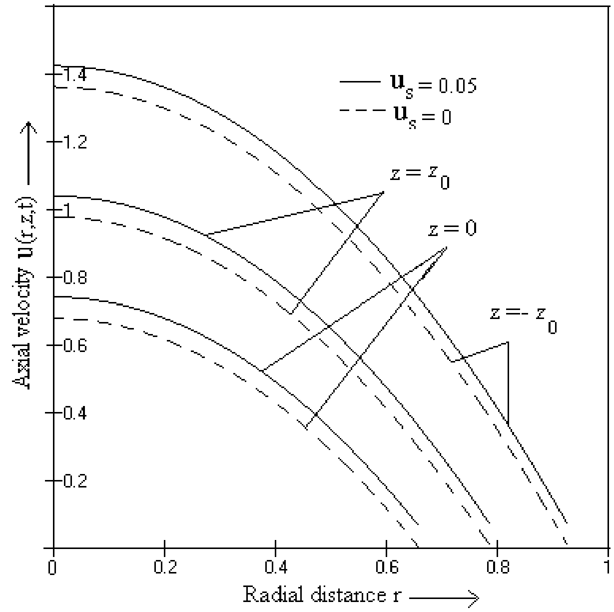


**Figure 2.** Variation of axial velocity with radial distance for slip and no-slip cases $t = 1,\ e = 1,\ \phi = 1^{\circ}$.

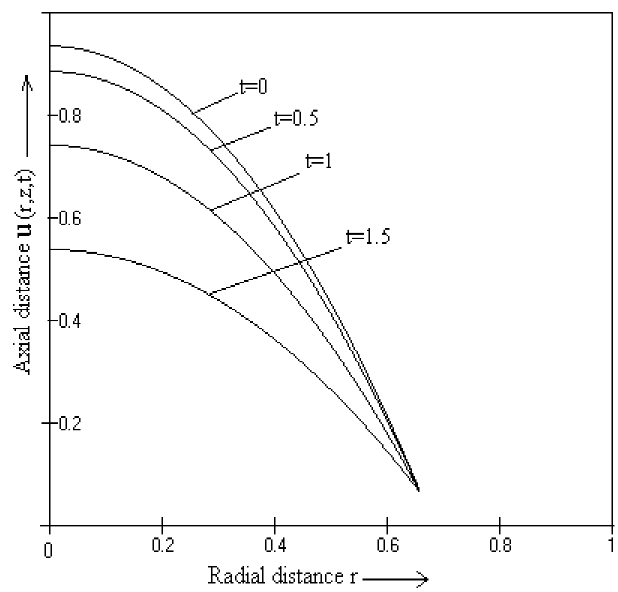


**Figure 3.** Variation of axial velocity with radial distance for different time $z = 0$, $e = 1,\ u_s = 0.05,\ \phi = 1^{\circ}$.

It is observed from Figure 2 that velocity profiles show a non-parabolic trend. Axial velocity is maximum at the axis and minimum at the stenotic wall. As expected, with slip at stenotic wall velocity is more than that with no-slip. Further it is observed from Figure 3 that velocity decreases as the non-dimensional time $t$ increases from $t = 0$ to $t = 1.5$ at the throat of the stenosis $(z = 0)$.

### 4.2. Volumetric flow rate

The variations of volumetric flow rate as obtained from equation (3.13) against axial distance in the stenotic region of the tapered artery for different slip velocities and for different tapering angles of the artery are represented in Figure 4 and Figure 5, respectively.

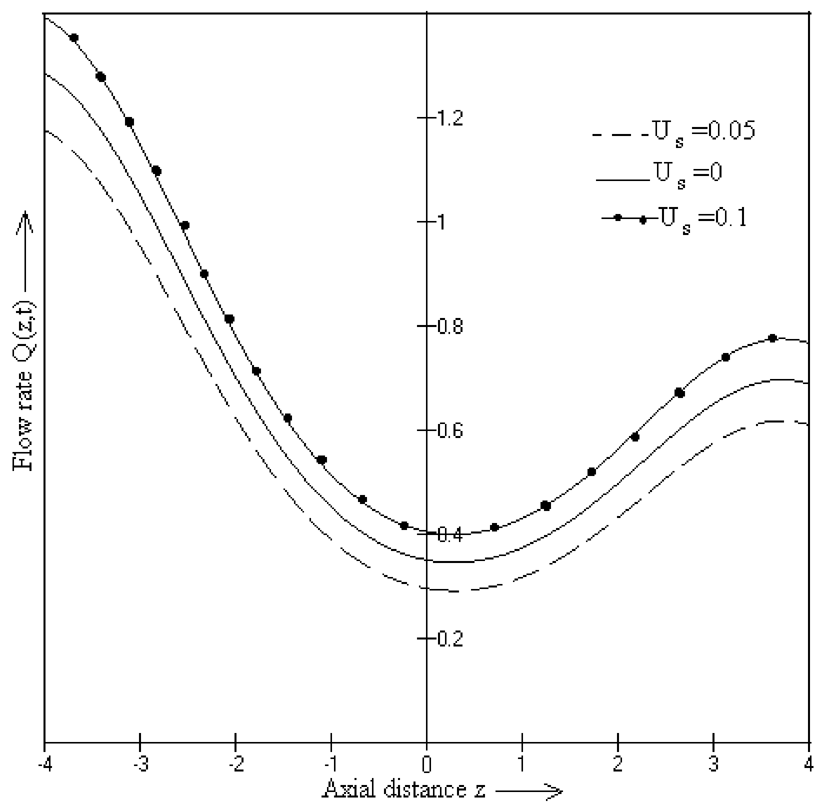


**Figure 4.** Variation of flow-rate with axial distance for different slip velocities $e = 1$, $t = 1$, $\phi = 1^{\circ}$.

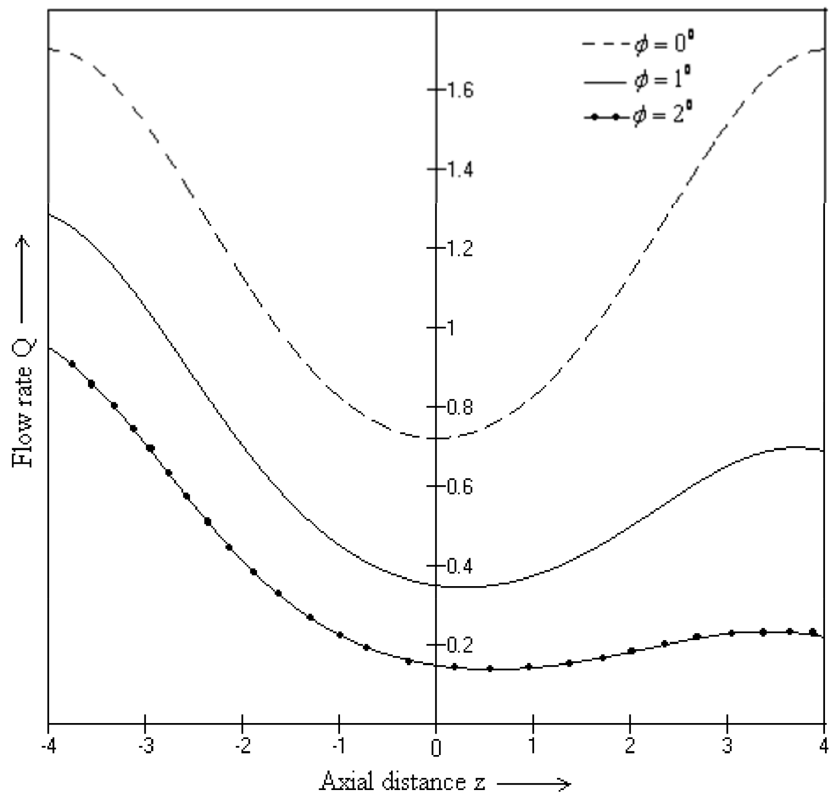


**Figure 5.** Variation of flow-rate with axial distance for different tapering angles $e = 1$, $t = 1$, $u_s = 0.05$.

From the figures it is observed that flow rate is maximum at the initiation of the stenosis $(z = -z_0)$ and minimum at the maximum constricted area, i.e., at the throat of the stenosis $(z = 0)$. The flow rate obtained with slip at stenotic wall is greater than that obtained with no-slip at wall and it further increases with the increase of velocity slip. Further, it is observed that though tapering of artery does not change the flow pattern but it affects the magnitudes of flow rate inside the artery. Flow rate obtained in the tapered artery is less in magnitude than that obtained in case of a uniform artery. Also an increase in tapering angle decreases the flow rate.

### 4.3. Wall shear stress

The variations of wall shear stress with axial distance for different tapering angles are given in Figure 6.

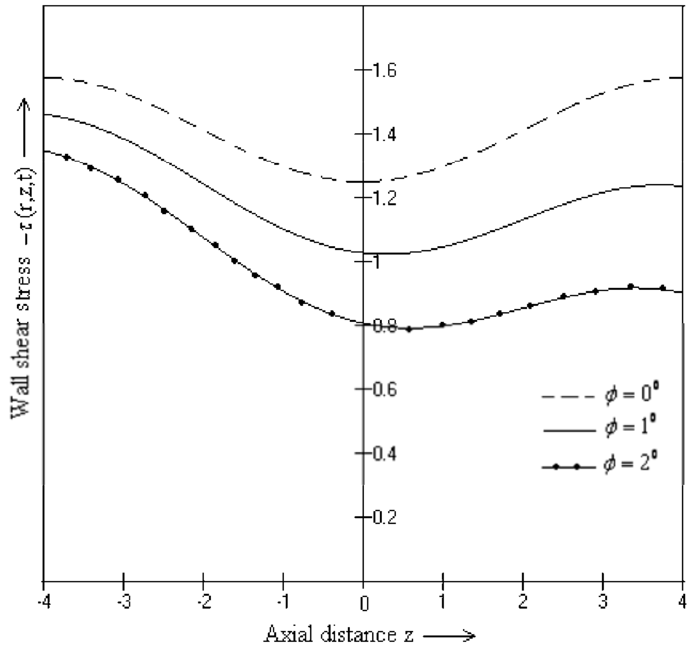


**Figure 6.** Variation of wall shear stress with axial distance for different tapering angles $e = 1$, $t = 1$, $u_s = 0.05$.

It is observed from the profile that wall shear stress in the tapered artery is more than that obtained in the uniform artery. Further an increase in tapering angle decreases the wall shear stress.

## 5. Conclusion

In this paper, the pulsatile flow of blood through a tapered artery with mild stenosis is modeled, using perturbation method and considering the body-fluid blood behaving as a Newtonian fluid. Analytic expressions for axial velocity, flow rate and wall shear stress are obtained and their variations with different flow parameters are shown in figures. It is observed that axial velocity and volumetric flow rate increase with the increase of velocity slip. Further, an increase in tapering angle increases the wall shear stress but reduces the flow-rate inside the artery. The axial velocity decreases with the increase in time from $t = 0$ to $t = 1.5$. As flow rate is increased due to the employment of a velocity slip at the constricted tapering wall, this clearly indicates that slip at a diseased artery wall could play a prominent role in blood flow modeling. It may be worthwhile to notice that an employment of slip at wall will accelerate the speed and flow rate on one hand and retarded the resistance to flow on the other. As a result, bore of the vessel will increase, stenosis size will be lowered and rate of flow will be greater than that of the earlier. This in turn will improve the better functioning of the diseased artery and so slip may be exploited to act as a device for curing atherosclerosis.

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