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ENERGY BALANCE METHOD TO SUB-HARMONIC RESONANCES OF NONLINEAR OSCILLATIONS WITH PARAMETRIC EXCITATION

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Abstract

Approximate analytic solutions of sub-harmonic resonances of nonlinear oscillations with parametric excitation are obtained using He's Energy Balance Method (EBM). Unlike perturbation methods, EBM does not depend on any small physical parameters at all. One iteration step can provide very accurate analytical approximate solutions for both small and large values of oscillation amplitude and parameter. Comparisons are made between Energy Balance Method result and numerical solution of the problem. These approximate solutions show excellent agreement with the numerical solution, and this method can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other sciences.

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1. Introduction

The study of nonlinear oscillators is of great interest in engineering and physical sciences and many analytical techniques have been developed for solving the second-order nonlinear differential equations that govern their motion. In order to study this phenomenon, we need to solve the governing equations. Because of nonlinearity of these equations, it is difficult to solve them analytically. There is a large variety of approximate methods for the determination of solutions of nonlinear second-order dynamical equations including the Perturbation technique [21, 22] which is useful if there exist small parameters in the nonlinear problems, standard and modified Lindstedt-Poincaré [1, 10, 23], the harmonic balance method [5, 17-19], parameter-expansion method [13, 28] and Parameterized perturbation method [7, 9]. Recently, some other approximate variational methods, including approximate energy method [2, 11, 12], variational iteration method [6, 8, 16, 26, 27] and variational approach [15, 20, 30, 31] have been used to solve nonlinear governing equations. These methods give successive approximations of high accuracy of the solution. The Energy Balance Method is one of the well-known methods to solve the nonlinear equations. This method is established by He [11, 12, 14] and has been used by many authors in [3, 4, 24, 25, 32].

In this paper, we will apply He's Energy Balance Method to solve periodic solutions for sub-harmonic resonances of nonlinear oscillations. By comparing the results of Energy Balance Method and numerical solution of the problems, we will show this method has high accuracy. Also, for strong nonlinear equations that obtaining the Hamiltonian is difficult or impossible, we can use new assumption to solve the equations easily.

2. He's Energy Balance Method

In He's Energy Balance Method, a variational principle for the nonlinear oscillation is established and then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. This method will be useful for differential equations with strong nonlinearity. To show this case, we consider a general nonlinear oscillator in the form:

$$X'' + f(X(t)) = 0, \tag{2.1}$$

in which X and t are generalized dimensionless displacement and time variables, respectively. The boundary conditions are:

$$X(0) = X_0, \qquad X'(0) = 0. \tag{2.2}$$

Its variational principle can be easily obtained:

$$J(X) = \int_0^t \left(-\frac{1}{2}X'^2 + F(X)\right)dt, \tag{2.3}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator, $F(X) = \int f(X)dX$.

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}X'^2 + F(X) = F(X_0) \tag{2.4}$$

or

$$R(t) = \frac{1}{2}X'^2 + F(X) - F(X_0) = 0. \tag{2.5}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, X_0. Assume that its initial approximate guess can be expressed as:

$$X(t) = X_0 \cos(\omega t). \tag{2.6}$$

Substituting (2.6) into X term of (2.5), yields

$$R(t) = \frac{1}{2}\omega^2 X_0^2 \sin^2 \omega t + F(X_0 \cos \omega t) - F(X_0) = 0. \tag{2.7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make H zero for all values of t by appropriate choice of ω. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(X_0) - F(X_0 \cos \omega t))}{X_0^2 \sin^2 \omega t}}. \tag{2.8}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\frac{2(F(X_0) - F(X_0 \cos \omega t))}{X_0^2 \sin^2 \omega t}}}. \tag{2.9}$$

3. He's Energy Balance Method to Sub-harmonic Resonances of Nonlinear Oscillations

In this section, we consider periodic solutions for sub-harmonic resonances of nonlinear oscillations with parametric excitation and use the Energy Balance Method to solve this equation in different states. The governing equation is [3]:

$$X'' + (1 + \varepsilon\cos(\lambda t))[\alpha X + \beta X^3] = 0, \tag{3.1}$$

and the boundary conditions for this equation are:

$$X(0) = X_0, \qquad X'(0) = 0, \tag{3.2}$$

where prime denotes derivation with respect to time. In equation (3.1), $X(t)$ is displacement, t is time variable and ε, α, β, λ are generalized dimensionless small parameters [3].

In boundary conditions (3.2), the value of X_0 is unknown and we will find the corresponding unknown amplitude X_0 with given ε, α, β and λ. To do this, we utilize EBM and show the authority of this method in solving nonlinear oscillators like this.

The first step to solve equation (3.1) is obtaining the Hamiltonian. So, we multiply the value of X' in equation (3.1):

$$X''X' + (1 + \varepsilon\cos(\lambda t))[\alpha XX' + \beta X^3 X'] = 0. \tag{3.3}$$

By integrating equation (3.3), we can readily obtain Hamiltonian formulation as follows:

$$\begin{aligned} H &= \frac{1}{2}X'^2 + \frac{1}{2}\alpha X^2 + \frac{1}{4}\beta X^4 + \varepsilon\int(\alpha XX'\cos(\lambda t) + \beta X^3 X'\cos(\lambda t))dt \\ &= \frac{1}{2}\alpha X_0^2 + \frac{1}{4}\beta X_0^4 \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} R(t) = {} & \frac{1}{2}X'^2 + \frac{1}{2}\alpha X^2 + \frac{1}{4}\beta X^4 + \varepsilon\int(\alpha XX'\cos(\lambda t) + \beta X^3 X'\cos(\lambda t))dt \\ & - \frac{1}{2}\alpha X_0^2 - \frac{1}{4}\beta X_0^4 = 0. \end{aligned} \tag{3.5}$$

Since the solution of equation (3.1) via boundary conditions (3.2) is periodic with the frequency ω, we consider parameter λ as follows:

$$\lambda = n\omega, \qquad n = 1, 2, 3, \ldots. \tag{3.6}$$

So, we rewrite the equation (3.5):

$$R(t) = \frac{1}{2}X'^2 + \frac{1}{2}\alpha X^2 + \frac{1}{4}\beta X^4 + \varepsilon\int(\alpha XX'\cos(n\omega t) + \beta X^3 X'\cos(n\omega t))dt$$

$$-\frac{1}{2}\alpha X_0^2 - \frac{1}{4}\beta X_0^4 = 0. \tag{3.7}$$

Substituting (2.6) into (3.7) and simplifying, we obtain:

$$R(t) = 0.5X_0^2\omega^2\sin^2(\omega t) + 0.5\alpha X_0^2\cos^2(\omega t) + 0.25\beta X_0^4\cos^4(\omega t)$$

$$+\frac{0.25\alpha\varepsilon X_0^2\omega\cos(2\omega t)\cos(n\omega t)}{2\omega + n\omega} - \frac{0.25\alpha\varepsilon X_0^2\omega\sin(2\omega t)\sin(n\omega t)}{2\omega + n\omega}$$

$$-\frac{0.25\alpha\varepsilon X_0^2\omega\cos(2\omega t)\cos(n\omega t)}{-2\omega + n\omega} - \frac{0.25\alpha\varepsilon X_0^2\omega\sin(2\omega t)\sin(n\omega t)}{-2\omega + n\omega}$$

$$+\frac{0.0625\beta\varepsilon X_0^4\omega\cos(4\omega t)\cos(n\omega t)}{4\omega + n\omega} - \frac{0.0625\beta\varepsilon X_0^4\omega\sin(4\omega t)\sin(n\omega t)}{4\omega + n\omega}$$

$$-\frac{0.125\beta\varepsilon X_0^4\omega\cos(2\omega t)\cos(n\omega t)}{-2\omega + n\omega} - \frac{0.125\beta\varepsilon X_0^4\omega\sin(2\omega t)\sin(n\omega t)}{-2\omega + n\omega}$$

$$+\frac{0.125\beta\varepsilon X_0^4\omega\cos(2\omega t)\cos(n\omega t)}{2\omega + n\omega} - \frac{0.125\beta\varepsilon X_0^4\omega\sin(2\omega t)\sin(n\omega t)}{2\omega + n\omega}$$

$$-\frac{0.0625\beta\varepsilon X_0^4\omega\cos(4\omega t)\cos(n\omega t)}{-4\omega + n\omega} - \frac{0.0625\beta\varepsilon X_0^4\omega\sin(4\omega t)\sin(n\omega t)}{-4\omega + n\omega}$$

$$-0.5\alpha X_0^2 - 0.25\beta X_0^4 = 0. \tag{3.8}$$

In this method according to basic idea of the Energy Balance Method, if $X = 0$, it shows the whole energy is in form of kinetic energy and if $X = \pi/2$, it shows the whole energy is in form of potential energy, in $X = \pi/4$ there is a balance between the potential energy and kinetic energy. So we can benefit from this point.

If we collocate at $\omega t = \pi/4$ and substitute $n = 1,$ we obtain:

$$R(t) = 0.249988 X_0^2 \omega^2 - 0.249988 \alpha X_0^2 - 0.187494 \beta X_0^4 + 4E - 14 X_0^2 \varepsilon\alpha$$
$$+ 0.033333 X_0^4 \varepsilon\beta = 0. \tag{3.9}$$

For determined values of α, β, ε, λ and n, we can obtain the corresponding unknown amplitude X_0 and the periodic solution with known frequency $\omega = \lambda/n.$ For an example, we consider the case of $\alpha = 0.9$, $\beta = 4$, $\varepsilon = 0.1$, $\lambda = 1$ and $n = 1$ (i.e., $\omega = 1$). With these values, if we solve equation (3.9), we will obtain the amplitude X_0 as follows:

$$X_0 = 0.184217. \tag{3.10}$$

So, in this case, we will have the solution of equation (3.1):

$$X(t) = 0.184217 \cos(t). \tag{3.11}$$

We have compared the result that has been obtained by the Energy Balance Method (equation (3.10)) and numerical solution in Figure 1. For other cases of α, β, ε, λ and n, the results show that this method is powerful to solve nonlinear oscillators. These comparisons are plotted in Figures 2-3.

4. Conclusion

In this paper, the Energy Balance Method (EBM) was employed to solve the periodic solutions for sub-harmonic resonances of nonlinear oscillations with parametric excitation. This method was applied for approaching amplitude of the system when the frequency is determined. Comparisons with the results of numerical solutions have been done by some figures. This example has shown that the approximate analytical solutions are in excellent agreement with the corresponding numerical solutions. The method can be easily extended to any nonlinear oscillator without any difficulty and, it is accurate, fast and reliable for such problems.

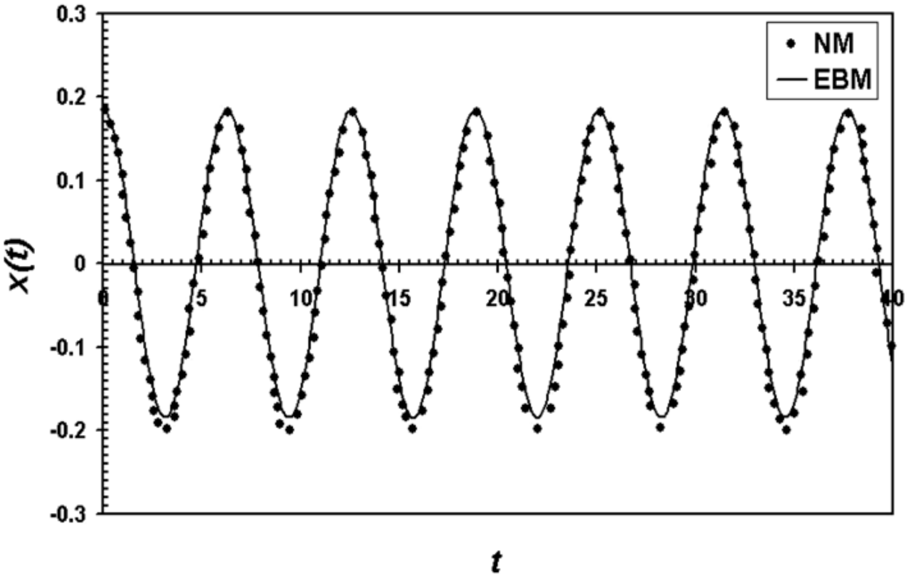

Figure 1. Comparison of the EBM solution with the numerical solution for the case that $\alpha = 0.9$, $\beta = 4$, $\lambda = 1$, $\varepsilon = 0.1$ and $n = 1$ $(\omega = 1)$.

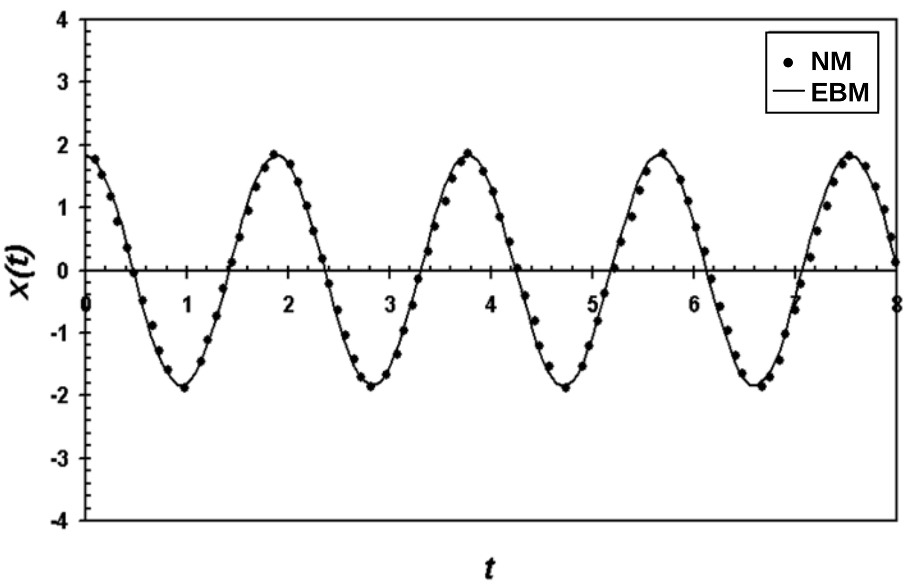

Figure 2. Comparison of the EBM solution with the numerical solution for the case that $\alpha = 1$, $\beta = 4$, $\lambda = 10$, $\varepsilon = 0.01$ and $n = 3$ $(\omega = 10/3)$.

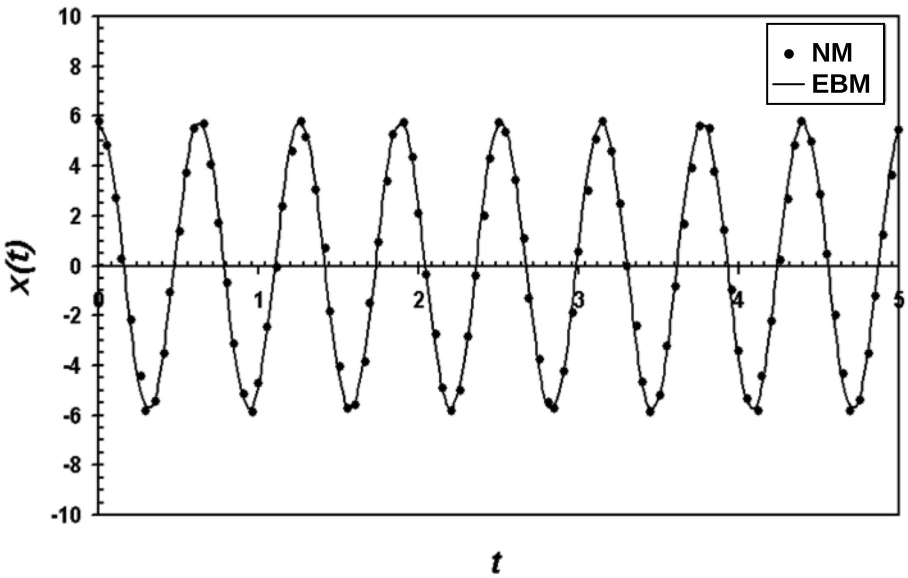

Figure 3. Comparison of the EBM solution with the numerical solution for the case that $\alpha = 1$, $\beta = 4$, $\lambda = 10$, $\varepsilon = 0.01$ and $n = 1$ $(\omega = 10)$.

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