Far East Journal of Theoretical Statistics
Volume 29, Number 1, 2009, Pages 1-7
Published Online: October 8, 2009
This paper is available online at http://www.pphmj.com

THE LIMITING SPECTRAL DENSITY OF LARGE DIMENSIONAL SAMPLE COVARIANCE MATRICES

SANG-IL CHOI

Department of Mathematics
Hanseo University
Seosan, Chungnam 356-820, Korea
e-mail: schoi@hanseo.ac.kr

Abstract

Let $B_n = \frac{1}{N} X_n X_n^T T_n$, where $X_n = [X_{ij}]_{n \times N}$ is with i.i.d. entries and T_n is an $n \times n$ symmetric nonnegative definite random matrix independent of the X_{ij}'s. Using the Stieltjes transform, it is shown that the limiting distribution of B_n has a continuous density function away from zero. In the present paper, it is derived that the limiting density function is analytic whenever it is positive and its behavior resembles a square root function on the boundary of its support.

1. Introduction

The spectra for sample covariance matrices of the form $\frac{1}{N} X_n X_n^T T_n$ are important in multivariate analysis. Let $X_n = [X_{ij}]_{n \times N}$ with $E| X_{11} - EX_{11} |^2 = 1.$ For each positive integer n, $n = n(N)$ and $n/N \to c > 0$ as $n \to +\infty$, and let T_n be an $n \times n$ symmetric nonnegative definite random matrix independent of the X_{ij}'s.

2000 Mathematics Subject Classification: 60E99, 26A46.

Keywords and phrases: eigenvalues of random matrices, spectral distribution, Stieltjes transform.

Received March 26, 2009

From previous work, the spectral distribution function of B_n, say F^{B_n} converges weakly to a nonrandom distribution function F as $n \to +\infty$ when F^{T_n} converges to a nonrandom distribution function H, [2].

For $z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : Im(z) > 0\}$, the Stieltjes transform of a distribution function G is defined by

$$m_G(z) = \int \frac{1}{\lambda - z} dG(\lambda).$$

From the inversion formula, distribution functions are uniquely determined by their Stieltjes transform.

In [1], it is shown that, for each $z \in \mathbb{C}^+$, $m = m_F(z)$ is the unique solution for $m \in \mathbb{C}^+$ to the equation $m = -\left(z - c \int \frac{\lambda}{1 + \lambda m} dH(\lambda) \right)^{-1}$.

Therefore, on $\mathbb{C}^+$, $m_F(z)$ has an inverse, given by

$$z(m) = -\frac{1}{m} + c \int \frac{\lambda}{1 + \lambda m} dH(\lambda), \quad m \in m_F(\mathbb{C}^+). \tag{1.1}$$

In [2], it is shown that, for all $x \in \mathbb{R}$ with $x \neq 0$, $\lim_{z \in \mathbb{C}^+ \to x} m_F(z) \equiv m_0(x)$ exists and the function m_0 is continuous on $\mathbb{R} - \{0\}$. Therefore, we have that, for all $x \neq 0$, the density function of F is

$$f(x) = \frac{1}{\pi} Im(m_0(x)),$$

which is continuous on $\mathbb{R} - \{0\}$.

Moreover, from (1.1), for every $x \neq 0$ for which $f(x) > 0$, $\pi f(x)$ is the imaginary part of the unique $m \in \mathbb{C}^+$ satisfying

$$x = -\frac{1}{m} + c \int \frac{\lambda}{1 + \lambda m} dH(\lambda). \tag{1.2}$$

This nonrandom distribution function F is depending on H and c. In case, $H = 1_{[1, \infty)}$ and $c = 1$, the density function $f(x) = \frac{1}{2\pi x} 1_{(0,4)}(x)\sqrt{x(4-x)}$, [2, 4].

For most cases of $x_0 \in \partial S_F$, the boundary of the support of F, for $x \in S_F$, $f(x)$ resembles $\sqrt{|x - x_0|}$ near x_0.

In this paper, we will analyze completely the case where H takes mass at two positive distinct values μ, σ^2 with $0 < \mu < \sigma^2$.

2. Derivation of the Density $f(x)$

First, we will find S_F by using (1.2). To find S_F, we have $x_0 \in I \subset S_F^c$, where I is an open interval, iff $x'(m_0) > 0$, where $m_0 = m_0(x_0)$, [2]. For $0 < p < 1$, and for $0 < \mu < \sigma^2$, let $H(x) = (1-p)1_{[\mu,\infty)}(x) + p1_{[\sigma^2,\infty)}(x)$. On

$$B = \left(-\infty, -\frac{1}{\mu}\right) \cup \left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right) \cup \left(-\frac{1}{\sigma^2}, 0\right) \cup (0, \infty),$$

from (1.2),

$$x(m) = -\frac{1}{m} + c\int \frac{\lambda}{1+\lambda m}\,dH(\lambda) = -\frac{1}{m} + c\left(\frac{1-p}{\frac{1}{\mu}+m} + \frac{p}{\frac{1}{\sigma^2}+m}\right). \tag{2.1}$$

Moreover,

$$x'(m) = \frac{1}{m^2} - c\left(\frac{1-p}{\left(\frac{1}{\mu}+m\right)^2} + \frac{p}{\left(\frac{1}{\sigma^2}+m\right)^2}\right). \tag{2.2}$$

We will only consider $c < 1$, the case for $c \geq 1$ can be analyzed in a similar way.

On $(0, \infty)$, since $x'(m) > 0$, $x(m)$ is increasing on $(0, \infty)$, and from (2.1), we have $x(m) \downarrow -\infty$ as $m \downarrow 0$, and $x(m) \uparrow 0$ as $m \uparrow \infty$. Therefore, $(-\infty, 0) \subset S_F^c$ and $0 \in S_F$.

On $\left(-\infty, -\frac{1}{\mu}\right)$, for m sufficiently large in the negative direction, $x(m) > 0$ and $x'(m) > 0$. The latter fact and (2.1) gives us $x(m) \downarrow 0$ as $m \to -\infty$. Moreover, for

m near $-\frac{1}{\mu}$, from (2.1), $x(m) < 0$, from (2.2), $x'(m) < 0$, and $x(m) \downarrow -\infty$ as $m \uparrow -\frac{1}{\mu}$. Therefore, $\{m : x'(m) > 0\} = (-\infty, m_1)$ for some $m_1 \in \left(-\infty, -\frac{1}{\mu}\right)$ with $x'(m_1) = 0$, and $(0, x_1) \subset S_F^c$ for $x_1 = x(m_1) > 0$ with $x_1 \in S_F$.

On $\left(-\frac{1}{\sigma^2}, 0\right)$, since

$$0 > m > -\frac{1}{\sigma^2} > -\frac{1}{\mu}, \quad \frac{c}{\frac{1}{\mu} + m} > 0 \quad \text{and} \quad cp\left(\frac{1}{\frac{1}{\sigma^2} + m} - \frac{1}{\frac{1}{\mu} + m}\right) > 0.$$

Thus, from (2.1), $x(m) > 0$. For m near $-\frac{1}{\sigma^2}$ from (2.2), $x'(m) < 0$ and, from (2.1), $x(m) \uparrow \infty$ as $m \downarrow -\frac{1}{\sigma^2}$. Moreover, for m near 0, from (2.2), $x'(m) > 0$ and, from (2.1), $x(m) \uparrow \infty$ as $m \uparrow 0$. Therefore, $\{m : x'(m) > 0\} = (m_4, 0)$ for some $m_4 \in \left(-\frac{1}{\sigma^2}, 0\right)$ s.t. $x'(m_4) = 0$ and $(x_4, \infty) \subset S_{F^c}$ for some $x_4 = x(m_4) > x_1$ s.t. $x_4 \in S_F$.

On $\left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right)$, there are two different cases. Let $g(m) = \frac{c(1-p)m^2}{\left(\frac{1}{\mu} + m\right)^2} + \frac{cpm^2}{\left(\frac{1}{\sigma^2} + m\right)^2}$.

If $g(m) < 1$ on $\left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right)$, then $\exists m_2, m_3$ s.t. $(m_2, m_3) \subset \left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right)$, $x'(m_2) = x'(m_3) = 0$, and $x'(m) > 0$, $\forall m \in (m_2, m_3)$. Let $x_2 = x(m_2)$ and $x_3 = x(m_3)$. Then $(x_2, x_3) \subset S_F^c$, $x_2, x_3 \in S_F$ and $x_1 < x_2 < x_3 < x_4$. Therefore, $S_F^c = (-\infty, 0) \cup (0, x_1) \cup (x_2, x_3) \cup (x_4, \infty)$, in other words, $S_F = \{0\} \cup [x_1, x_2] \cup [x_3, x_4]$.

If $g(m) \geq 1$ on $\left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right)$, then $x(m)$ is decreasing on $\left(-\frac{1}{\mu}, -\frac{1}{\sigma^2}\right)$. Therefore, $S_F^c = (-\infty, 0) \cup (0, x_1) \cup (x_4, \infty)$, in other words, $S_F = \{0\} \cup [x_1, x_4]$.

The following argument will give us the density of F. From (2.1), we have, for $x \neq 0$ and $m = m_0(x) \in \mathbb{C}^+$,

$$x = \frac{(c-1)\mu\sigma^2 m^2 + (\mu(c(1-p)-1) + \sigma^2(cp-1))m - 1}{\mu\sigma^2 m\left(\frac{1}{\mu} + m\right)\left(\frac{1}{\sigma^2} + m\right)}.$$

Therefore,

$$m^3 + Am^2 + Bm + C = 0, \tag{2.3}$$

where $A = \frac{x(\mu + \sigma^2) - (c-1)\mu\sigma^2}{x\mu\sigma^2}$, $B = \frac{x - \mu(c(1-p)-1) - \sigma^2(cp-1)}{x\mu\sigma^2}$, $C = \frac{1}{x\mu\sigma^2}$.

Let $\overline{m} = m + \frac{A}{3}$. Then

$$\overline{m}^3 + 3p(x)\overline{m} + 2q(x) = 0, \tag{2.4}$$

where $p(x) = -\frac{A^2}{9} + \frac{B}{3}$ and $q(x) = \frac{A^3}{27} - \frac{AB}{6} + \frac{C}{2}$. Notice the difference between the roots of (2.3) and the roots of (2.4) is a real number.

Let $u(x) = (-q(x) + \sqrt{q^2(x) + p^3(x)})^{1/3}$ and $v(x) = (-q(x) - \sqrt{q^2(x) + p^3(x)})^{1/3}$ with the convention that square and cube roots of complex numbers are the principal roots. Then, from the formula for the cubic equation, we have

$$\overline{m} = u(x) + v(x) \quad \text{or} \quad -\frac{u(x) + v(x)}{2} \pm \frac{\sqrt{3}i}{2}(u(x) - v(x)).$$

Notice, if $q^2(x) + p^3(x) > 0$, then (2.4) will have one real root and two conjugate imaginary roots, if $q^2(x) + p^3(x) = 0$, then (2.4) will have three real roots of which at least two are equal, and, if $q^2(x) + p^3(x) < 0$, then (2.4) will have three different real roots.

Since $m = \overline{m} - \frac{A}{3}$, we have

$$m = \begin{pmatrix} u(x) + v(x) - \dfrac{x(\mu + \sigma^2) - (c-1)\mu\sigma^2}{3x\mu\sigma^2}, \text{ or} \\ -\dfrac{u(x)+v(x)}{2} + \dfrac{\sqrt{3}i}{2}(u(x) - v(x)) - \dfrac{x(\mu + \sigma^2) - (c-1)\mu\sigma^2}{3x\mu\sigma^2}, \text{ or} \\ -\dfrac{u(x)+v(x)}{2} - \dfrac{\sqrt{3}i}{2}(u(x) - v(x)) - \dfrac{x(\mu + \sigma^2) - (c-1)\mu\sigma^2}{3x\mu\sigma^2} \end{pmatrix}.$$

Let $\overline{p}(x) = x^4((q(x))^2 + (p(x))^3)$ and $\overline{q}(x) = x^3(q(x))$. Then $\overline{p}(x)$ is a fourth degree polynomial with negative leading coefficient $-\frac{(\mu - \sigma^2)^2}{108(\mu\sigma^2)^4}$, and $\overline{q}(x)$ is a third degree polynomial. If $\overline{p}(x) < 0$, then all three m's are real and unequal. Therefore, $Im(m) = 0$ and the density $f(x) = 0$ on $\{x \neq 0 : \overline{p}(x) < 0\}$, in other words, $\{x \neq 0 : \overline{p}(x) < 0\} \subset S_F^c$. Moreover, from the graph of (2.1), for $x \neq 0 \in S_F^c$, we have three different m's s.t. $x = x(m)$, and these m's must satisfy (2.3). Therefore, $\{x \neq 0 : \overline{p}(x) < 0\} = (S_F \cup \{0\})^c$.

Now, for $x \neq 0$ s.t. $f(x) > 0$, we must have $Im(m(x)) > 0$. Therefore, $\overline{p}(x) > 0$, and, since $u(x) \geq v(x)$, we must take

$$m = -\frac{u(x)+v(x)}{2} + \frac{\sqrt{3}i}{2}(u(x) - v(x)) - \frac{x(\mu + \sigma^2) - (c-1)\mu\sigma^2}{3x\mu\sigma^2},$$

and hence

$$Im(m(x)) = \frac{\sqrt{3}}{2}(u(x) - v(x)).$$

Therefore, for $x \neq 0$,

$$\begin{aligned} f(x) &= \frac{1}{\pi} Im(m(x)) \\ &= \frac{\sqrt{3}}{2\pi}((-q(x) + \sqrt{q^2(x) + p^3(x)})^{1/3} - (-q(x) - \sqrt{q^2(x) + p^3(x)})^{1/3}) \\ &= \frac{\sqrt{3}}{2\pi}\left(\frac{(\overline{q}(x) + x\sqrt{\overline{p}(x)})^{1/3} - (\overline{q}(x) - x\sqrt{\overline{p}(x)})^{1/3}}{x}\right), \end{aligned}$$

thus

$$f(x) = \frac{\sqrt{3}}{2\pi}\left(\frac{2\sqrt{\overline{p}(x)}}{((\overline{q}(x) + x\sqrt{\overline{p}(x)})^{2/3} + (\overline{q}(x) + x\sqrt{\overline{p}(x)})^{1/3}(\overline{q}(x) - x\sqrt{\overline{p}(x)})^{1/3} + (\overline{q}(x) - x\sqrt{\overline{p}(x)})^{2/3})}\right),$$

for $x \in \{x : \overline{p}(x) > 0\}$, and $f(x) = 0$, otherwise.

Notice, from the early argument in this chapter for S_F, if $g(m) > 1$, then $\{x \neq 0 : \overline{p}(x) > 0\}$ is one interval, and if $g(m) \leq 1$, then $\{x \neq 0 : \overline{p}(x) > 0\}$ is a union of two intervals, i.e., $\{x \neq 0 : \overline{p}(x) > 0\} = (x_1, x_4)$ or $(x_1, x_2) \cup (x_3, x_4)$. Moreover, $f(x)$ resembles square root function near the boundary of the support of F. This is the complete analysis of the case where H takes mass at two distinct positive values.

References

[1] V. A. Marčenko and L. A. Pastur, Distribution of eigenvalues for some sets of random matrices, Mathematics of the USSR-Sbornik 1(4) (1967), 457-483.

[2] J. W. Silverstein and Sang-Il Choi, Analysis of the limiting spectral distribution of large-dimensional random matrices, J. Multivariate Anal. 54(2) (1995), 295-309.

[3] J. W. Silverstein and P. L. Combettes, Signal detection via spectral theory of large dimensional random matrices, IEEE Trans. Signal Process. 40(8) (1992), 2100-2105.

[4] Y. Q. Yin, Limiting spectral distribution for a class of random matrices, J. Multivariate Anal. 20(1) (1986), 50-68.