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ON THE DETERMINANTS OF SQUARE MATRIX VALUED SOLUTIONS OF SCHRÖDINGER SYSTEMS

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Abstract

This paper considers the Wronskian of fundamental solutions, the matrix Wronskian of a pair of fundamental solutions, and coupled and uncoupled fundamental solutions for the Schrödinger differential systems.

1. Introduction

For the Schrödinger system there are two Wronskians: the Wronskian of a fundamental solution which is the determinant of that solution and the Wronskian of a pair of two fundamental solutions which is the determinant of the matrix Wronskian in terms of the derivatives of that matrix solutions.

The Wronskian of a solution vanishes everywhere or never does. Its proof follows that for first order system theory, however we shall give a proof for seeking independence of this work.

For the matrix Wronskian there are many definitions (see, e.g., [1-6]) used for study of the boundness of the eigenfunctions of the given system.

There are many definitions because of seeking to include the most general

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potential (the matrix variable coefficient of the unknown variable of the Schrödinger differential equation system), i.e., the non selfadjoint potentials.

When the potential is selfadjoint, the eigenvalues of the bounded eigenfunctions are the zeroes of the matrix Wronskian and conversely but on other circumstances this may not be true.

The coupled or uncoupled fundamental solutions are independent of any matrix Wronskian so that we can figure out that bounded eigenfunctions may come from zeroes of a matrix Wronskian or where it never vanishes in the non selfadjoint case.

We can characterize the eigenvalues of Schrödinger spectral problem system as zeroes of the limit of their determinants under suitable conditions. This characterization turns out to be independent of any determinant of fundamental solutions (Wronskian).

2. On Some Matrix Eigenfunctions Properties

2.1. The Schrödinger eigenvalue problem system on an open interval

For a no null complex square matrix-valued function $u = u(x) = \| u_{ij}(x) \|$, $a_- < x < a_+$, called the *matrix-valued potential* (a matrix of complex valued functions defined on the open interval $I_0 =]a_-, a_+[$ which is called the *domain of u*), we consider the eigenvalue problem system

$$(\varphi'' + k^2\varphi)(x) = (u\varphi)(x), \quad a_- < x < a_+, \quad (\varphi'' + k^2\varphi \overset{I_0}{\equiv} u\varphi), \tag{1}$$

where k and $\varphi = \varphi(x, k)$ are, respectively, fixed complex numbers (the set of all these numbers conform the spectral complex variable) and the unknown variable φ, also depending upon the eigenvalue k (this dependence will not be explicit throughout the entire work, if it is not necessary), underlies in the space of the two times differentiable complex matrix-valued functions (matrices of two times differentiable complex valued functions).

The solution to the eigenvalue problem system in (1), i.e., the two times differentiable matrix-valued function solving the Schrödinger eigenvalue problem system in I_0 will be called a *matrix eigenfunction* corresponding to the eigenvalue k or just an eigenfunction. Its column size is that of u but its column number is fixed but arbitrary.

Remarks. (1) The size of u is the same at any $a \in I_0$ as well as its column size $n_r(u)$ and its column number $n_c(u)$ in fact $n_c(u) = n_r(u)$.

(2) The extremes $a_\pm$ of the interval $I_0 =]a_-, a_+[$ may be non-finite, i.e., either $a_\pm = \pm\infty$, respectively.

2.2. Linear matrix combinations of matrix-valued function pairs

Let Φ_i, $i = 1, 2$, be complex matrix-valued functions defined on an interval of the real line. An expression $\Phi_1 A_1 + \Phi_2 A_2$, where A_1, A_2 are fixed matrices of suitable size, is called a *linear matrix combination* of the complex matrix-valued functions Φ_i, $i = 1, 2$. The matrix-valued functions $\Phi_i A_i$, $i = 1, 2$, will be called the *matrix multiple* of Φ_i, $i = 1, 2$, and the fixed matrices A_i, $i = 1, 2$, respectively, the matrix coefficients of the linear matrix combination.

Remarks. (1) $\Phi_i A_i$ is a linear combination of columns of Φ_i. Thus the columns of their linear matrix combination are a linear combination of columns of Φ_i, $i = 1, 2$.

(2) The column numbers of Φ_i, $i = 1, 2$, are fixed on the corresponding interval and could be different from each other so that the matrices fixed A_i, $i = 1, 2$ are no square one in general (even if they are a square one, it will not be required that they are invertible).

(3) The column size of the matrix A_i, $n_r(A_i) = n_c(\Phi_i)$ (the column number of the matrix-valued function Φ_i, which is the same at any number in the interval), $i = 1, 2$, respectively. On the other hand, $n_c(A_1) = n_c(A_2) = n_c(\Phi_1 A_1 + \Phi_2 A_2)$ (column number of the given linear matrix combination).

Definitions. (1) Two matrix-valued functions are called *coupled* in an interval if they have a linear matrix combination which is the null matrix constant in the interval but the corresponding matrix coefficients are not all of them the null matrix. In this case, the corresponding matrix coefficients will be called the *coupling matrix coefficients*.

(2) If two matrix-valued functions are not coupled in an interval, then they are called *uncoupled* in that interval.

Remarks. (1) The homogeneous system $\begin{array}{l}\Phi_1 c_1 + \Phi_2 c_2 = 0 \\ \Phi_1' c_1 + \Phi_2' c_2 = 0\end{array}$ in the matrix unknowns c_i, $i = 1, 2$ at any real number in an interval is uniquely solved whenever $2n_r(\Phi_1) > \sum_{i=1}^{2} n_c(\Phi_i)$ (overdetermined case). This implies that Φ_i, $i = 1, 2$, are uncoupled in *I*. Indeed, if $\Phi_1 c_1 + \Phi_2 c_2 \overset{I}{\equiv} 0$, then $\Phi_1' c_1 + \Phi_2' c_2 \overset{I}{\equiv} 0$, then c_i, $i = 1, 2$ conform a solution of the system at any $a \in I$, therefore $c_1 = 0$, $c_2 = 0$, because of the uniqueness.

(2) The above system can be written in the form $\begin{bmatrix}\Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2'\end{bmatrix}\begin{bmatrix}c_1 \\ c_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$. The matrix-valued function $\begin{bmatrix}\Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2'\end{bmatrix}$ is called the *Wronskian matrix*. In the case, no overdetermined, if Φ_i, $i = 1, 2$, are uncoupled in *I*, then the corresponding homogeneous system (Wronskian system) has a unique solution at each $a \in I$.

(3) In the square case, there is just one solution at any real number in the interval if and only if the determinant $\begin{vmatrix}\Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2'\end{vmatrix}$ (called the *Wronskian of* Φ_i, $i = 1, 2$) is no null at any real number in the interval.

(4) It is null at any real number in the interval if and only if there are infinite number of functions of the Wronskian system at any real number in the interval: the Wronskian of Φ_i, $i = 1, 2$ is null at $a \in I$ if and only if there are infinite number of pairs of complex column matrices c_i, $i = 1, 2$ (no both null) such that $\Phi_1 c_1 + \Phi_2 c_2 \,|_a = \Phi_1' c_1 + \Phi_2' c_2 \,|_a = 0$.

(5) A linear matrix combination of eigenfunctions is itself an eigenfunction.

(6) A pair of uncoupled matrix-valued functions generates a linear space which is not the null one. In our case, for some eigenvalues, any corresponding eigenfunction will be written as a linear combination of a suitable eigenfunctions which are necessarily uncoupled.

2.3. Eigenfunctions satisfying an initial value problem system

We shall assume that *I* is an open subinterval of I_0 and introduce a complex matrix-valued function v defined on I_0 which is independent of any eigenfunction or solution or the potential. An eigenfunction of the auxiliary (non-homogeneous differential equation system) eigenvalue problem system with v as the non-homogeneous term will be called a *non-eigenfunction* or just a *solution*. When $v \equiv 0$ in I_0, we recover the eigenvalue problem system in (1). We may replace v by uv or if μ is any matrix-valued function on an interval, we can set $v(x) = \mu(a)$ (v is the constant matrix-valued function of value $\mu(a)$) or $v(x) = (x - a)\mu'(a)$ for a on the interval (in these cases, v is two times differentiable and $v'' \equiv 0$). It can also turn out to be that $v \equiv -\mu$ under certain conditions.

Lemma 1. *A pair of solutions of* $\varphi'' + k^2\varphi \overset{I}{\equiv} u\varphi + v$, $\varphi(a) = \varphi'(a) = 0$ *are the same solutions in* I_0. *In addition, if* $v \equiv 0$ *in* I_0, *then any eigenfunction* $\eta \equiv 0$ *in* I_0.

Corollary. *Let* μ_i, $i = 1, 2$, *be non-eigenfunctions with their Wronskian system being no overdetermined. They are uncoupled in I if and only if their Wronskian system is solved uniquely at some* $a \in I$.

Proof. If the given non-eigenfunctions μ_i, $i = 1, 2$ are uncoupled in I, in view of the above Remarks (2), the Wronskian system is uniquely solved at any $a \in I$. Conversely, assume that for some $a_0 \in I$, there are no both null matrices c_i, $i = 1, 2$, such that $\mu_1 c_1 + \mu_2 c_2 |_{a_0} = \mu_1' c_1 + \mu_2' c_2 |_{a_0} = 0$. By virtue of Lemma 1, $\mu_1 c_1 + \mu_2 c_2 \overset{I}{\equiv} 0$, then the Wronskian system is not uniquely solved at any $a \in I$.

Remark. The Wronskian is null at some $a_0 \in I$ if and only if it is null at any $a \in I$ in view of the above Remarks (4) and Lemma 1.

Proof of Lemma 1. Since two solutions of (homogeneous or non-homogeneous) matrix linear differential equation agree providing their initial conditions coincide, the lemma follows at once.

Proposition. *Assume that v has second derivative* $v'' \equiv 0$ *on* I_0. *Let* μ *be a solution of* $\varphi'' + k^2(\varphi + v) \overset{I}{\equiv} u(\varphi + v)$, $(\varphi + v)|_a = (\varphi + v)'|_a = 0$, *then* $\mu \equiv -v$ *on* I_0.

Proof. Since $v'' \equiv 0,$ the hypothesis on μ implies that $\mu + v$ solves the initial problem system $\varphi'' + k^2\varphi \overset{I}{\equiv} u\varphi,\ \varphi|_a = \varphi'|_a = 0.$ By the second part of Lemma 1 (homogeneous case), we have that $\mu + v \equiv 0$ on I_0 proving the claim.

Proposition. *Let v be as in the above proposition. If η is an eigenfunction of the eigenvalue problem system in* (1) *such that $\eta + v|_a = (\eta + v)'|_a = 0,$ at some $a \in I,$ then $uv \equiv k^2 v$ on $I_0.$*

Remark. For $k = 0,$ $uv \equiv 0$ on $I_0,$ then $v = 0$ if $\det u \neq 0.$

Proof. Since by hypothesis η is an eigenfunction of the eigenvalue problem system in (1) and $v'' \equiv 0$ on $I_0,$ $\eta + v$ solves $\varphi'' + k^2(\varphi + v) \overset{I}{\equiv} u(\varphi + v),\ (\varphi + v)|_a = (\varphi + v)'|_a = 0.$ Thus we can apply the above proposition with $\mu = \eta$ so that $\eta \equiv -v$ on $I_0.$ Therefore, in this case, $-v$ is an eigenfunction of the eigenvalue problem in (1) which implies that $k^2 v \equiv uv$ on I_0 proving the claim. $\square$

If a is an extreme of *I*, then $u(a) = u|_a = \lim_a u.$ Our next result needs that $u = 0$ at some element of $\bar{I} = I \bigcup \{\text{the extremes of } I\}.$ In addition to include the eigenvalue 0, we shall require that $\det u \neq 0$ at some real number in *I*.

Lemma 2. *Assume that the no null potential u satisfies the above both conditions. Then any eigenfunction η of the eigenvalue problem system in* (1) *with either $\eta = 0$ or $\eta' = 0$ at some real number in I must be the null eigenfunction* $(\eta \equiv 0$ *in* $I_0).$

Proof. Let η be an eigenfunction of the eigenvalue problem system in (1) that corresponds to an eigenvalue *k*. Assume that $\eta(a) = 0$ at some $a \in I.$ Set $v(x) = -(x - a)\eta'(a).$ Hence $v'' = 0$ and $\eta + v = (\eta + v)' = 0$ at $x = a.$ In view that η is an eigenfunction of the eigenvalue problem in (1), the above proposition implies that $u\eta'(a) \equiv k^2\eta'(a)$ on $I_0.$ If $k \neq 0,$ the condition $u|_b = 0$ for some $b \in \bar{I}$ implies that $\eta'(a) = 0.$ The additional condition $\det u \neq 0$ at some $b' \in I$ implies the same result for $k = 0$ (see Remark after the last proposition). Hence in view of the second part of the Lemma 1, $\eta \equiv 0$ in I_0, proving the claim in this case.

If $\eta'(a) = 0$, set $v(x) = -\eta(a)$ and proceed as before. □

In the following proposition and Lemma 3, it will be assumed that Wronskian system of the given eigenfunctions η_i, $i = 1, 2$ of the eigenvalue problem system in (1) is solved uniquely at some $a \in I$.

Proposition. *Let* η_i, $i = 0, 1, 2$, *be given eigenfunctions. Assume that there exist complex matrices* A_i, B_i, $i = 1, 2$, *such that* $\begin{matrix} \eta_0(a) = \eta_1(a)A_1 + \eta_2(a)A_2 \\ \eta_0(b) = \eta_1(b)B_1 + \eta_2(b)B_2 \end{matrix}$ *for* $a, b \in I$, *then* $A_i = B_i$, $i = 1, 2$.

Proof. By hypothesis, $\eta_0 - \eta_1 A_1 - \eta_2 A_2$, $\eta_0 - \eta_1 B_1 - \eta_2 B_2$ are eigenfunctions satisfying $\eta_0 - \eta_1 A_1 - \eta_2 A_2 |_a = \eta_0 - \eta_1 B_1 - \eta_2 B_2 |_b = 0$. By Lemma 2, $\eta_0 \overset{I}{\equiv} \eta_1 A_1 + \eta_2 A_2 \overset{I}{\equiv} \eta_1 B_1 + \eta_2 B_2$. Thus $A_i - B_i$, $i = 1, 2$, is a solution for the Wronskian system at any real number of *I*. By hypothesis, it is uniquely solved at some real number of *I*. Hence $A_i = B_i$, $i = 1, 2$.

Lemma 3. *For any eigenfunction* η, *there are unique complex matrices*, A_i, $i = 1, 2$, *such that* $\eta \equiv \eta_1 A_1 + \eta_2 A_2$ *on* I_0.

Proof. Let η, η_i, $i = 1, 2$, be any eigenfunction and a pair of given eigenfunctions, respectively. We shall prove that the system

$$\begin{aligned} \eta(a) &= \eta_1(a)c_1 + \eta_2(a)c_2, \\ \eta'(a) &= \eta_1'(a)c_1 + \eta_2'(a)c_2, \end{aligned} \tag{2}$$

has a solution at some $a \in I$. This non-homogeneous system is either underdetermined or square since the column size of $\eta(a)$ is lower than or equal to the size of c_1 (see the first set of Remarks (3) in this subsection). In the first case, it is granted at least one solution because it is undetermined. In the later, the determinant of the nonhomogeneous system in (2) is the Wronskian which is no null at some $a \in I$ since the square Wronskian system is solved uniquely at some $a \in I$ (see the set of Remarks (3) after the definitions in this subsection) by hypothesis.

Thus we can apply the first part of Lemma 1 with $v \equiv 0$, so that $\eta \equiv \eta_1 A_1 + \eta_2 A_2$ in I_0 for unique complex matrices A_i, $i = 1, 2$ by the above proposition.

3. Fundamental Solutions

Definition 4. The eigenfunctions ϱ_i, $i = 1, 2$ are fundamental solutions on the interval I_0 if and only if for any eigenfunction η, there exist unique complex matrices A_i, $i = 1, 2$ such that $\eta \equiv \varrho_1 A_1 + \varrho_2 A_2$ in I_0.

Theorem 5. *Two eigenfunctions with either overdetermined Wronskian system or nowhere vanishing Wronskian or uncoupled are fundamental solutions.*

Proof. This follows at once in view of Lemma 3 and Remarks (1) after the definitions in Subsection 2.2.

3.1. Square integrable eigenfunctions

Proposition 6. *Assume that there are open intervals* I_i, $i = 1, 2$, *such that* $\int_{I_i} \| \varrho_i c_i \|^2 < +\infty$ *and the set* $\overline{(I_0 - (I_1 \cup I_2))}$ *is bounded and contained in* I_0. *Then* $\int_{I_0} \| \varrho_1 c_1 + \varrho_2 c_2 \|^2 < +\infty$ *if and only if* $\int_{I_0} \| \varrho_i c_i \|^2 < +\infty$, $i = 1, 2$.

Proof. Since the columns of the linear matrix combination $\varrho_1 c_1 + \varrho_2 c_2$ are a linear combination of columns of ϱ_i, $i = 1, 2$, and a column matrix-valued function is square integrable if and only if each entry is a square integrable scalar function, we can assume that c_i, ϱ_i, $i = 1, 2$ are scalars and scalar valued functions, respectively, where for some index *i* the corresponding scalar may be zero. In this case, the corresponding scalar function may not be square integrable.

Assume $\varrho_1 c_1 + \varrho_2 c_2 \in L^2(I_0)$. We shall prove, for example, that $c_1 \varrho_1 \in L^2(I_0)$. We have that $\int_{I_0} | \varrho_1 c_1 |^2 = \int_{I_0 - (I_1 \cup I_2)} | \varrho_1 c_1 |^2 + \int_{I_1 \cup I_2} | \varrho_1 c_1 |^2$. The first integral in the right hand side is finite since the eigenfunction $\eta_1 c_1$ is continuous on the bounded subset $\overline{I_0 - (I_1 \cup I_2)}$ of I_0. Since $\int_{I_1 \cap I_2} | \varrho_1 c_1 |^2 \leq \int_{I_1} | \varrho_1 c_1 |^2 < \infty$, we just need to prove that $\varrho_1 c_1 \in L^2(I_2)$ to show that the latter is also finite, which follows from the facts that $L^2(I_0)$ is a vectorial space, the relation $c_1 \varrho_1 = (c_1 \varrho_1 +$

$c_2\varrho_2) - c_2\varrho_2$ and the hypothesis on each term of the right hand side on it $(I_2 \subsetneqq I_0)$. We can similarly proceed to prove that $\int_{I_0} |\varrho_2 c_2|^2 < +\infty$.

If both $\varrho_i c_i \in L^2(I_0)$, $i = 1, 2$, then $\varrho_1 c_1 + \varrho_2 c_2 \in L^2(I_0)$ because $L^2(I_0)$ is a vectorial space. □

3.2. Coupled case

This case occurs when there are no fundamental solutions (when the Wronskian system is overdetermined this case cannot occur). If the system is square, couple case occurs if and only if the Wronskian identically vanishes but also it may occur in the undetermined case.

Proposition 7. *If* $\int_{I_i} \| \eta_i c_i \|^2 < +\infty$ *and* $\eta_1 c_1 \overset{I_0}{\equiv} \eta_2 c_2$ *for some complex matrices* c_i, $i = 1, 2$ *and* $\overline{I_0 - (I_1 \cup I_2)}$ *is bounded*, *then*

$$\int_{I_0} \| \eta_1 c_1 \|^2 = \int_{I_0} \| \eta_2 c_2 \|^2 < +\infty.$$

Proof. We have that $\int_{I_0} \| \eta_i c_i \|^2 \leq \left(\sum_{i=1}^{2} \int_{I_i} + \int_{I_0 - (I_1 \cup I_2)} + \int_{I_1 \cap I_2} \right) \| \eta_i c_i \|^2$

for $i = 1, 2$. By hypothesis, the result now follows at once from the continuity of η_i and the boundness of $\overline{I_0 - (I_1 \cup I_2)}$. □

4. Eigenfunction Determinants

We shall assume in this section that the eigenfunctions involved are square matrix-valued.

Lemma 8. *The zeroes of an eigenfunction determinant are isolated.*

Proof. Assume that the lemma is false. Thus there will be an integer *m*, a real number sequence $(a_n)_n$ such that determinant $\det(\eta(a_n)) = 0$ for all integers $n \geq 1$, $n \neq m$, with the determinant $\det(\eta(a_m)) \neq 0$, and $\lim_{n \to \infty} a_n = a$. Hence, we have a unitary column matrix c_n with $\eta(a_n) c_n = 0$ for all integers $n \geq 1$. By continuity, $\eta(a) c = 0$, where $c = \lim_{n \to \infty} c_n$. On the other hand, for each *n* there is a linear isometry T_n such that $c = T_n(c_n)$. Thus there is a T_n-eigenvectors basis for the column matrix space. Hence c_n is a linear combination of eigenvectors of T_n which are in the kernel of $\eta(a_n)$. Thus $\eta(a_n)c = 0$. Henceforth $(\eta c)'(a) = \lim_{n\to\infty} \frac{\eta(a_n) - \eta(a)}{a_n - a} c = \lim_{n\to\infty} \frac{0}{a_n - a} c = 0$. Therefore, ηc is a solution whose column matrix values of itself and its derivative at a are the null column matrix. In view of Lemma 1 with $v \equiv 0$, we have that $\eta c \equiv 0$ on the interval, i.e., the determinant $\det(\eta) \equiv 0$ on the interval, in particular, $\det \eta(a_m) = 0$ which contradicts the choice of the integer m. This shows that the claim is true. □

Theorem 9. *Under the assumption on the no null potential u in Lemma* 2, *the eigenfunction determinant det* η *never vanishes on* I_0.

Proof. Assume that the claim is false. Hence there is a real number x_0, where $\eta(x_0)c = 0$, for some no null complex column matrix *c*. In view of Lemma 2 for the eigenfunction ηc, $\eta c \equiv 0$ in I_0, i.e., $\det \eta \equiv 0$ which contradicts the fact that the zeroes of the eigenfunction determinant $\det \eta$ are isolated (the last lemma) proving the claim for any eigenvalue. □

Corollary 10. $\det \eta'$ *never vanishes on* I_0.

Proof. Use Lemma 2 in the case that $\eta'(x_0)c = 0$ and proceed as in the Theorem 9. □

5. The Scalar Case: The Eigenvalue Problem

5.1. A preliminary result

Proposition 11. *Let a and f be a real number of* I_0 *and an absolutely integrable and differentiable complex valued function with bounded derivative on the open interval* $I_a \subset I_0$, *respectively, such that* $f(a) = 0$, $f \neq 0$ *on* $I_a - \{a\}$ *and* $a \in \bar{I}_a$. *Then there is not a real number* $a_0 \in I_a - \{a\}$ *such that* $\lim_a \left(f \int_{a_0} \frac{1}{f^2} \right) = 0$.

Proof. Assume that the issue is false, i.e., $f \int_{a_0} \frac{1}{f^2} \Big|_a = 0$ for some real number $a_0 \in I_a - \{a\}$. By hypothesis, $\int_{a_0} \frac{1}{f^2}$ exists on the open set $I_a - \{a\}$. Hence, if $I_a \ni b < a_0$, we have that $\left|\int_b^x \frac{1}{f^2}\right| = \left|\int_b^{a_0} \frac{1}{f^2} + \int_{a_0}^x \frac{1}{f^2}\right| \le \left|\int_b^{a_0} \frac{1}{f^2}\right| + \left|\int_{a_0}^x \frac{1}{f^2}\right|$. Combining this with $f(x)\int_b^{a_0} \frac{1}{f^2}\Big|_a = 0$ which is true because $f|_a = 0$, we conclude that $f(x)\int_b^x \frac{1}{f^2}\Big|_a = 0$. If $I_a \ni b > a_0$, then $\int_b^x \frac{1}{f^2} = \int_{a_0}^x \frac{1}{f^2} - \int_{a_0}^b \frac{1}{f^2}$. Hence we can proceed as before to obtain that $\left(f\int_b \frac{1}{f^2}\right)\Big|_a = 0$ for any real $b \in I_a$.

Note that $\int_b f = f^3\int_b \frac{1}{f^2} - 3\int_b f'f^2\int_b \frac{1}{f^2}$, by an integration by parts. The above discussion implies that the first term is zero. On the other hand, $|f'|$ is bounded and $|f|$ is integrable on I_a by hypothesis, *f* is bounded on a subinterval of I_a (the latter, if necessary, can be replaced by the former) because $\lim_a f = 0$, implies that the limit as $x \to a$ can pass under second integral sign so that we can conclude that $\int_b^a f = 0$, $b \in I_a$ which, after replacing b by x and then differentiating, implies that $f \equiv 0$ on a subinterval of I_a which contradicts the assumption on *f* on the interval I_a proving the claim. □

5.2. Scalar eigenfunctions coupled

We consider scalar eigenfunctions corresponding to a no null scalar potential u decaying to zero, i.e., $u = 0$ at some of the extremes of I_0.

Theorem 12. *Let* σ, ρ *be scalar eigenfunctions of eigenvalue problem in* (1) *such that* $\sigma|_a = \rho|_a = 0$, *where a is an extreme of* I_0. *If, in addition,* σ *is absolutely integrable on some subinterval of* I_0 *so that one of its extremes is also a, then there is a scalar c such that* $\rho \equiv c\sigma$ *on* I_0.

Proof. Let σ, ρ be scalar eigenfunctions. By virtue of Theorem 9, they never vanish in the interval I_0. In this case, by a straightforward calculation, we can verify that $\sigma\int_{a_0}\sigma^{-2}$ is an eigenfunction. Note

$$\det\begin{pmatrix} \sigma & \sigma\int\frac{1}{\sigma^2} \\ \sigma' & \left(\sigma\int\frac{1}{\sigma^2}\right)' \end{pmatrix} = 1.$$

In view of the theory for linear second order differential equations, they are linear independent, i.e., uncoupled on the interval. Hence $\rho = c\sigma + d\sigma\int\sigma^{-2}$, for some complex numbers c, d. By hypothesis, $0 = \rho - c\sigma|_a = d\sigma\int_{a_0}^{a}\sigma^{-2}$ for any fixed real number a_0. Since $\sigma' = \int_{a_0}\sigma'' + \sigma'(a_0) = \int_{a_0}(-k^2 + u)\sigma + \sigma'(a_0)$ and by hypothesis σ is absolutely integrable on some subinterval of I_0, where a is one of its extremes, we can apply the Proposition 11 to conclude that $\sigma\int_{a_0}^{a}\sigma^{-2}$ cannot be zero, provided that $a_0 \neq a$ belongs to the subinterval, therefore $d = 0$ which implies that $\rho = c\sigma$ proving the claim. □

6. Consequences for Potentials on Real Line or Half Line

6.1. Matrix Jost functions on the real line

If, for example, u is continuous on the real line and satisfies the Faddeev condition $\int_{-\infty}^{+\infty}(1 + |x|)\|u(x)\|dx < +\infty$, where $0 \leq \|u(x)\|$ means any norm on the square matrix space for the square complex matrix $u(x)$, $-\infty < x < +\infty$, then we can assume that the Schrödinger square matrix-valued eigenvalue problem in (1) for the real line possesses eigenfunctions ϕ_+, ψ_+ called the *Jost solutions*. Its determinants never vanish on the real line whenever the potential decays to the null matrix as the spatial variable goes to either $\pm\infty$ and the potential determinant does not identically vanish on the real line in view of Theorem 9.

Theorem 13. *Let u be no null matrix-valued potential with Jost functions*, *either* $u|_{\pm\infty} = 0$ *and* $\det u \neq 0$ *at some real number of the real line*.

(1) *If they are uncoupled*, *then their Wronskian never vanishes in the real line*.

In this case, they are fundamental solutions on the real line.

(2) *They are coupled if and only if their Wronskian identically vanishes on the real line.*

Proof. (1) is a direct consequence of Theorem 5 for the square case. (2) follows from the Corollary of Lemma 1 and Remark after it. □

The Faddeev condition also implies that the Jost functions satisfy the boundary conditions $\lim_{x \to -\infty} e^{ikx}\phi_+ = \lim_{x \to +\infty} e^{-ikx}\psi_+ = \mathbb{I}$ (the matrix identity), respectively.

Corollary. *In addition, assume that they satisfy the boundary conditions*:

(1) *If* $\phi_+ c = \psi_+ d$, *then* $\int_{-\infty}^{+\infty} \| \phi_+ c \|^2 = \int_{-\infty}^{+\infty} \| \psi_+ d \|^2 < +\infty.$

(2) *In order to have a square integrable eigenfunction, the corresponding eigenvalue must be a zero of either* $\det(\lim_{+\infty} \phi_+)$ *or* $\det(\lim_{-\infty} \psi_+)$ *whenever* ϕ_+, ψ_+ *are uncoupled.*

Proof. The boundary conditions imply that the Jost functions ϕ_+, ψ_+ are square integrable at $]-\infty, 1]$, $[1, +\infty[$, respectively. Thus Proposition 7 implies at once the first part.

We can also quote Proposition 6 in view that the Jost functions are fundamental solutions by virtue of Theorem 13, since they are uncoupled by hypothesis. Hence an eigenfunction $\eta = \phi_+ c + \psi_+ d$ is square integrable if and only if $\phi_+ c$, $\psi_+ d$ are square integrable on the real line. Thus $\lim_{+\infty} \phi_+ c = \lim_{-\infty} \psi_+ d = 0.$ Hence one of the determinants $\det \lim_{+\infty} \phi_+$ or $\det \lim_{-\infty} \psi_+$ is null. □

Remark. If, for example, $\lim_{+\infty} \phi_+$, either fails to exist or is different from zero, then $c = 0$ which implies that $\eta = \psi_+ d$ is square integrable. In the other case, $\eta = \phi_+ c$ is square integrable. If both limits fail to exist or are no null (both matrix multiples of the Jost functions are no square integrable), then η is null solution.

6.1.1. The scalar case

Theorem 14. *Let u be a no null scalar potential such that either* $u|_{\pm\infty} = 0$. *If the potential has Jost functions satisfying the boundary conditions, then each of the* *following conditions implies the other three*:

(1) *Either* ϕ_+ *or* ψ_+ *is square integrable*.

(2) *There exists a real number c such that* $\psi_+ \equiv c\phi_+$ *in the real line*.

(3) *Either* $\lim\limits_{-\infty} \psi_+ = 0$ *or* $\lim\limits_{+\infty} \phi_+ = 0$.

(4) *The Wronskian identically vanishes on the real line*.

Proof. ($1 \Rightarrow 2$) If ϕ_+ is square integrable, then $\lim\limits_{+\infty} \phi = 0$. By the boundary conditions, we can apply Theorem 12 to conclude that there is a real number c such that $\psi_+ = c\phi_+$. If ψ_+ is square integrable, the proof is analogous.

($2 \Rightarrow 3$) Assume that $\psi_+ = \phi_+ c$. In view of the boundary conditions, we can apply Corollary of Theorem 13 to conclude that $\psi_+ = \phi_+ c$ is square integrable. Hence $\lim\limits_{-\infty} \psi_+ = \lim\limits_{+\infty} \phi_+ = 0$.

($3 \Rightarrow 1$) Assume, for example, $\lim\limits_{+\infty} \phi_+ = 0$. By the boundary conditions, we also have $\lim\limits_{+\infty} \psi_+ = 0$ and absolutely integrable on an unbounded interval of positive numbers. Thus by virtue of Theorem 12, $\psi_+ = c\phi_+$. Now by Corollary of Theorem 13, we have that ψ_+, ϕ_+ are integrable.

This proves that (1), (2), (3) are equivalent conditions. We already know that (2) and (4) are equivalent (see (2) in Theorem 13), which completes the proof of Theorem 14. □

6.2. The boundary problem on the half line

If, for example, the matrix-valued potential V is continuous on the half line $(0 < x < +\infty)$ satisfying the decaying condition $\int_0^{+\infty} x\| V(x) \| dx < +\infty$, then the Schrödinger eigenvalue problem system in (1) for the half line possesses eigenfunctions E and $E^{(1)}$ (the notation and hypothesis on V have been taken from [1]) bounded by the boundary conditions $\lim\limits_{x\to+\infty} e^{i\lambda x} E(x) = \lim\limits_{x\to+\infty} e^{-i\lambda x} E^{(1)}(x) = \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix.

By the boundary conditions, $\mathbb{I}^{-1} = (\lim\limits_{+\infty} e^{-i\lambda} E^{(1)})^{-1} = \lim\limits_{+\infty} e^{i\lambda} E^{(1)^{-1}}$. Thus if

$Ec_1 + E^{(1)}c_2 \equiv 0$, $\lim_{x\to+\infty} e^{i\lambda x}c_2 \equiv 0$, therefore, $c_2 = 0$ because $\operatorname{Im}\lambda < 0$. Since $\lim_{x\to+\infty} e^{i\lambda x}E(x) = \mathbb{I}$ (the identity matrix), c_1 is also the null matrix. Thus E, $E^{(1)}$ *are fundamental solutions on the half line for any eigenvalue*.

The decaying condition on the matrix potential implies that $\lim_{+\infty} V = 0$ (the null square matrix). Thus in view of Theorem 9 and its Corollary 10, *the fundamental solutions and their derivatives are invertible at each real positive number*. In view of Theorem 4 for any eigenfunction η, there are unique matrices c, d such that $\eta = Ec + E^{(1)}d$. By the boundary conditions, $d = 0$ therefore $\eta = Ec$. Since E is square integrable in the half line whenever the corresponding eigenvalue is non real, *any eigenfunction corresponding to a non real eigenvalue is square integrable on the half line*.

7. Some Consequences

The square integrable matrix-valued eigenfunctions on the real line come from either coupled or uncoupled matrix Jost functions (Theorem 13 and its Corollary) while the square integrable eigenfunctions on the half line come only from those where they are fundamental solutions. The situation in the scalar case is a little different: the scalar eigenfunctions are square integrable on the real line if and only if they are scalar multiple of each other (Theorem 14 and Theorem 13's Corollary) while those in the half line just come from fundamental solutions. In any case, the corresponding eigenvalues can be characterized as zeroes of Jost solutions' determinants.

When the matrix Jost functions are fundamental solutions, i.e., the Wronskian is never vanishing on the real line, then the corresponding eigenvalue is a zero of either $\det(\lim_{+\infty} \phi_+)$ or $\det(\lim_{-\infty} \psi_+)$ (Theorem 13's Corollary) while if they are coupled, i.e., the Wronskian is identically null on the real line, then this eigenvalue is a zero of both of these determinants since they are square integrable on the real line by virtue of Proposition 7 and the boundary conditions.

The eigenvalue, where the Jost functions on the real line are fundamental solutions so that the corresponding eigenfunction is square integrable, i.e., the matrix multiples of their linear matrix combination are square integrable, is a common zero of $\det(\lim_{+\infty} \phi_+)$ and $\det(\lim_{-\infty} \psi_+)$ if and only if matrix coefficients are both no null. The eigenvalue is not a common zero if and only if just one of matrix multiples of the Jost functions $\phi_+ c$ or $\psi_+ d$ is no square integrable, which occurs if and only if one of their determinants is not zero (for example, if $\det(\lim_{+\infty} \phi_+) \neq 0,$ then $c = 0$ and the second matrix multiple is integrable).

The eigenvalue, where the Wronskian is identically null, i.e., where they are coupled, is a common zero of $\det(\lim_{+\infty} \phi_+)$ and $\det(\lim_{-\infty} \psi_+)$ since they are both square integrable in this case.

Since each eigenfunction on the half line with non real eigenvalue is square integrable, to get similar scalar functions of the spectral variable whose zeroes give the eigenvalues for matrix potentials on the half line, it is necessary to introduce the following boundary-initial problem system called the *non-singular problem* [1],

$$\varphi'' + \lambda^2 \varphi = V\varphi, \qquad \varphi(0) = 0, \qquad \lim_{x \to +\infty} e^{i\lambda x} \varphi(x) = d,$$

where d is a column matrix necessarily in the kernel of $E(0)$. The condition at 0^+ is called the *Dirichlet boundary conditions* [5]. In fact, the eigenfunctions solving this problem system are exactly the (square integrable) eigenfunctions whose corresponding eigenvalues are the zeroes of its determinant and conform exactly the point spectrum of the Schrödinger matrix operator $-\partial^2 + V$ on some dense subspace of the square integrable matrix-valued functions on the half line $(\mathbf{L}_2]0, +\infty[)$ [1, 4, 5]. The Schrödinger matrix operator $-\partial^2 + u$ has empty continuous spectrum because the set of eigenvalues corresponding to square integrable eigenfuction is discrete [2] since they are the zeroes of an analytic function. It is finite for self-adjoint matrix potentials (see, e.g., [6]).

The matrix Jost functions on the real line together with their determinants become analytic in a plane region containing all complex numbers with non null positive imaginary part provided that the matrix potential exponentially decays at $\pm\infty$. Since the eigenvalues corresponding to square integrable eigenfunctions are zeroes in the upper half plane of these determinants, they conform a finite set in this case.

The matrix operator $-\partial^2 + u$ defined on the real line, for example, with

$$u = \begin{pmatrix} -U^2 & iU_x \\ iU_x & -U^2 \end{pmatrix}, \quad \begin{pmatrix} -iU\overline{U} & iU_x \\ iU_x & -iU\overline{U} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -(1/4)U_x^2 & -(1/2)U_{xx} \\ (1/2)U_{xx} & -(1/4)U_x^2 \end{pmatrix},$$

where U is a solution on the real line with exponential decay of the KdV, nonlinear Schrödinger or sine-Gordon equation $U_t + 6U^2U_x + U_{xxx} = 0$, $iU_t - U_{xx} + 2\overline{U}U^2 = 0$ or $U_{xx} - U_{tt} = \sin U$ [6], respectively, has empty continuous and finite point spectrum. Therefore it can be now recovered by the inverse scattering method in the reflectionless case (exponential decaying solutions of the three and four dimensional AKNS systems can be also enclosed).

The scalar Jost functions on the real line are always coupled therefore square integrable. Thus the corresponding eigenvalue is a common zero of $\lim\limits_{+\infty} \phi_+$ and $\lim\limits_{-\infty} \psi_+$. This is in contrast to the scalar square integrable eigenfunctions on the half line with Dirichlet boundary conditions since the corresponding Jost functions are fundamental on the half line solutions for any eigenvalue.

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