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SIMPLE PROOFS OF THE FORMULAS OF ROOTS OF ALGEBRAIC EQUATIONS OF DEGREES 4 AND 3

KAZUYUKI HATADA

Department of Mathematics
Faculty of Education
Gifu University
1-1, Yanagido, Gifu City
GIFU 501-1193, Japan
e-mail: hatada@gifu-u.ac.jp

Abstract

We give a comprehensible and quick proof of Euler's (resp. Ferro and Tartaglia's) formula for roots of algebraic equations of degree 4 (resp. 3). We use certain polynomial identities.

1. Roots of Algebraic Equations of Degree 4

Let A be a commutative ring, and let $x, y, z, u \in A$. One has

$$\begin{vmatrix} x & y & z & u \\ y & x & u & z \\ z & u & x & y \\ u & z & y & x \end{vmatrix}$$

$$= u^4 - 2u^2x^2 - 2u^2y^2 - 2u^2z^2 + 8uxyz + x^4 - 2x^2y^2 - 2x^2z^2 + y^4 - 2y^2z^2 + z^4$$

$$= x^4 + (-2y^2 - 2z^2 - 2u^2)x^2 + 8yzux + (u^4 - 2u^2y^2 - 2u^2z^2 + y^4 - 2y^2z^2 + z^4)$$

$$= (x + y + z + u)(x + y - z - u)(x - y + z - u)(x - y - z + u).$$

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We use this. In Section 1, we give our method. We describe Euler's original method in Remark 1. There have been no publications of our method. Let K be a field and let $a, b, c \in K$. Let us treat any polynomial $x^4 + ax^2 + bx + c \in K[x]$. If the system of equations

$$\begin{aligned} a &= -2(y^2 + z^2 + u^2), \\ b &= 8yzu, \\ c &= u^4 - 2u^2y^2 - 2u^2z^2 + y^4 - 2y^2z^2 + z^4 \\ &= (y^2 + z^2 + u^2)^2 - 4(y^2z^2 + z^2u^2 + y^2u^2) \end{aligned} \tag{1}$$

has a solution $(y, z, u) = (y_0, z_0, u_0) \in K'^3$ with some field extension K' of K, we have

$$\begin{aligned} & x^4 + ax^2 + bx + c \\ &= x^4 + (-2y_0^2 - 2z_0^2 - 2u_0^2)x^2 + 8y_0z_0u_0x \\ &\quad + (u_0^4 - 2u_0^2y_0^2 - 2u_0^2z_0^2 + y_0^4 - 2y_0^2z_0^2 + z_0^4) \\ &= (x + y_0 + z_0 + u_0)(x + y_0 - z_0 - u_0)(x - y_0 + z_0 - u_0)(x - y_0 - z_0 + u_0). \end{aligned}$$

Then all the roots of $x^4 + ax^2 + bx + c = 0$ are given by

$$-y_0 - z_0 - u_0, \quad -y_0 + z_0 + u_0, \quad y_0 - z_0 + u_0, \quad y_0 + z_0 - u_0.$$

Assume $\mathrm{char}.K \neq 2$. The equations (1) are equivalent to

$$\begin{aligned} y^2 + z^2 + u^2 &= \frac{-a}{2}, \\ y^2z^2 + z^2u^2 + y^2u^2 &= \frac{1}{16}a^2 - \frac{1}{4}c, \\ yzu &= \frac{b}{8}. \end{aligned} \tag{2}$$

We will find a triple (y_0, z_0, u_0) satisfying (2). Note $y^2z^2u^2 = \left(\frac{b}{8}\right)^2$. We consider

$$T^3 + \frac{a}{2}T^2 + \left(\frac{1}{16}a^2 - \frac{1}{4}c\right)T - \left(\frac{b}{8}\right)^2 = 0. \tag{3}$$

Let α, β, γ denote all the roots of (3).

Case 1 of $b \neq 0$.

Let $\sqrt{\alpha}$ (resp. $\sqrt{\beta}$) denote any square root of α (resp. β). Put $\tau = \dfrac{b}{8\sqrt{\alpha}\sqrt{\beta}}$.

Then $\alpha\beta\tau^2 = \left(\dfrac{b}{8}\right)^2 = \alpha\beta\gamma$. Therefore, $\tau = \sqrt{\gamma}$ or $\tau = -\sqrt{\gamma}$. Hence we can take $(y_0, z_0, u_0) = \left(\sqrt{\alpha}, \sqrt{\beta}, \dfrac{b}{8\sqrt{\alpha}\sqrt{\beta}}\right)$. And all the roots of $x^4 + ax^2 + bx + c = 0$ are gotten.

Case 2 of $b = 0$.

Let $\sqrt{\alpha}$ (resp. $\sqrt{\beta}$) denote any square root of α (resp. β). We may assume $\gamma = 0$. We see $yzu = 0$ is equivalent to $y^2z^2u^2 = 0$. We can take $(y_0, z_0, u_0) = (\sqrt{\alpha}, \sqrt{\beta}, 0)$. And all the roots of $x^4 + ax^2 + bx + c = 0$ are gotten.

Now we treat $X^4 + AX^3 + BX^2 + CX + D = 0$ where $\{A, B, C, D\} \subset K$ and K is a field with char.$K \neq 2$. Put $x = X + \dfrac{A}{4}$. As a polynomial, we have $X^4 + AX^3 + BX^2 + CX + D = x^4 + ax^2 + bx + c$ with $a = \dfrac{-3}{8}A^2 + B$, $b = \dfrac{1}{8}A^3 - \dfrac{1}{2}AB + C$ and $c = \dfrac{-3}{256}A^4 + \dfrac{1}{16}A^2B - \dfrac{1}{4}AC + D$. Let (y_0, z_0, u_0) denote the triple defined above for $x^4 + ax^2 + bx + c = 0$. We have proven

Euler's formula. All the roots of $X^4 + AX^3 + BX^2 + CX + D = 0$ are

$$-\frac{1}{4}A - y_0 - z_0 - u_0,\ -\frac{1}{4}A - y_0 + z_0 + u_0,\ -\frac{1}{4}A + y_0 - z_0 + u_0,\ -\frac{1}{4}A + y_0 + z_0 - u_0.$$

Remark 1. Euler's original method of solving $x^4 + ax^2 + bx + c = 0$ where $\{a, b, c\} \subset K$ and K is a field with char.$K \neq 2$ is as follows. Write $x = r + s + t$. Then one has

$$\begin{aligned} & x^4 + ax^2 + bx + c \\ &= (r^2 + s^2 + t^2)^2 + 4(r^2s^2 + s^2t^2 + t^2r^2) + a(r^2 + s^2 + t^2) + c \\ &\qquad + 4\left(r^2 + s^2 + t^2 + \frac{a}{2}\right)(rs + st + tr) + (8rst + b)(r + s + t) \\ &= 0. \end{aligned} \tag{4}$$

As a sufficient condition satisfied by r, s, t for (4), one has

$$\begin{aligned}&(r^2+s^2+t^2)^2+4(r^2s^2+s^2t^2+t^2r^2)+a(r^2+s^2+t^2)+c=0,\\ &r^2+s^2+t^2=-\frac{a}{2},\\ &rst=-\frac{b}{8}.\end{aligned} \tag{5}$$

From the first equation of (5) one has

$$r^2s^2+s^2t^2+t^2r^2=\frac{a^2}{16}-\frac{c}{4}.$$

Hence $r^2,\ s^2,\ t^2$ are all of the roots of

$$W^3+\frac{a}{2}W^2+\left(\frac{a^2}{16}-\frac{c}{4}\right)W-\frac{b^2}{64}=0.$$

This is the same equation as (3). If a, b, c are distinct letters which are algebraically independent over the prime field of K, then one has distinct four roots of $x^4+ax^2+bx+c=0$. Let α, β, γ denote the roots of (3). Put $s=\pm\sqrt{\beta}$, $t=\pm\sqrt{\gamma}$ and $r=-\frac{b}{8st}$. They satisfy (5). Put

$$S=\left\{-\frac{b}{8st}+s+t \mid s\in\{\pm\sqrt{\beta}\},\ \ t\in\{\pm\sqrt{\gamma}\}\right\}.$$

The number of S is 4. Any element of S satisfies $x^4+ax^2+bx+c=0$. Hence all the roots of $x^4+ax^2+bx+c=0$ are equal to all the elements of S.

Euler's original method has difficulty in finding the polynomial identity

$$\begin{aligned}&(r+s+t)^4+a(r+s+t)^2+b(r+s+t)+c\\ &=(r^2+s^2+t^2)^2+4(r^2s^2+s^2t^2+t^2r^2)+a(r^2+s^2+t^2)+c\\ &\qquad+4\left(r^2+s^2+t^2+\frac{a}{2}\right)(rs+st+tr)+(8rst+b)(r+s+t)\end{aligned}$$

in (4). Therefore our method is more comprehensible than Euler's.

Algebraic equations of degree 4 were solved first by Ferrari. His method was introduced in Cardano's ARS MAGNA. It is described in [1].

Remark 2. In [2] we have given another proof of Euler's formula. In [2], putting $X^4 + AX^3 + BX^2 + CX + D = (X^2 + pX + q)(X^2 + rX + s)$, we have gotten $p^6 - 3Ap^5 + (3A^2 + 2B)p^4 - A(A^2 + 4B)p^3 + (2A^2B + AC + B^2 - 4D)p^2 - A(AC + B^2 - 4D)p - A^2D + ABC - C^2 = 0$. Therefore, if $A = 0$, we have $p^6 + 2Bp^4 + (B^2 - 4D)p^2 - C^2 = 0$. From this we have given Euler's formula.

In [3], by the method of [2] we have shown that the discriminant of $X^4 + AX^3 + BX^2 + CX + D$ is equal to the discriminant of $X^3 + 2aX^2 + (a^2 - 4c)X - b^2$. Here $a = \frac{-3}{8}A^2 + B$, $b = \frac{1}{8}A^3 - \frac{1}{2}AB + C$ and $c = \frac{-3}{256}A^4 + \frac{1}{16}A^2B - \frac{1}{4}AC + D$.

2. Roots of Algebraic Equations of Degree 3

Let A be a commutative ring, and let $x, y, z \in A$. One has

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = x^3 - 3xyz + y^3 + z^3$$

$$= (x + y + z)(x^2 - xy - xz - yz + y^2 + z^2).$$

We use this. In Section 2 we give our method. We describe Ferro and Tartaglia's method in Remark 3. There have been no publications of our method. Let K be a field with $\mathrm{char}.K \neq 3$ and let $p, q \in K$. Let us treat any polynomial $x^3 + px + q \in K[x]$. Let ω denote a primitive cubic root $\in \overline{K}$ of $1 \in K$. $(\omega^2 + \omega + 1 = 0.)$ We have $(x+y+z)(x^2 - xy - xz - yz + y^2 + z^2) = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$. If the system of equations

$$\begin{aligned} p &= -3yz \\ q &= y^3 + z^3 \end{aligned} \tag{6}$$

has a solution $(y, z) = (y_1, z_1) \in K'^2$ with some field extension K' of $K(\omega)$, we have

$$x^3 + px + q = (x + y_1 + z_1)(x + \omega y_1 + \omega^2 z_1)(x + \omega^2 y_1 + \omega z_1).$$

Hence all the solutions of the equation $x^3 + px + q = 0$ are given by

$$-y_1 - z_1, \ \ -\omega y_1 - \omega^2 z_1, \ \ -\omega^2 y_1 - \omega z_1.$$

We will find a pair (y_1, z_1) satisfying (6). By (6), $p^3 = -27y^3z^3$. We consider $t^2 - qt - \frac{p^3}{27} = 0$. Let δ and ε denote all the roots of this. $\left(\delta + \varepsilon = q \text{ and } \delta\varepsilon = -\frac{p^3}{27}.\right)$ Let $\sqrt[3]{\delta}$ (resp. $\sqrt[3]{\varepsilon}$) denote any cubic root of δ (resp. ε). Put $y_1 = \sqrt[3]{\delta}$. There is a unique element z_1 of $\{\sqrt[3]{\varepsilon}, \sqrt[3]{\varepsilon}\omega, \sqrt[3]{\varepsilon}\omega^2\}$ such that $y_1 z_1 = -\frac{p}{3}$. We have (y_1, z_1).

Now we treat $X^3 + PX^2 + QX + R = 0$ where $\{P, Q, R\} \subset K$ and K is a field with $\mathrm{char}.K \neq 3$. Put $x = X + \frac{P}{3}$. As a polynomial we have $X^3 + PX^2 + QX + R = x^3 + px + q$ with $p = Q - \frac{1}{3}P^2$, $q = \frac{2}{27}P^3 - \frac{1}{3}QP + R$. Let (y_1, z_1) denote the pair defined above for $x^3 + px + q = 0$. We have proven

S. Ferro and N. Tartaglia's formula. Let $\mathrm{char}.K \neq 3$. All the roots of $X^3 + PX^2 + QX + R = 0$ are

$$-\frac{1}{3}P - y_1 - z_1, \ \ -\frac{1}{3}P - \omega y_1 - \omega^2 z_1 \text{ and } -\frac{1}{3}P - \omega^2 y_1 - \omega z_1.$$

Remark 3. We explain Ferro and Tartaglia's method to solve $x^3 + px + q = 0$ where $\{p, q\} \subset K$ and K is a field with $\mathrm{char}.K \neq 3$. Let p and q be algebraically independent over the prime field of K. Put $x = u + v$. Then

$$(u + v)^3 + p(u + v) + q = u^3 + v^3 + q + (3uv + p)(u + v) = 0.$$

As a sufficient condition satisfied by u and v for $x^3 + px + q = 0$, one has $u^3 + v^3 + q = 0$ and $3uv = -p$. Therefore $u^3 + v^3 = -q$ and $u^3v^3 = -\frac{p^3}{27}$. Let φ and ψ denote all the two roots of $t^2 + qt - \frac{p^3}{27} = 0$. Let $\alpha = \sqrt[3]{\varphi} =$ any cubic root of φ. Let $\omega \in \overline{K}$ denote any primitive cubic root of unity 1. Let β be a unique element of $\{\sqrt[3]{\psi}, \sqrt[3]{\psi}\omega, \sqrt[3]{\psi}\omega^2\}$ such that $\alpha\beta = -\frac{p}{3}$. Therefore, as $u + v$ one has $\alpha + \beta$, $\alpha\omega + \beta\omega^2$ and $\alpha\omega^2 + \beta\omega$. They are three. Hence they are all the roots of $x^3 + px + q = 0$. This method was introduced in Cardano's book ARS MAGNA. Since this method does not explain why one has to put $x = u + v$, our method given in this section is more comprehensible than Ferro and Tartaglia's.

Remark 4. In [4], the formulas of roots of algebraic equations of degrees 4 and 3 are treated systematically by Field Theory and Galois Theory. The treatment is not so easy as our methods in the present article, but theoretical.

Remark 5. The usual method of solving $x^2 + ax + b = 0$ where $\{a, b\} \subset K$ and K is a field with $\text{char}.K \neq 2$ is regarded as an application of $X^2 - Y^2 = (X + Y)(X - Y)$ since $X^2 + aX + b = \left(X + \frac{a}{2}\right)^2 - \sqrt{\frac{a^2}{4} - b}^{\,2}$.

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