International Journal of Numerical Methods and Applications
Volume 2, Number 1, 2009, Pages 1-46
Published Online: September 14, 2009
This paper is available online at http://www.pphmj.com

ADJOINT VARIABLE METHOD FOR DRAG REDUCTION UNDER OSEEN FLOW

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Abstract

To decrease the drag of an object under Oseen flow, a shape optimization system based on the adjoint variable method is presented. The adjoint variable method is based on a constrainted variational principle, and consists of the state equation, the adjoint equation and the sensitivity equation. Comparing to the initial shape, the optimal shape (rugby ball) can be reduced by about 25% under Oseen equation.

1. Introduction

The study of the optimal shape to reduce the surface force under a constant volume in viscous flow (as shown in Figure 1) started in 1971 [2, 15, 19-21]. Most studies focused on the theoretical formulation. The reduction of the surface force was tried by minimizing the energy dissipation. These optimization problems only could be applied in the case the surface can be described by a continuous function. In 1980's, in order to construct complicated optimal structures which cannot be described by continuous functions, the optimal shape was obtained by using the computer analysis technique [3, 17]. In 2000's, due to advances in high performance

2000 Mathematics Subject Classification: 76D07.

Keywords and phrases: shape optimization, CFD, FEM, adjoint variable method, Oseen equation.

Received July 7, 2009

computing, using complex high-resolution meshes became possible [4, 7, 13]. When using such mesh, besides the main optimization techniques (e.g., the adjoint variable method), auxiliary techniques such as mesh smoothing or deformation become necessary.

The adjoint variable method is a sensitivity analysis method based on the Lagrange-Multiplier Method. The Lagrange-Multiplier Method is based on the calculus of variations under constrains, which is based on the variational principle. In the stationary point of the action I, the first variation δI produces no change even if the variables of the functional (I) would change. In the adjoint variable method, the Lagrange function L is similarly formulated and the optimal shape can be constructed by obtaining the stationary point $(\delta L = 0)$ based on the variational principle [5, 16].

In this paper, the cost function is defined as the surface force and the adjoint variable method based on the continuous sensitivity equation is formulated. The shape optimization algorithm is divided into five phases: the state equation (the Oseen equation), the adjoint equation, the sensitivity equation, the mesh relocation and the constraint volume. The state variable data is saved at every time step in the state equation phases, and the adjoint variable data is saved at every time step in the adjoint equation phases. By using the saved data, the sensitivity is calculated. By using this algorithm containing such techniques, the smooth sensitivity distribution can be constructed and the optimal shape can be robustly converged to form the arbitrary shape. We verify the results obtained using the proposed shape optimization technique under Oseen flow by comparison to the literature (Pironneau's results under Stokes flow) [15].

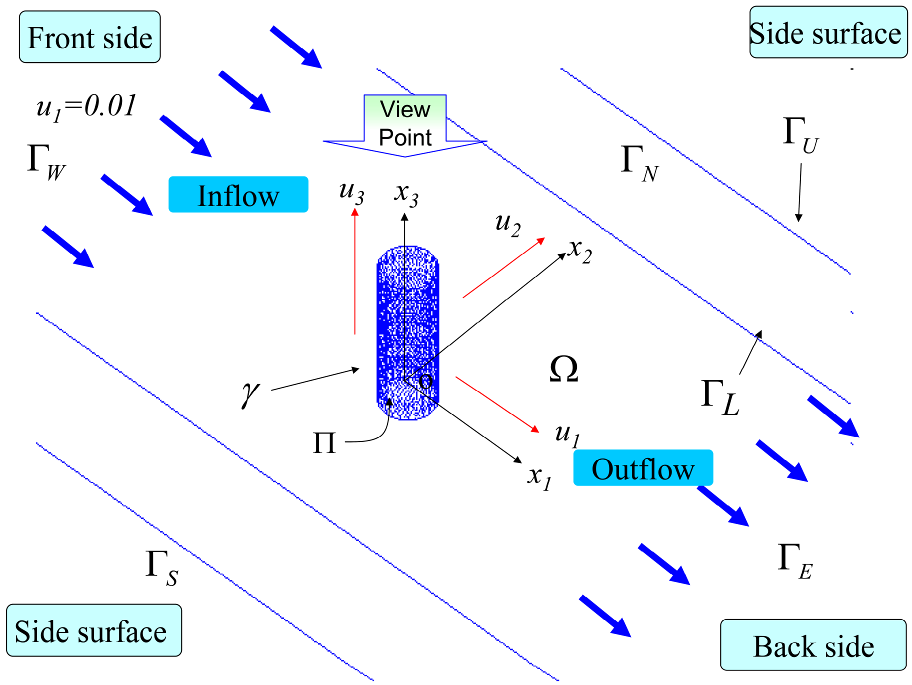

Figure 1. Computational domain and boundary conditions.

2. Adjoint Variable Method

2.1. Problem

To minimize the cost function under constraints, we formulated the Lagrange function by introducing the adjoint variables. The adjoint variable method is based on the variational method. By introducing Lagrange multipliers called adjoint variables, the constrained optimization of the cost function is transformed to the unconstrained optimization of the Lagrange function. A circular cylinder is placed in the computational domain Ω, as shown in Figure 1. Γ is the *N*-*S*-*E*-*W*-*U*-*L* boundary at the north side, the south side, the east side, the west side, the upper side and the lower side. γ represents the surface of the object under optimization. A fluid flows in on the boundary Γ_w and flows out on the boundary Γ_E. The origin of coordinates is at the centre of the cylinder. Here, the equation is defined as follows:

$$\Psi = \Gamma_E + \Gamma_W + \Gamma_S + \Gamma_N + \Gamma_U + \Gamma_L + \gamma = \Gamma + \gamma. \tag{1}$$

The domain $\prod$ shows the internal domain in the object. The variables *t* and

(x_1, x_2, x_3) show the time and the space coordinates, respectively. In this paper, the cost function is defined as:

$$J = -\int_{t_s}^{t_e}\int_{\gamma}\left\{-pn_1 + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right)n_1 + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)n_2\right.$$

$$\left. + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)n_3\right\}d\gamma dt = -\int_{t_s}^{t_e}\int_{\gamma} T_1 d\gamma dt \in \mathbf{R}^1. \tag{2}$$

The cost function is the traction force on the surface γ with respect to x_1 direction. The variable n_i $(i = 1, 2, 3)$ shows the normal vector on the surface. The variables u_1, u_2, u_3 and p show the velocities and the pressure, respectively. The variables T_1, T_2 and T_3 show the tractions with respect to the x_1, x_2 and x_3 directions, respectively. The constants t_s and t_e show the start of the test time and the end of the test time in the optimization. The constant Re represents the Reynolds number as follows:

$$\mathrm{Re} = \frac{\rho L U_1}{\mu}. \tag{3}$$

The constants L, U_1, ρ and μ represent the representative length, the representative flow, the density and the viscosity coefficient, respectively. The equations considered in this paper are dimensionless. We formulated the Lagrange function by introducing the adjoint variables as follows:

$$L = J + B + V + F \in \mathbf{R}^1, \tag{4}$$

$$F = \int_{t_s}^{t_e}\int_{\Omega}\lambda_1(t, \mathbf{x})\, f_1(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))d\Omega dt + \int_{t_s}^{t_e}\int_{\Omega}\lambda_2(t, \mathbf{x})\, f_2(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))d\Omega dt$$

$$+ \int_{t_s}^{t_e}\int_{\Omega}\lambda_3(t, \mathbf{x})\, f_3(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))d\Omega dt$$

$$+ \int_{t_s}^{t_e}\int_{\Omega}\lambda_4(t, \mathbf{x})\, f_4(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))d\Omega dt \in \mathbf{R}^1, \tag{5}$$

$$B = \int_{t_s}^{t_e}\int_{\Gamma_N + \Gamma_s}\lambda_5(t, \mathbf{x})T_1(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_N + \Gamma_S}\lambda_6(t, \mathbf{x})u_2(t, \mathbf{x})d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S}\lambda_7(t,\,\mathbf{x})T_3(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L}\lambda_8(t,\,\mathbf{x})T_1(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L}\lambda_9(t,\,\mathbf{x})T_2(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L}\lambda_{10}(t,\,\mathbf{x})u_3(t,\,\mathbf{x})d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{11}(t,\,\mathbf{x})T_1(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{12}(t,\,\mathbf{x})T_2(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{13}(t,\,\mathbf{x})T_3(t,\,\mathbf{x},\,\mathbf{W}(t,\,\mathbf{x}))d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W}\lambda_{14}(t,\,\mathbf{x})(u_1(t,\,\mathbf{x})-0.01)d\gamma dt+\int_{t_s}^{t_e}\int_{\Gamma_W}\lambda_{15}(t,\,\mathbf{x})u_2(t,\,\mathbf{x})d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W}\lambda_{16}(t,\,\mathbf{x})u_3(t,\,\mathbf{x})d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\lambda_{17}(t,\,\mathbf{x})u_1(t,\,\mathbf{x})d\gamma dt+\int_{t_s}^{t_e}\int_{\gamma}\lambda_{18}(t,\,\mathbf{x})u_2(t,\,\mathbf{x})d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\lambda_{19}(t,\,\mathbf{x})u_3(t,\,\mathbf{x})d\gamma dt\in\mathbf{R}^1, \tag{6}$$

$$V=\kappa\int_{t_s}^{t_e}\int_{\Pi}d\Pi dt\in\mathbf{R}^1, \tag{7}$$

where the variable F shows the constraint function by the governing equation and the variable B shows the constraint function by the boundary condition for the governing equation. The variable λ_1 shows the adjoint pressure corresponding to the pressure p and $\lambda_2\sim\lambda_4$ show the adjoint velocities corresponding to the flow velocities $u_1 \sim u_3$. The variables $\lambda_5 \sim \lambda_{19}$ represent the undetermined adjoint variables. For example, the variables λ_5, λ_6 and λ_7 show the introduced adjoint variables to set the boundary conditions $T_1 = 0$, $u_2 = 0$ and $T_3 = 0$ on the boundary Γ_N and boundary Γ_S in Figure 1, respectively. The control variables used to deform the shape are the coordinates of the node points on the surface in the analytical model. The objective function consists of the cost function (J) and the function (V) which represents the constant volume constraint on the object. The shape is deformed to minimize this objective function. The function $\boldsymbol{f}(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))$, consists of the continuity equation $f_1(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))$ and the Oseen equations $f_2(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))$, $f_3(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))$ and $f_4(t, \mathbf{x}, \mathbf{W}(t, \mathbf{x}))$. The stationary conditions (the state equation, the adjoint equation and the sensitivity equation) are derived by using the first variation. The Lagrange function is formulated as follows:

$$
\begin{aligned}
L = J + B + V &+ \int_{t_s}^{t_e}\int_\Omega \lambda_1\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) d\Omega dt \\
&+ \int_{t_s}^{t_e}\int_\Omega \lambda_2\Bigg[-\frac{\partial u_1}{\partial t} - \frac{\partial p}{\partial x_1} - U_1\frac{\partial u_1}{\partial x_1} - U_2\frac{\partial u_1}{\partial x_2} - U_3\frac{\partial u_1}{\partial x_3} \\
&+ \frac{1}{\mathrm{Re}}\left\{2\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)\right\}\Bigg] d\Omega dt \\
&+ \int_{t_s}^{t_e}\int_\Omega \lambda_3\Bigg\{-\frac{\partial u_2}{\partial t} - \frac{\partial p}{\partial x_2} - U_1\frac{\partial u_2}{\partial x_1} - U_2\frac{\partial u_2}{\partial x_2} - U_3\frac{\partial u_2}{\partial x_3} \\
&+ \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + 2\frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)\right\}\Bigg\} d\Omega dt \\
&+ \int_{t_s}^{t_e}\int_\Omega \lambda_4\Bigg\{-\frac{\partial u_3}{\partial t} - \frac{\partial p}{\partial x_3} - U_1\frac{\partial u_3}{\partial x_1} - U_2\frac{\partial u_3}{\partial x_2} - U_3\frac{\partial u_3}{\partial x_3} \\
&+ \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) + 2\frac{\partial}{\partial x_3}\frac{\partial u_3}{\partial x_3}\right\}\Bigg\} d\Omega dt \in \mathbf{R}^1. \quad (8)
\end{aligned}
$$

The constants U_1, U_2 and U_3 denote the representative flow (constants). In order to derive the adjoint equation and the sensitivity equation, equation (8) can be transformed as follows (see appendix A for details regarding this transformation):

$$
\begin{aligned}
L = & -\int_{t_s}^{t_e}\int_{\gamma} T_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\Omega} p\,\frac{\partial\lambda_{k+1}}{\partial x_k}\,d\Omega dt \\
& + \int_{t_s}^{t_e}\int_{\Omega} u_i\left\{\frac{\partial\lambda_{i+1}}{\partial t} - \frac{\partial\lambda_1}{\partial x_i} + U_j\,\frac{\partial\lambda_{i+1}}{\partial x_j} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_i} + \frac{\partial\lambda_{i+1}}{\partial x_j}\right)\right\} d\Omega dt \\
& + \int_{t_s}^{t_e}\int_{\psi} \lambda_{i+1}T_i d\psi dt - \int_{t_s}^{t_e}\int_{\psi} u_i S_i d\psi dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S} \lambda_5 T_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S} \lambda_6 u_2 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S} \lambda_7 T_3 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L} \lambda_8 T_1 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L} \lambda_9 T_2 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L} \lambda_{10} u_3 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_E} \lambda_{11} T_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_E} \lambda_{12} T_2 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_E} \lambda_{13} T_3 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_W} \lambda_{14} u_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_W} \lambda_{15} u_2 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\Gamma_W} \lambda_{16} u_3 d\gamma dt + \int_{t_s}^{t_e}\int_{\gamma} \lambda_{17} u_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\gamma} \lambda_{18} u_2 d\gamma dt \\
& + \int_{t_s}^{t_e}\int_{\gamma} \lambda_{19} u_3 d\gamma dt - \int_{\Omega} [\lambda_{i+1}u_i]_{t_s}^{t_e} d\Omega + \kappa\int_{t_s}^{t_e}\int_{\prod} d\textstyle\prod dt. \qquad (9)
\end{aligned}
$$

The functions S_1-S_3 are as follows:

$$S_l = \lambda_{l+1}U_j n_j - \lambda_1 n_l + \frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{l+1}}{\partial x_j} + \frac{\partial\lambda_{j+1}}{\partial x_l}\right)n_j = 0, \quad l = 1, 2, 3. \qquad (10)$$

In the case of a subscript being used more than one time in the same term, it should be interpreted according to the summation convention.

2.2. State equation

We express the function that minimizes the Lagrange function (equation (9)) using the adjoint variables $\lambda(t, \boldsymbol{x})$. The comparison function $\Lambda(t, \boldsymbol{x})$ is defined as follows [5, 16]:

$$\Lambda_l(\alpha, t, x_1, x_2, x_3) = \lambda_l(t, x_1, x_2, x_3) + \alpha\eta_l(x_1, x_2, x_3), \quad l = 1, 2, 3, 4 \ \text{ in } \Omega, \tag{11}$$

where α is the parameter and $\eta(\boldsymbol{x})$ is an arbitrary differentiable function. The comparison function $\Lambda(t, \boldsymbol{x})$ represents the domain around the variable λ_l by changing the parameter α. Replacing the variable λ_l by the variable Λ_l, equation (9) becomes $L(\lambda_l + \alpha\eta_l)$. When the parameter α in $L(\lambda_l + \alpha\eta_l)$ is zero, it becomes equivalent with equation (9). When the variable λ_l gives the stationary condition with respect to the Lagrange function L, the function $L(\lambda_l + \alpha\eta_l)$ should satisfy the stationary condition with respect to the arbitrary function η_l. To derive the variable λ_l which gives the extremal value with respect to the function $L(\lambda_l + \alpha\eta_l)$, the function $L(\lambda_l + \alpha\eta_l)$ is differentiated with respect to the parameter α and the parameter α in $L(\lambda_l + \alpha\eta_l)$ is set to zero as follows:

$$\delta L = \left\{\lim_{\alpha\to 0} \frac{L(\lambda_l + \alpha\eta_l) - L(\lambda_l)}{\alpha}\right\}\alpha = \left[\frac{\partial L(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0}, \quad l = 1 \sim 19. \tag{12}$$

The integral (equation (9)) is minimized with respect to α for the value $\alpha = 0$ and $\lambda(t, \boldsymbol{x})$ is the actual minimizing function. All three variables α, t and x are independent. By using the fundamental lemma of the calculus of variations, the state equation is derived from the above equation as follows:

$$\left[\frac{\partial L(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0} = \left[\frac{\partial J(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial B(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial V(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial F(\lambda_l + \alpha\eta_l)}{\partial\alpha}\right]_{\alpha=0}, \quad l = 1 \sim 19. \tag{13}$$

The state equations consist of the continuum and the Oseen equations. The Oseen equations consist of the time derivative term, the convective term, the pressure term and the diffusion term as follows:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \ (= f_1) \ \text{ in } \Omega, \tag{14}$$

$$-\frac{\partial u_1}{\partial t}-\frac{\partial p}{\partial x_1}-\left(U_1\frac{\partial u_1}{\partial x_1}+U_2\frac{\partial u_1}{\partial x_2}+U_3\frac{\partial u_1}{\partial x_3}\right)$$

$$+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)\right\}$$

$$=0\,(=f_2)\ \text{ in }\ \Omega, \tag{15}$$

$$-\frac{\partial u_2}{\partial t}-\frac{\partial p}{\partial x_2}-\left(U_1\frac{\partial u_2}{\partial x_1}+U_2\frac{\partial u_2}{\partial x_2}+U_3\frac{\partial u_2}{\partial x_3}\right)$$

$$+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)\right\}$$

$$=0\,(=f_3)\ \text{ in }\ \Omega, \tag{16}$$

$$-\frac{\partial u_3}{\partial t}-\frac{\partial p}{\partial x_3}-\left(U_1\frac{\partial u_3}{\partial x_1}+U_2\frac{\partial u_3}{\partial x_2}+U_3\frac{\partial u_3}{\partial x_3}\right)$$

$$+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_3}+\frac{\partial u_3}{\partial x_3}\right)\right\}$$

$$=0\,(=f_4)\ \text{ in }\ \Omega. \tag{17}$$

In the state equations, the boundary conditions are shown in Table 1.

Table 1. Boundary conditions

Domains	State equations	Adjoint equations
Γ_W	$u_1 = 0.01,\ u_2 = 0,\ u_3 = 0$	$\lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0$
Γ_E	$\boldsymbol{T} = 0$	$\boldsymbol{S} = 0$
Γ_N, Γ_S	$u_2 = 0,\ T_1 = 0,\ T_3 = 0$	$\lambda_3 = 0,\ S_1 = 0,\ S_3 = 0$
Γ_U, Γ_L	$u_3 = 0,\ T_1 = 0,\ T_2 = 0$	$\lambda_4 = 0,\ S_1 = 0,\ S_2 = 0$
γ	$u_1 = 0,\ u_2 = 0,\ u_3 = 0$	$\lambda_2 = 1,\ \lambda_3 = 0,\ \lambda_4 = 0$

2.3. Adjoint equation

The comparison function $W_l\ (\alpha, t, x_1, x_2, x_3)$ is defined as follows:

$$W_l(\alpha, t, x_1, x_2, x_3) = w_l(t, x_1, x_2, x_3) + \alpha\eta_{l+19}(x_1, x_2, x_3),\quad l = 1, 2, 3, 4\ \text{ in }\ \Omega. \tag{18}$$

The adjoint variable is calculated by solving the adjoint equation, the stationary condition which is obtained by taking the first variation of the Lagrange function with respect to the state variable w_l as follows:

$$\left[\frac{\partial L(w_l + \alpha\eta_{l+19})}{\partial\alpha}\right]_{\alpha=0} = \left[\frac{\partial J(w_l + \alpha\eta_{l+19})}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial B(w_l + \alpha\eta_{l+19})}{\partial\alpha}\right]_{\alpha=0}$$
$$+ \left[\frac{\partial V(w_l + \alpha\eta_{l+19})}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial F(w_l + \alpha\eta_{l+19})}{\partial\alpha}\right]_{\alpha=0}. \quad (19)$$

The adjoint equation is derived as follows (see appendix B for details regarding this transformation):

$$\frac{\partial\lambda_2}{\partial x_1} + \frac{\partial\lambda_3}{\partial x_2} + \frac{\partial\lambda_4}{\partial x_3} = 0 \quad \text{in } \Omega, \quad (20)$$

$$-\frac{\partial\lambda_2}{\partial\tau} - \frac{\partial\lambda_1}{\partial x_1} + U_1\frac{\partial\lambda_2}{\partial x_1} + U_2\frac{\partial\lambda_2}{\partial x_2} + U_3\frac{\partial\lambda_2}{\partial x_3} + \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial\lambda_2}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_1}\right)\right.$$
$$\left. + \frac{\partial}{\partial x_2}\left(\frac{\partial\lambda_3}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\lambda_4}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_3}\right)\right\} = 0 \quad \text{in } \Omega, \quad (21)$$

$$-\frac{\partial\lambda_3}{\partial\tau} - \frac{\partial\lambda_1}{\partial x_2} + U_1\frac{\partial\lambda_3}{\partial x_1} + U_2\frac{\partial\lambda_3}{\partial x_2} + U_3\frac{\partial\lambda_3}{\partial x_3} + \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial\lambda_2}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_1}\right)\right.$$
$$\left. + \frac{\partial}{\partial x_2}\left(\frac{\partial\lambda_3}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\lambda_4}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_3}\right)\right\} = 0 \quad \text{in } \Omega, \quad (22)$$

$$-\frac{\partial\lambda_4}{\partial\tau} - \frac{\partial\lambda_1}{\partial x_3} + U_1\frac{\partial\lambda_4}{\partial x_1} + U_2\frac{\partial\lambda_4}{\partial x_2} + U_3\frac{\partial\lambda_4}{\partial x_3} + \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial\lambda_2}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_1}\right)\right.$$
$$\left. + \frac{\partial}{\partial x_2}\left(\frac{\partial\lambda_3}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\lambda_4}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_3}\right)\right\} = 0 \quad \text{in } \Omega. \quad (23)$$

In the adjoint equations, the boundary conditions are shown in Table 1.

2.4. Sensitivity equation

Adjoint formulations can employ either a discrete or a continuous approach. In the discrete approach, the sensitivity, the gradient of the Lagrange function, must be calculated using only nodal information. Therefore, programming techniques for computing the sensitivities, such as automatic differentiation [12], are demanded. In this study, the adjoint formulation is based on the continuous approach so such methods are not needed. The comparison function $X_l(\alpha, x_1, x_2, x_3)$ is defined as follows:

$$X_l(\alpha,\, x_1,\, x_2,\, x_3) = x_l(x_1,\, x_2,\, x_3) + \alpha\eta_{l+23}(x_1,\, x_2,\, x_3), \quad l = 1,\, 2,\, 3 \text{ in } \Omega. \tag{24}$$

The first variation of the Lagrange function with respect to x represents the sensitivity equation. The first variation is as follows:

$$\left[\frac{\partial L(x_l + \alpha\eta_{l+23})}{\partial\alpha}\right]_{\alpha=0} = \left[\frac{\partial J(x_l + \alpha\eta_{l+23})}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial B(x_l + \alpha\eta_{l+23})}{\partial\alpha}\right]_{\alpha=0}$$
$$+ \left[\frac{\partial F(x_l + \alpha\eta_{l+23})}{\partial\alpha}\right]_{\alpha=0} + \left[\frac{\partial V(x_l + \alpha\eta_{l+23})}{\partial\alpha}\right]_{\alpha=0}$$
$$l = 1,\, 2,\, 3 \in \mathbf{R}^3. \tag{25}$$

The above equation is as following (see appendix C for details regarding this transformation):

$$\left\{\lambda_2(U_1 n_1 + U_2 n_2 + U_3 n_3) + \left(-\lambda_1 + 2\mu\frac{\partial\lambda_2}{\partial x_1}\right)n_1\right.$$
$$\left. + \mu\left(\frac{\partial\lambda_2}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_1}\right)n_2 + \mu\left(\frac{\partial\lambda_2}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_1}\right)n_3\right\}\frac{\partial u_1}{\partial x_1}$$
$$+ \left\{\lambda_3(U_1 n_1 + U_2 n_2 + U_3 n_3) + \mu\left(\frac{\partial\lambda_3}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_2}\right)n_1\right.$$
$$\left. + \left(-\lambda_1 + 2\mu\frac{\partial\lambda_3}{\partial x_2}\right)n_2 + \mu\left(\frac{\partial\lambda_3}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_2}\right)n_3\right\}\frac{\partial u_2}{\partial x_1}$$
$$+ \left\{\lambda_4(U_1 n_1 + U_2 n_2 + U_3 n_3) + \mu\left(\frac{\partial\lambda_4}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_3}\right)n_1\right.$$
$$\left. + \mu\left(\frac{\partial\lambda_4}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_3}\right)n_2 + \left(-\lambda_1 + 2\mu\frac{\partial\lambda_4}{\partial x_3}\right)n_3\right\}\frac{\partial u_3}{\partial x_1} - \kappa n_1 = G_1 \text{ on } \gamma, \tag{26}$$

$$\left\{\lambda_2(U_1 n_1 + U_2 n_2 + U_3 n_3) + \left(-\lambda_1 + 2\mu\frac{\partial\lambda_2}{\partial x_1}\right)n_1\right.$$
$$\left. + \mu\left(\frac{\partial\lambda_2}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_1}\right)n_2 + \mu\left(\frac{\partial\lambda_2}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_1}\right)n_3\right\}\frac{\partial u_1}{\partial x_2}$$
$$+ \left\{\lambda_3(U_1 n_1 + U_2 n_2 + U_3 n_3) + \mu\left(\frac{\partial\lambda_3}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_2}\right)n_1\right.$$
$$\left. + \left(-\lambda_1 + 2\mu\frac{\partial\lambda_3}{\partial x_2}\right)n_2 + \mu\left(\frac{\partial\lambda_3}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_2}\right)n_3\right\}\frac{\partial u_2}{\partial x_2}$$

$$+\left\{\lambda_4(U_1n_1+U_2n_2+U_3n_3)+\mu\left(\frac{\partial\lambda_4}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_3}\right)n_1\right.$$

$$\left.+\mu\left(\frac{\partial\lambda_4}{\partial x_2}+\frac{\partial\lambda_3}{\partial x_3}\right)n_2+\left(-\lambda_1+2\mu\frac{\partial\lambda_4}{\partial x_3}\right)n_3\right\}\frac{\partial u_3}{\partial x_2}-\kappa n_2=G_2 \quad \text{on } \gamma, \tag{27}$$

$$\left\{\lambda_2(U_1n_1+U_2n_2+U_3n_3)+\left(-\lambda_1+2\mu\frac{\partial\lambda_2}{\partial x_1}\right)n_1\right.$$

$$\left.+\mu\left(\frac{\partial\lambda_2}{\partial x_2}+\frac{\partial\lambda_3}{\partial x_1}\right)n_2+\mu\left(\frac{\partial\lambda_2}{\partial x_3}+\frac{\partial\lambda_4}{\partial x_1}\right)n_3\right\}\frac{\partial u_1}{\partial x_3}$$

$$+\left\{\lambda_3(U_1n_1+U_2n_2+U_3n_3)+\mu\left(\frac{\partial\lambda_3}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_2}\right)n_1\right.$$

$$\left.+\left(-\lambda_1+2\mu\frac{\partial\lambda_3}{\partial x_2}\right)n_2+\mu\left(\frac{\partial\lambda_3}{\partial x_3}+\frac{\partial\lambda_4}{\partial x_2}\right)n_3\right\}\frac{\partial u_2}{\partial x_3}$$

$$+\left\{\lambda_4(U_1n_1+U_2n_2+U_3n_3)+\mu\left(\frac{\partial\lambda_4}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_3}\right)n_1\right.$$

$$\left.+\mu\left(\frac{\partial\lambda_4}{\partial x_2}+\frac{\partial\lambda_3}{\partial x_3}\right)n_2+\left(-\lambda_1+2\mu\frac{\partial\lambda_4}{\partial x_3}\right)n_3\right\}\frac{\partial u_3}{\partial x_3}-\kappa n_3=G_3 \quad \text{on } \gamma. \tag{28}$$

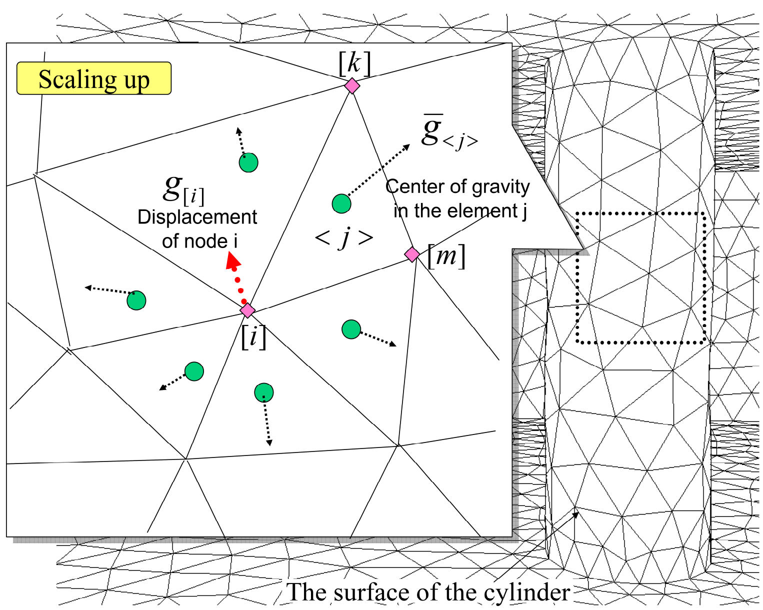

Figure 2. Smoothing on the surface of the object.

3. Techniques Used in the Shape Optimization Algorithm

3.1. Smoothing

In shape optimization, meshes have been mainly analyzed by using 2D low resolution models. Shapes were optimized by controlling the nodes located on the surface. This approach is called the *mesh point approach* [9]. Unfortunately, in the case of 3D high resolution unstructured grids (especially using tetrahedral elements), the mesh point approach does not work so well. As a node on the boundary in a finite element is moved along the sensitivity, the smooth shape is lost and an irregular shape is constructed [14]. As the surface becomes irregular, it partly causes a numerical vibration in the fluid analysis and negative volumes of mesh elements. In this study, the smoothing method is applied to the surface shape, as this method can be easily implemented [1],

$$\bar{g}_{\langle j\rangle,(l)} = \frac{g_{[i],(l)} + g_{[k],(l)} + g_{[m],(l)}}{3}, \quad l = 1, 2, 3, \ldots \text{ on } \gamma, \tag{29}$$

$$g_{[i],(l+1)} = \frac{\sum_j \bar{g}_{\langle j\rangle,(l)} A_{\langle j\rangle}}{\sum_j A_{\langle j\rangle}}, \quad l = 1, 2, 3, \ldots \text{ on } \gamma. \tag{30}$$

The surface is shown in Figure 2. $[i]$, $[k]$, $[m]$ show the node numbers. The smoothing method is one of the methods where the averaged movement amount of an element is converted to the movement amount of a node. The variable $g_{[i]}$ shows the movement amount of a node point $[i]$. The variable $g_{\langle j\rangle}$ represents the movement amount at gravity position of element $\langle j\rangle$. The variable $A_{\langle j\rangle}$ stands for the area of an element $\langle j\rangle$. The lower subscript (l) is the iteration number. By updating the movement amount iteratively $(l = l + 1)$, the deformed surface mesh is constructed.

3.2. The relocation of nodes in a mesh

The relocation of nodes to deform from the initial shape to the optimal shape causes negative volume in some elements around the surface. Therefore, the nodes in the computational domain should be relocated according to the relocation of nodes on the surface. To robustly relocate nodes, the biharmonic equation consisting of the fourth derivative is applied to deforming the mesh [8],

$$\nabla^4 \Theta(\mathbf{x}) = 0 \quad \text{in } \Omega, \tag{31}$$

$$\Theta(\mathbf{x}) = \beta \cdot g_{(k)} \quad \text{on} \quad \gamma, \tag{32}$$

$$\Theta(\mathbf{x}) = 0 \quad \text{on} \quad \Gamma_W, \Gamma_N, \Gamma_E, \Gamma_S. \tag{33}$$

The variable $\Theta(\mathbf{x})$ shows the movement of nodes. The variable $g_{(k)}$ shows the movement on the surface with respect to the shape step (k). β shows the coefficient. If $g_{(k)}$ is large, it often causes negative volume. Therefore, the coefficient β is empirically set to a small value.

3.3. Constant volume constraint

By using only the sensitivity from the initial shape to the optimal shape, due to the volume becoming negative, the object may cause an unrealistic deformation. This problem can be overcome by considering constraints. The function $h(\mathbf{x})$ is defined as follows:

$$h(\mathbf{x}_{(k)}) = V(\mathbf{x}_{(k)}) - V(\mathbf{x}_{(0)}), \quad k = 0, 1, \ldots \in \mathbf{R}^1 \quad \text{on} \quad \gamma, \tag{34}$$

where $V(\mathbf{x}_{(k)})$ represents the volume with respect to the shape step (k). Minimizing the function $h(\mathbf{x})$ means satisfying the constant volume constraint. To minimize $h(\mathbf{x})$ while maintaining the surface shape, the surface shape is deformed along an outward normal vector of the object as follows:

$$\mathbf{x}_{(k),(j+1)} = \mathbf{x}_{(k),(j)} + \kappa\mathbf{n} = \mathbf{x}_{(k),(j)} + \alpha h(\mathbf{x}_{(k),(j)})\mathbf{n}, \quad j = 0, 1, \ldots \in \mathbf{R}^3 \quad \text{on} \quad \gamma. \tag{35}$$

The lower subscript (j) is the iteration number of the mesh deformation. As this (j) is increased, the volume of the deformed shape gets closer to the volume of the initial shape. $\mathbf{n}(\mathbf{x}_{(k)})$ remains constant while the subscript (j) is increased. The deformation amount is set to be small by multiplying a coefficient α in the second term. The constant volume mechanism is shown in Figure 3. In the beginning, the shape is deformed using sensitivity analysis based on the adjoint variable method. In case the deformed volume is smaller than the initial volume, the volume is slowly increased along an outward normal vector of the object surface by expanding the shape. In case that the deformed volume is larger than the initial volume, the deformed volume is slowly decreased along an inward normal vector of the object surface by suppressing the shape. In other words, this algorithm is repeated until the deformed shape is in good agreement with the initial volume.

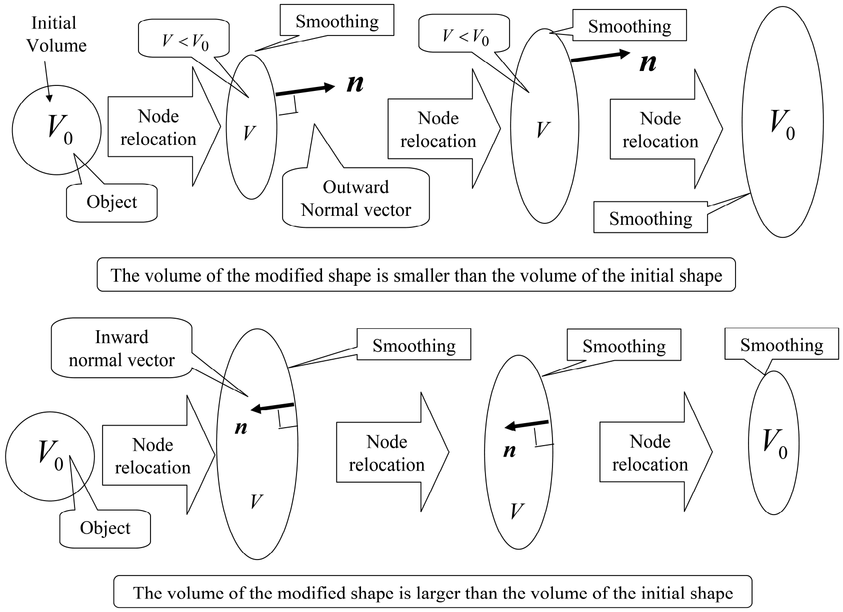

Figure 3. Volume constant.

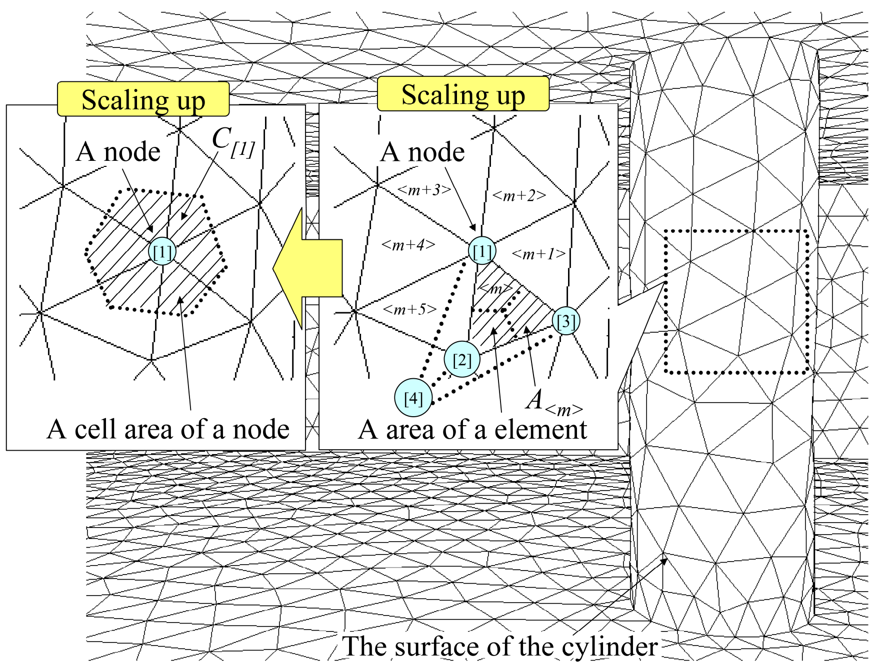

Figure 4. An element (four node tetrahedron) in the mesh.

4. Discretization

In this section, we describe the discretization of the sensitivity equation and the gradient method. More about the discretization of the state equation used here can be found described in literature [22]. The discretization of the adjoint equation is done using the same approach as for the discretization of the state equation.

4.1. Sensitivity equation

In this paper, the discretization method used is the finite element method. The shape function of the first order triangular elements is defined as follows:

$$N_{[i]} = a_{[i]} + b_{[i]}x_1 + c_{[i]}x_2 + d_{[i]}x_3. \tag{36}$$

$[i]$ shows the node number. $a_{[i]}, b_{[i]}, c_{[i]}, d_{[i]}$ show the coefficients (see [22]). The interpolation function of the velocity is defined as follows:

$$u_{l,\langle m\rangle} = N_{[1]}u_{l,[1]} + N_{[2]}u_{l,[2]} + N_{[3]}u_{l,[3]} + N_{[4]}u_{l,[4]}, \quad l = 1, 2, 3. \tag{37}$$

$\langle m\rangle$ shows the number of elements. [1], [2], [3], [4] show the internal numbers of nodes with respect to the number of elements $\langle m\rangle$. The value of the partial differentiation is obtained by equation (37). For example, the value of the partial differentiation u_1 with respect to x_1 is obtained as follows:

$$\left(\frac{\partial u_1}{\partial x_1}\right)_{\langle m\rangle} = b_{[1]}u_{1,[1]} + b_{[2]}u_{1,[2]} + b_{[3]}u_{1,[3]} + b_{[4]}u_{1,[4]}. \tag{38}$$

In the calculation of the sensitivity, the value of the elements has to be transferred to the value of the nodes. The value of the number of element $\langle m\rangle$ is divided by three in proportion to the element area $\langle m\rangle$.

For example, the node [1] is connected to six elements as shown in Figure 4. [1], [2] and [3] show the node numbers on the surface γ. [4] shows the node number in the domain Ω. By selecting the three nodes [1], [2] and [3] related with the surface, the part value of the node [1] can be obtained by the connectivity information. The part value is then added to the number [1], [2] and [3],

$$\left(\frac{\partial u_1}{\partial x_1}\right)_{\langle m\rangle} \cdot \frac{A_{\langle m\rangle}}{3}, \tag{39}$$

where $A_{\langle m\rangle}$ indicates the surface area in element $\langle m\rangle$. If the area of element $\langle m\rangle$ is large, the value of the partial differentiation for element $\langle m\rangle$ is dominant at node [1],

$$\frac{A_{\langle m\rangle}}{3}. \tag{40}$$

The partial differentiation of u_1 with respect to x_1 is obtained as follows:

$$\left(\frac{\partial u_1}{\partial x_1}\right)_{[1]} = \frac{\sum_{n=0}^{5}\left(\frac{\partial u_1}{\partial x_1}\right)_{\langle m+n\rangle}\frac{A_{\langle m+n\rangle}}{3}}{\sum_{n=0}^{5}\frac{A_{\langle m+n\rangle}}{3}}, \tag{41}$$

where the value of the node [1] can be obtained by summing parts of the value after finishing the process equations (39)-(41). The other terms in the sensitivity equations are obtained in a similar way. The sensitivity $\boldsymbol{G}$ at every node is calculated. G_l, for example, is determined as follows:

$$\begin{aligned}
G_{1,[1]} = &\Big\{\lambda_{2,[1]}(U_1 n_{1,[1]} + U_2 n_{2,[1]} + U_3 n_{3,[1]}) - \lambda_{1,[1]} n_{1,[1]} \\
&+ 2\mu\left(\frac{\partial\lambda_2}{\partial x_1}\right)_{[1]} n_{1,[1]} + \mu\left\{\left(\frac{\partial\lambda_2}{\partial x_2}\right)_{[1]} + \left(\frac{\partial\lambda_3}{\partial x_1}\right)_{[1]}\right\} n_{2,[1]} \\
&+ \mu\left\{\left(\frac{\partial\lambda_2}{\partial x_3}\right)_{[1]} + \left(\frac{\partial\lambda_4}{\partial x_1}\right)_{[1]}\right\} n_{3,[1]}\Big\}\left(\frac{\partial u_1}{\partial x_1}\right)_{[1]} + \Big\{\lambda_{3,[1]}(U_1 n_{1,[1]} + U_2 n_{2,[1]} \\
&+ U_3 n_{3,[1]}) - \lambda_{1,[1]} n_{2,[1]} + \mu\left\{\left(\frac{\partial\lambda_2}{\partial x_2}\right)_{[1]} + \left(\frac{\partial\lambda_3}{\partial x_1}\right)_{[1]}\right\} n_{1,[1]} \\
&+ 2\mu\left(\frac{\partial\lambda_3}{\partial x_2}\right)_{[1]} n_{2,[1]} + \mu\left\{\left(\frac{\partial\lambda_3}{\partial x_3}\right)_{[1]} + \left(\frac{\partial\lambda_4}{\partial x_2}\right)_{[1]}\right\} n_{3,[1]}\Big\}\left(\frac{\partial u_2}{\partial x_1}\right)_{[1]} \\
&+ \Big\{\lambda_{4,[1]}(U_1 n_{1,[1]} + U_2 n_{2,[1]} + U_3 n_{3,[1]}) - \lambda_{1,[1]} n_{3,[1]} \\
&+ \mu\left\{\left(\frac{\partial\lambda_2}{\partial x_3}\right)_{[1]} + \left(\frac{\partial\lambda_4}{\partial x_1}\right)_{[1]}\right\} n_{1,[1]} + \mu\left\{\left(\frac{\partial\lambda_4}{\partial x_2}\right)_{[1]} + \left(\frac{\partial\lambda_3}{\partial x_3}\right)_{[1]}\right\} n_{2,[1]} \\
&+ 2\mu\left(\frac{\partial\lambda_4}{\partial x_3}\right)_{[1]} n_{3,[1]}\Big\}\left(\frac{\partial u_3}{\partial x_1}\right)_{[1]}.
\end{aligned} \tag{42}$$

The adjoint variables $\lambda_{1,[1]}$ etc. are calculated by using the saved files containing the adjoint and state variables. This approach is also applied to the drag on the surface and the normal vector on the surface.

4.2. Gradient method

Variables G_1-G_3 from equations (26)-(28) represent the sensitivity. The shape is modified so that the sensitivity becomes zero. The superscripts (1) to (n) show the start of the test time and the end of the test time, respectively. The surface in the shape step $(k+1)$ can be obtained as follows:

$$\begin{bmatrix} x_{l,(k+1),[1]} \\ x_{l,(k+1),[2]} \\ \vdots \\ x_{l,(k+1),[i]} \end{bmatrix} = \begin{bmatrix} x_{l,(k),[1]} \\ x_{l,(k),[2]} \\ \vdots \\ x_{l,(k),[i]} \end{bmatrix} + \begin{bmatrix} \beta(G^{(1)}_{l,(k),[1]} + G^{(2)}_{l,(k),[1]} + \cdots + G^{(n)}_{l,(k),[1]}) \\ \beta(G^{(1)}_{l,(k),[2]} + G^{(2)}_{l,(k),[2]} + \cdots + G^{(n)}_{l,(k),[2]}) \\ \vdots \\ \beta(G^{(1)}_{l,(k),[i]} + G^{(2)}_{l,(k),[i]} + \cdots + G^{(n)}_{l,(k),[i]}) \end{bmatrix}$$

$$= \begin{bmatrix} x_{l,(k),[1]} \\ x_{l,(k),[2]} \\ \vdots \\ x_{l,(k),[i]} \end{bmatrix} + \beta \begin{bmatrix} g_{l,(k),[1]} \\ g_{l,(k),[2]} \\ \vdots \\ g_{l,(k),[i]} \end{bmatrix}, \quad l = 1, 2, 3, \quad k = 0, 1, \ldots \text{ on } \gamma. \tag{43}$$

The value β is decided based on a heuristic search method [18]. $\beta \cdot g$ represents the amount of the movement on the surface. The value β is empirically decided as not to produce negative volumes in the mesh.

5. Algorithm

As described before, we used Oseen's approximation to derive the first variation in the convective term. Before executing the algorithm, the Oseen equations are solved until the flow field reaches steady state.

In the first phase of the algorithm, the state variables ($\boldsymbol{W}$) are calculated by using the state equations. The state equations are solved from the test of start time to the test of end time. All the nodal values of the state variables ($\boldsymbol{W}$) are stored at every time step.

In the second phase of the algorithm, the adjoint variables $(\boldsymbol{\lambda})$ are calculated by equations (20)-(23) from the test of end time to the test of start time. The adjoint equations, which include the time derivative, are also solved until the adjoint flow field reaches the steady state. All the nodal values of the adjoint variables $(\boldsymbol{\lambda})$ are saved at every time step. This data is stored as files.

In the third phase, the sensitivity at every time step is calculated by using the saved files containing the adjoint and state variables. The sensitivity represents the displacement of the nodes on the surface of the object and must have a small value in order to robustly converge to the optimal coordinates and to avoid collapse of the mesh topology.

In the fourth phase, the shape is modified by using the time integral sensitivity. The optimization method is the gradient method. After that, the nodes of the mesh are relocated according to the time integral sensitivity. The node relocation is performed by using the biharmonic equation.

In the fifth phase, the shape is modified in order to satisfy the constraint of constant volume.

In case the shape converges to the optimum, the result is outputted. In the case that the shape does not converge to the optimum, the algorithm returns to the first phase.

6. Shape Optimization Objects in Flow

6.1. Calculation model and conditions

The mesh is shown in Figure 5. The mesh resolution is 15009 nodes and 67855 elements. The radius and the height of the cylinder are 0.5 and 0.3, respectively. We choose the element type to be a 4-node tetrahedron. The $P1$-$P1$ element with linear shape functions for velocity and pressure is used. Therefore, the tractions on the boundary Γ_E are treated as "$p = 0$" and the adjoint tractions on the boundary Γ_E are treated as "$\lambda_1 = 0$" while computing the shape optimization. The inflow velocity in the boundary Γ_E is 0.01. The representative flow is set as $(U_1, U_2, U_3) = (0.01, 0.0, 0.0)$ in the adjoint and sensitivity equations. The test of the start time and the test of the end time are set as 1600(s) and 1601(s), respectively. In the adjoint analysis, the time condition in the test of the end time is set as $\boldsymbol{\lambda}(t_e) = 0$.

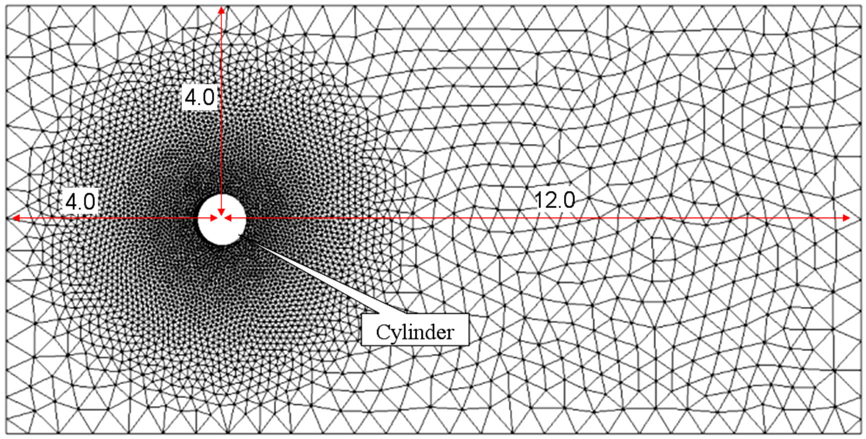

Figure 5. The computational domain and boundary conditions.

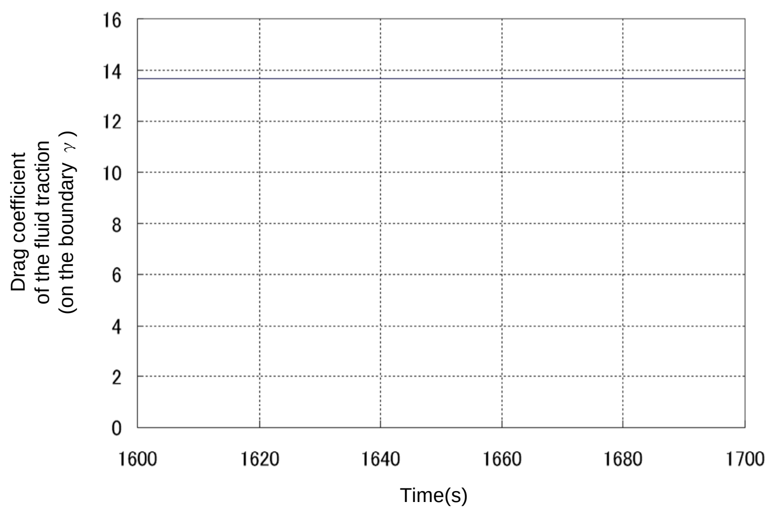

Figure 6. The drag coefficient.

6.2. Calculation results

Figure 6 shows the time history of the surface force (the cylinder). The horizontal axis and the vertical axis show the time and the drag coefficient, respectively. The drag coefficient is constant with respect to time because the flow is slow. This coefficient is 13.65, and this value is almost in agreement with literature [11].

The sensitivity distribution is shown in Figure 7. The sensitivity with respect to the cylinder is not present on the upstream side and the downstream side, while it is present on the side of the cylinder. The sensitivity distribution disappears with the converge to the optimal shape. Figure 7 shows the shape variation with respect to the shape step. In order to reduce the surface force, the cylinder shape deforms to an elliptical shape. Furthermore, as deformation continues, the elliptical shape converges to the optimal shape which has sharp ends both on the upstream and the downstream side. This optimal shape is similar to Pironneau's results [2, 3, 7, 13, 15, 17, 19, 20].

The cost function is shown in Figure 8. The horizontal axis shows the shape step. The vertical axis shows the normalized cost function with respect to the initial cost function. Comparing to the surface force of the initial shape, the surface force of the optimal shape is reduced by 25%. This ratio is almost in agreement with the literature [17].

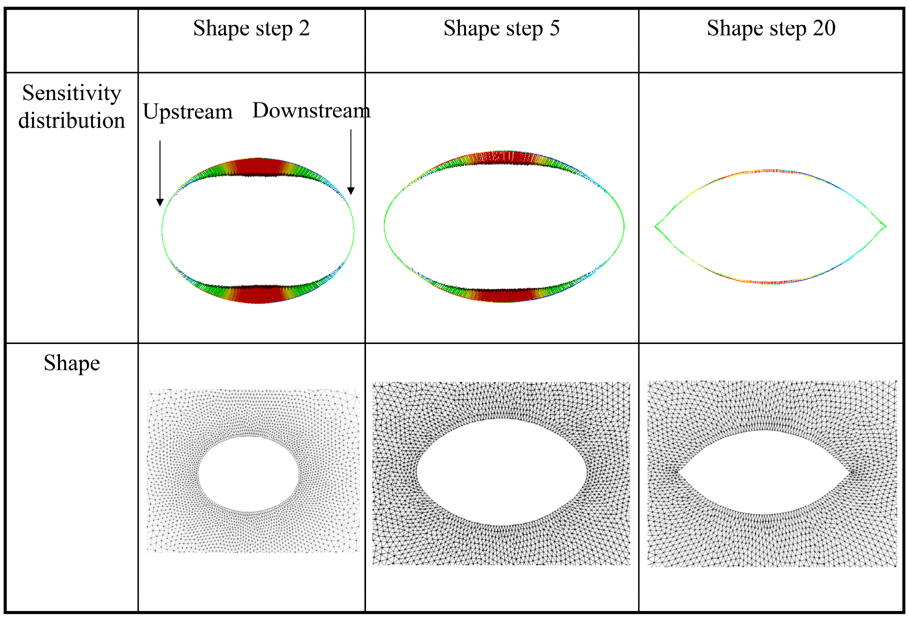

Figure 7. The sensitivity distribution and the shape deformation with respect to shape steps.

7. Conclusion

In order to reduce the surface force under the Oseen equation, the adjoint variable method is formulated. The stationary condition (the state equation, the adjoint equation, the sensitivity equation and the boundary condition) is derived from the Lagrange function. To derive the boundary condition by the first variation, the natural boundary condition and the fundamental boundary condition could be identified by utilizing the arbitrary function η. Under the constraint of the Oseen equation, the optimal shape was successfully constructed by this algorithm. We confirmed that this shape is almost in agreement with the Pironneau's result under Stoke's flow.

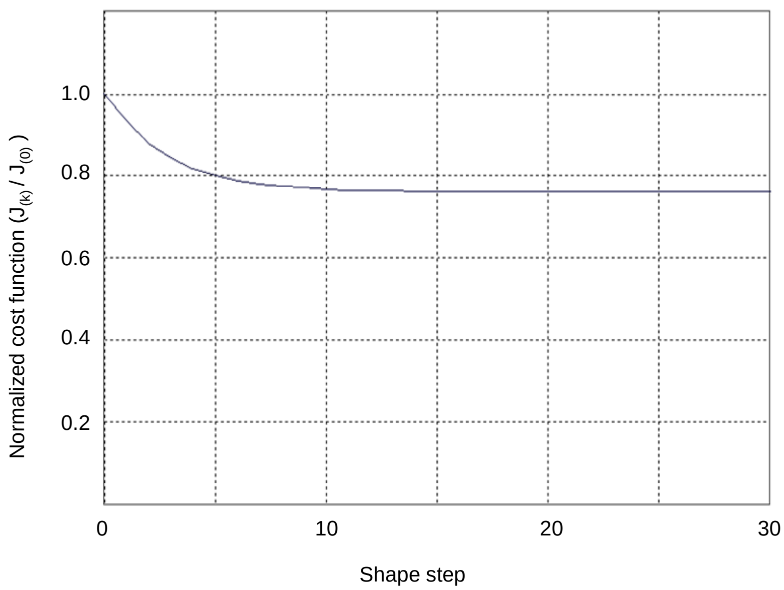

Figure 8. Cost function.

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A. Transformation of the Lagrange Function

The Lagrange function (equation (8)) is transformed to derive the adjoint equation and the sensitivity equation. By applying the Gauss-Green theorem, equation (8) becomes as follows:

$$\int_{t_s}^{t_e}\int_{\Omega}\lambda_1\frac{\partial u_k}{\partial x_k}d\Omega dt = \int_{t_s}^{t_e}\int_{\psi}\lambda_1 u_i n_i d\psi dt - \int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_1}{\partial x_i}u_i d\Omega dt. \tag{44}$$

The time derivative term (the fifth term-the seventh term) in equation (8) is as follows:

$$\int_{t_s}^{t_e}\int_{\Omega}\lambda_{i+1}\left(-\frac{\partial u_i}{\partial t}\right)d\Omega dt = -\int_{\Omega}[\lambda_{i+1}u_i]_{t_s}^{t_e}d\Omega + \int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_{i+1}}{\partial t}u_i d\Omega dt. \tag{45}$$

The pressure term (the fifth term-the seventh term) in equation (8) is as follows:

$$\int_{t_s}^{t_e}\int_{\Omega}\lambda_{i+1}\left(-\frac{\partial p}{\partial x_i}\right)d\Omega dt = -\int_{t_s}^{t_e}\int_{\psi}\lambda_{i+1}pn_i d\psi dt + \int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_{k+1}}{\partial x_k}pd\Omega dt. \tag{46}$$

The convective term (the fifth term-the seventh term) in equation (8) is as follows:

$$-\int_{t_s}^{t_e}\int_{\Omega}\lambda_{i+1}U_j\frac{\partial u_i}{\partial x_j}d\Omega dt = -\int_{t_s}^{t_e}\int_{\psi}\lambda_{i+1}U_j u_i n_j d\psi dt + \int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_{i+1}}{\partial x_j}U_j u_i d\Omega dt. \tag{47}$$

The viscous term (the fifth term-the seventh term) in equation (8) is as follows:

$$\begin{aligned}
&\int_{t_s}^{t_e}\int_{\Omega}\lambda_{i+1}\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)d\Omega dt\\
&= \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\lambda_{i+1}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)n_j d\psi dt - \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_{i+1}}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)d\Omega dt\\
&= \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\lambda_{i+1}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)n_j d\psi dt - \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\frac{\partial \lambda_{i+1}}{\partial x_j}(u_j n_i + u_i n_j)d\psi dt\\
&\quad + \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\Omega}\left(\frac{\partial}{\partial x_i}\frac{\partial \lambda_{i+1}}{\partial x_j}u_j + \frac{\partial}{\partial x_j}\frac{\partial \lambda_{i+1}}{\partial x_j}u_i\right)d\Omega dt.
\end{aligned} \tag{48}$$

The fourth term and the fifth term in equation (8) is as follows:

$$-\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\frac{\partial \lambda_{i+1}}{\partial x_j}(u_i n_j + u_j n_i)d\psi dt$$

$$= -\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\left(\frac{\partial\lambda_{i+1}}{\partial x_j}u_i n_j + \frac{\partial\lambda_{j+1}}{\partial x_i}u_i n_j\right)d\psi dt$$

$$= -\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\left(\frac{\partial\lambda_{i+1}}{\partial x_j} + \frac{\partial\lambda_{j+1}}{\partial x_i}\right)u_i n_j d\psi dt. \tag{49}$$

Equation (9) is derived after the transformation and the arrangement.

B. Derivation of the Adjoint Equations

Using the first variation with respect to the state variable $\boldsymbol{w} = (p, u_1, u_2, u_3)$, equations are derived. These equations are called as *adjoint equations*. The index $l = 1$ in equation (18) is as follows:

$$W_1(\alpha, t, x_1, x_2, x_3) = p(t, x_1, x_2, x_3) + \alpha\eta_{20}(x_1, x_2, x_3). \tag{50}$$

To derive the first variation L with respect to p, the function η_{20} is introduced. This function is the continuity condition, an arbitrary differentiable function which is defined on the set of spatial coordinates over the analytical domain. Its boundary depends on the boundary p. Arbitrary functions are defined with respect to each state variable. In the Lagrange function (equation (8)), the first term in the right side is as follows:

$$\left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\gamma}T_i(p + \alpha\eta_{20})d\gamma dt\right]_{\alpha=0}$$

$$= \left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\gamma}\left\{-(p + \alpha\eta_{20})n_1 + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right)n_j\right\}d\gamma dt\right]_{\alpha=0}. \tag{51}$$

Each variable is independent. Exchange of the order between the integral domain x and the differential α is possible,

$$\left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\gamma}T_1(p + \alpha\eta_{20})d\gamma dt\right]_{\alpha=0}$$

$$= \left[-\int_{t_s}^{t_e}\int_{\gamma}\frac{\partial}{\partial\alpha}\left\{-(p + \alpha\eta_{20})n_1 + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right)n_j\right\}d\gamma dt\right]_{\alpha=0}$$

$$= \int_{t_s}^{t_e}\int_{\gamma}n_1\eta_{20}d\gamma dt. \tag{52}$$

The second term in equation (8) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega}(p+\alpha\eta_{20})\frac{\partial\lambda_{k+1}}{\partial x_k}d\Omega dt\right]_{\alpha=0}$$

$$=\left[\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial\lambda_{k+1}}{\partial x_k}\eta_{20}d\Omega dt\right]_{\alpha=0}=\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial\lambda_{k+1}}{\partial x_k}\eta_{20}d\Omega dt. \tag{53}$$

The other term in equation (8) is also a similar approach. Equations (51)-(53) are summarized as follows:

$$\left[\frac{\partial L(p+\alpha\eta_{20})}{\partial\alpha}\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\gamma}n_1\eta_{20}d\gamma dt+\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial\lambda_{k+1}}{\partial x_k}\eta_{20}d\Omega dt-\int_{t_s}^{t_e}\int_{\psi}\lambda_{i+1}n_i\eta_{20}d\Gamma dt$$

$$-\int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S}\lambda_5 n_1\eta_{20}d\Gamma dt-\int_{t_s}^{t_e}\int_{\Gamma_N+\Gamma_S}\lambda_7 n_3\eta_{20}d\Gamma dt$$

$$-\int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L}\lambda_8 n_1\eta_{20}d\Gamma dt-\int_{t_s}^{t_e}\int_{\Gamma_U+\Gamma_L}\lambda_9 n_2\eta_{20}d\Gamma dt$$

$$-\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{11}n_1\eta_{20}d\Gamma dt-\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{12}n_2\eta_{20}d\Gamma dt$$

$$-\int_{t_s}^{t_e}\int_{\Gamma_E}\lambda_{13}n_3\eta_{20}d\Gamma dt=0. \tag{54}$$

By the fundamental lemma of the calculus of variations, equation (54) becomes as follows:

$$\frac{\partial\lambda_2}{\partial x_1}+\frac{\partial\lambda_3}{\partial x_2}+\frac{\partial\lambda_4}{\partial x_3}=0 \quad \text{in } \Omega, \tag{55}$$

$$n_1-\lambda_2 n_1-\lambda_3 n_2-\lambda_4 n_3=0 \quad \text{on } \gamma. \tag{56}$$

The above equation should be consistently satisfied with respect to the arbitrary surface. Therefore, the boundary γ is set as follows:

$$(\lambda_2 \quad \lambda_3 \quad \lambda_4)=(1 \quad 0 \quad 0) \quad \text{on } \gamma. \tag{57}$$

Similarly, the boundary Γ_E and Γ_S are as follows:

$$-(\lambda_2+\lambda_5)n_1-\lambda_3 n_2-(\lambda_4+\lambda_7)n_3=0 \quad \text{on} \quad \Gamma_N,\, \Gamma_S. \tag{58}$$

The above equation is also as follows:

$$\lambda_3=0, \quad \lambda_5=-\lambda_2, \quad \lambda_7=-\lambda_4 \quad \text{on} \quad \Gamma_N,\, \Gamma_S. \tag{59}$$

The other boundary condition is similarly derived as follows:

$$\lambda_8=-\lambda_2, \quad \lambda_9=-\lambda_3, \quad \lambda_4=0 \quad \text{on} \quad \Gamma_U,\, \Gamma_L, \tag{60}$$

$$\lambda_{11}=-\lambda_2, \quad \lambda_{12}=-\lambda_3, \quad \lambda_{13}=-\lambda_4 \quad \text{on} \quad \Gamma_E, \tag{61}$$

$$\lambda_2=0, \quad \lambda_3=0, \quad \lambda_4=0 \quad \text{on} \quad \Gamma_W. \tag{62}$$

The first variation with respect to u_1 is derived. Equation (18) in the index $1=2$ is as follows:

$$W_2(\alpha,\, x_1,\, x_2,\, x_3)=u_1(t,\, x_1,\, x_2,\, x_3)+\alpha\eta_{21}(x_1,\, x_2,\, x_3) \quad \text{in} \quad \Omega. \tag{63}$$

The function η_{21} is the arbitrary function to take the first variation. However, the function η_{21} has to satisfy $u_1=0$ on the boundary γ, Γ_w. In the boundary γ, the velocity u_1 has to be zero. Therefore, $u_1+\alpha\eta_{21}$ (the perturbation $\alpha\eta_{21}$ is added to u_1) has to be zero as well. In this boundary, W_2 and u_1 are zero and the function η_{21} is naturally derived as follows:

$$\eta_{21}(x_1,\, x_2,\, x_3)=0 \quad \text{on} \quad \gamma,\, \Gamma_W. \tag{64}$$

The same approach is used for the boundary Γ_w. Using the above equation, the first term in equation (9) becomes as follows:

$$\left[\frac{\partial J(u_1+\alpha\eta_{21})}{\partial\alpha}\right]_{\alpha=0}=\left[\frac{\partial J(u_1+\alpha\cdot 0)}{\partial\alpha}\right]_{\alpha=0}=0 \quad \text{on} \quad \gamma. \tag{65}$$

The third term in equation (9) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_\Omega (u_1+\alpha\eta_{21})\left\{\frac{\partial\lambda_2}{\partial t}-\frac{\partial\lambda_1}{\partial x_1}+U_j\frac{\partial\lambda_2}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_j}\right)\right\}d\Omega dt\right]_{\alpha=0}$$

$$+\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_\Omega u_2\left\{\frac{\partial\lambda_3}{\partial t}-\frac{\partial\lambda_1}{\partial x_2}+U_j\frac{\partial\lambda_3}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_2}+\frac{\partial\lambda_3}{\partial x_j}\right)\right\}d\Omega dt\right]_{\alpha=0}$$

$$+\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} u_3\left\{\frac{\partial\lambda_4}{\partial t}-\frac{\partial\lambda_1}{\partial x_3}+U_j\frac{\partial\lambda_4}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_3}+\frac{\partial\lambda_4}{\partial x_j}\right)\right\}d\Omega dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Omega}\left\{\frac{\partial\lambda_2}{\partial t}-\frac{\partial\lambda_1}{\partial x_1}+U_j\frac{\partial\lambda_2}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_j}\right)\right\}\eta_{21}d\Omega dt. \tag{66}$$

The fifth term is as follows:

$$-\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\psi}(u_1+\alpha\eta_{21})S_1 d\Omega dt\right]_{\alpha=0}=-\int_{t_s}^{t_e}\int_{\psi}S_1\eta_{21}d\Omega dt. \tag{67}$$

The function S_1 represents the adjoint traction. The twenty-first term in equation (9) is as follows:

$$-\left[\frac{\partial}{\partial\alpha}\int_{\Omega}[\lambda_2(t,x_1,x_2,x_3)\{u_1(t,x_1,x_2,x_3)+\alpha\eta_{21}(x_1,x_2,x_3)\}\right.$$

$$\left.+\lambda_3(t,x_1,x_2,x_3)u_2(t,x_1,x_2,x_3)+\lambda_4(t,x_1,x_2,x_3)u_3(t,x_1,x_2,x_3)]_{t_s}^{t_e}d\Omega\right]_{\alpha=0}$$

$$=-\left[\int_{\Omega}[\lambda_2(t,x_1,x_2,x_3)\eta_{21}(x_1,x_2,x_3)]_{t_s}^{t_e}d\Omega\right]_{\alpha=0}$$

$$=-\int_{\Omega}\{\lambda_2(t_e,x_1,x_2,x_3)-\lambda_2(t_s,x_1,x_2,x_3)\}\eta_{21}(x_1,x_2,x_3)d\Omega. \tag{68}$$

In a similar way, the other term is obtained by taking the first variation. Using equations (64)-(68), equation (9) is as follows:

$$\left[\frac{\partial L(u_1+\alpha\eta_{21})}{\partial\alpha}\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Omega}\left\{\frac{\partial\lambda_2}{\partial t}-\frac{\partial\lambda_1}{\partial x_1}+U_j\frac{\partial\lambda_2}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_j}\right)\right\}\eta_{21}d\Omega dt$$

$$-\int_{\Omega}\{\lambda_2(t_e,x_1,x_2,x_3)\eta_{21}(x_1,x_2,x_3)-\lambda_2(t_s,x_1,x_2,x_3)\eta_{21}(x_1,x_2,x_3)\}d\Omega$$

$$-\int_{t_s}^{t_e}\int_{\psi}\left\{\lambda_2U_jn_j-\lambda_1n_1+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}+\frac{\partial\lambda_2}{\partial x_j}\right)n_j\right\}\eta_{21}d\psi dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W}\lambda_{14}\eta_{21}d\psi dt+\int_{t_s}^{t_e}\int_{\gamma}\lambda_{17}\eta_{21}d\psi dt. \tag{69}$$

By the fundamental lemma of the calculus of variations, the domain Ω is as follows:

$$\frac{\partial\lambda_2}{\partial t} - \frac{\partial\lambda_1}{\partial x_1} + U_j \frac{\partial\lambda_2}{\partial x_j} + \frac{1}{\mathrm{Re}} \frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_j} \right) = 0 \quad \text{on } \Omega. \tag{70}$$

Equation (70) causes inverse diffusion problems because the viscosity term in equation (70) has the same positive sign as the time derivative [6, 10]. Inverse diffusion problems cause numerical oscillations and cannot converge. For stability reasons [12], the backward time is defined as follows:

$$t = -\tau \quad \text{on } \Omega. \tag{71}$$

The second term is as follows:

$$\lambda_2(t_e, x_1, x_2, x_3) - \lambda_2(t_s, x_1, x_2, x_3) = 0 \quad \text{in } \Omega. \tag{72}$$

Using equation (64), the third term in equation (69) is as follows:

$$\lambda_2 U_i n_i - \lambda_1 n_1 + \frac{1}{\mathrm{Re}}\left\{ \left(\frac{\partial\lambda_2}{\partial x_1} + \frac{\partial\lambda_2}{\partial x_1} \right) n_1 + \left(\frac{\partial\lambda_2}{\partial x_2} + \frac{\partial\lambda_3}{\partial x_1} \right) n_2 + \left(\frac{\partial\lambda_2}{\partial x_3} + \frac{\partial\lambda_4}{\partial x_1} \right) n_3 \right\}$$

$$= S_1 = 0 \quad \text{on } \Gamma_N, \Gamma_S, \Gamma_U, \Gamma_L, \Gamma_E. \tag{73}$$

The other term is derived as follows:

$$\lambda_{14} = 0 \quad \text{on } \Gamma_W, \qquad \lambda_{17} = 0 \quad \text{on } \gamma. \tag{74}$$

The adjoint equation in the domain and the boundary with respect to the variable u_1 are derived. Adjoint equations with respect to the variable u_2 and u_3 are also derived in a similar way.

C. Derivation of the Sensitivity Equation

By using equations (59)-(62), (74) etc., the Lagrange function (equation (9)) becomes as follows:

$$L = -\int_{t_s}^{t_e}\int_\gamma T_1 d\gamma dt + \int_{t_s}^{t_e}\int_\Omega p \frac{\partial\lambda_{k+1}}{\partial x_k} d\Omega dt$$

$$+ \int_{t_s}^{t_e}\int_\Omega u_i \left\{ \frac{\partial\lambda_{i+1}}{\partial t} - \frac{\partial\lambda_1}{\partial x_i} + U_j \frac{\partial\lambda_{i+1}}{\partial x_j} + \frac{1}{\mathrm{Re}} \frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_i} + \frac{\partial\lambda_{i+1}}{\partial x_j} \right) \right\} d\Omega dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_i S_i d\psi dt + \int_{t_s}^{t_e}\int_{\Gamma_W+\gamma} \lambda_2 T_1 d\gamma dt + \int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma} \lambda_3 T_2 d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma} \lambda_4 T_3 d\gamma dt - \int_{\Omega} [\lambda_{i+1} u_i]_{t_s}^{t_e} d\Omega + \kappa \int_{t_s}^{t_e}\int_{\Pi} d\Pi dt \in \mathbf{R}^1. \quad (75)$$

The sensitivity is calculated by solving the sensitivity equations, the stationary conditions which are obtained by taking the first variation of the Lagrange function with respect to the spatial coordinate *x*. In the beginning, the first variation is derived with respect to x_1 as follows:

$$X_1(\alpha, x_1, x_2, x_3) = x_1 + \alpha\eta_{24}(x_1, x_2, x_3), \quad X_2(\alpha, x_1, x_2, x_3) = x_2,$$

$$X_3(\alpha, x_1, x_2, x_3) = x_3 \quad \text{in } \Omega. \quad (76)$$

The boundary Γ is fixed as $X_1 = x_1$. Therefore, the function η_{24} is as follows:

$$\eta_{24}(x_1, x_2, x_3) = 0 \quad \text{on } \Gamma. \quad (77)$$

The first term in (75) is as follows:

$$\left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\gamma} T_1(X_1(\alpha, x_1, x_2, x_3))d\gamma dt\right]_{\alpha=0}$$

$$= \left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\gamma}\left\{-p(X_1(\alpha, x_1, x_2, x_3))n_1(X_1(\alpha, x_1, x_2, x_3))\right.\right.$$

$$+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_i}\right)$$

$$\left.\left.\times n_i(X_1(\alpha, x_1, x_2, x_3))\right\}d\gamma dt\right]_{\alpha=0}. \quad (78)$$

The integral domain γ^*, Ω^*, Ψ^* shows the domain with respect to *X*. The boundary γ^* depends on the parameter α. Exchange of the order between the integral domain *x* and the differential α is impossible. Therefore, an integral domain is converted. The surface domain *d*γ on the object, and the projected areas with respect to the x_1x_2-plane, the x_2x_3-plane and the x_1x_3-plane are shown in Figure 9. The surface domain $d\gamma^*$ containing the perturbation, and the projected areas with respect to the X_1X_2-plane, the X_2X_3-plane and the X_1X_3-plane are shown in Figure 9.

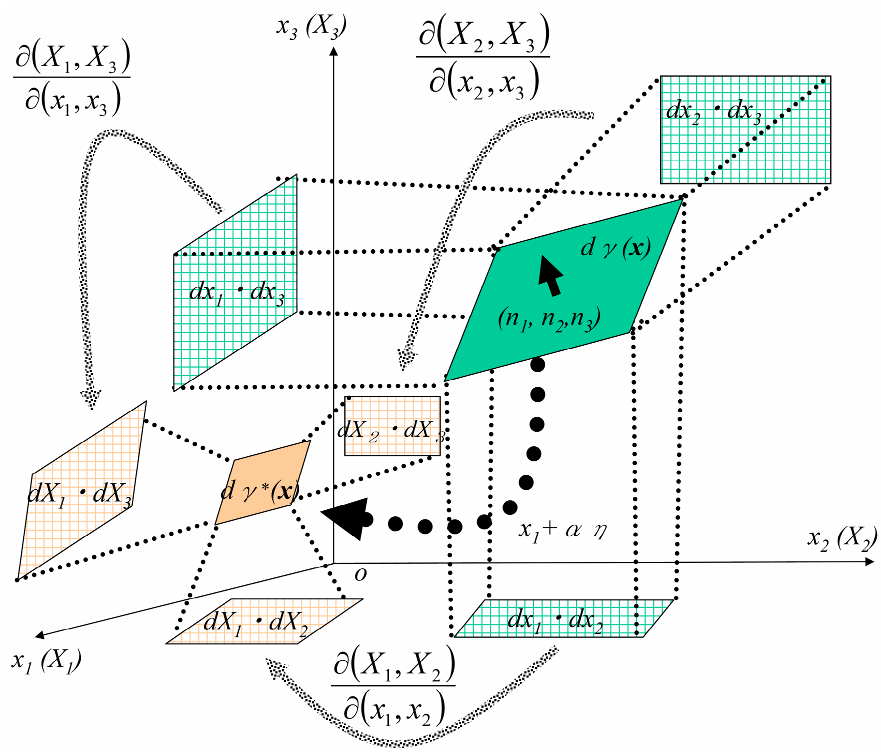

Figure 9. The small surface area γ and the projected area.

The unit normal vector (n_1, n_2, n_3) on the domain $d\gamma$ is defined. The relation becomes as follows:

$$n_1(x_1, x_2, x_3)d\gamma = dx_2dx_3, \quad n_2(x_1, x_2, x_3)d\gamma = dx_3dx_1,$$

$$n_3(x_1, x_2, x_3)d\gamma = dx_1dx_2, \tag{79}$$

$$n_1(x_1 + \alpha\eta_{24}, x_2, x_3)d\gamma^* = dX_2dX_3, \quad n_2(x_1 + \alpha\eta_{24}, x_2, x_3)d\gamma^* = dX_3dX_1,$$

$$n_3(x_1 + \alpha\eta_{24}, x_2, x_3)d\gamma^* = dX_1dX_2. \tag{80}$$

The Jacobian to transform the coordinate from the projected area on the x_3x_1-plane to the X_3X_1-plane is as follows [5, 16]:

$$\left|\frac{\partial(X_3, X_1)}{\partial(x_3, x_1)}\right| = \begin{vmatrix} \frac{\partial X_3}{\partial x_1} & \frac{\partial X_1}{\partial x_1} \\ \frac{\partial X_3}{\partial x_3} & \frac{\partial X_1}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 0 & 1 + \alpha\frac{\partial \eta_{24}}{\partial x_1} \\ 1 & \alpha\frac{\partial \eta_{24}}{\partial x_3} \end{vmatrix} = 1 + \alpha\frac{\partial \eta_{24}}{\partial x_1}. \tag{81}$$

The Jacobian to transform the coordinate from the projected area on the x_1x_2-plane to the X_1X_2-plane is as follows [5, 16]:

$$\left|\frac{\partial(X_1,\, X_2)}{\partial(x_1,\, x_2)}\right| = \begin{vmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_2}{\partial x_1} \\ \dfrac{\partial X_1}{\partial x_2} & \dfrac{\partial X_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 + \alpha \dfrac{\partial \eta_{24}}{x_1} & 0 \\ \alpha \dfrac{\partial \eta_{24}}{\partial x_2} & 1 \end{vmatrix} = 1 + \alpha \frac{\partial \eta_{24}}{\partial x_1}. \tag{82}$$

The Jacobian to transform the coordinate from the projected area on the x_2x_3-plane to the X_2X_3-plane is as follows [5, 16]:

$$\left|\frac{\partial(X_2,\, X_3)}{\partial(x_2,\, x_3)}\right| = \begin{vmatrix} \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_3}{\partial x_2} \\ \dfrac{\partial X_2}{\partial x_3} & \dfrac{\partial X_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{83}$$

Equation (78) is as follows:

$$\left[-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma^*} T_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))d\gamma^* dt\right]_{\alpha=0}$$

$$= \Bigg[-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma^*}\left\{-p(X_1(\alpha,\, x_1,\, x_2,\, x_3)) + \frac{2}{\mathrm{Re}}\frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1}\right\} dX_2 dX_3 dt$$

$$-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma^*}\frac{1}{\mathrm{Re}}\left(\frac{\partial u_2(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_2}\right) dX_3 dX_1 dt$$

$$-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma^*}\frac{1}{\mathrm{Re}}\left(\frac{\partial u_3(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_3}\right) dX_1 dX_2 dt\Bigg]_{\alpha=0}$$

$$= \Bigg[-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma}\Big\{-p(X_1(\alpha,\, x_1,\, x_2,\, x_3))$$

$$+ \frac{2}{\mathrm{Re}}\frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1}\Big\}\left|\frac{\partial(X_2,\, X_3)}{\partial(x_2,\, x_3)}\right| dx_2 dx_3 dt$$

$$-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma}\frac{1}{\mathrm{Re}}\left(\frac{\partial u_2(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_2}\right)$$

$$\times\left|\frac{\partial(X_3,\, X_1)}{\partial(x_3,\, x_1)}\right| dx_3 dx_1 dt - \frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma}\frac{1}{\mathrm{Re}}\Big(\frac{\partial u_3(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_1}$$

$$+ \frac{\partial u_1(X_1(\alpha,\, x_1,\, x_2,\, x_3))}{\partial X_3}\Big)\left|\frac{\partial(X_1,\, X_2)}{\partial(x_1,\, x_2)}\right| dx_1 dx_2 dt\Bigg]_{\alpha=0}$$

$$= \left[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \left\{ -p(X_1(\alpha, x_1, x_2, x_3)) + \frac{2}{\mathrm{Re}} \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \right\} n_1 d\gamma dt \right.$$

$$- \frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left\{ \frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_2} \right\}$$

$$\times \left(1 + \alpha \frac{\partial \eta_{24}}{\partial x_1} \right) n_2 d\gamma dt - \frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left\{ \frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \right.$$

$$\left. \left. + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_3} \right\} \left(1 + \alpha \frac{\partial \eta_{24}}{\partial x_1} \right) n_3 d\gamma dt \right]_{\alpha = 0}. \tag{84}$$

The first term in (84) is as follows:

$$\left[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \left\{ -p(X_1(\alpha, x_1, x_2, x_3)) \right. \right.$$

$$\left. \left. + \frac{2}{\mathrm{Re}} \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \right\} n_1(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \right]_{\alpha = 0}$$

$$= \left[-\int_{t_s}^{t_e} \int_{\gamma} \left\{ -\frac{\partial p(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha} \right. \right.$$

$$\left. + \frac{2}{\mathrm{Re}} \frac{\partial}{\partial X_1} \left(\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha} \right) \right\}$$

$$\left. \times n_1(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \right]_{\alpha = 0} + \left[-\int_{t_s}^{t_e} \int_{\gamma} \left\{ -p(X_1(\alpha, x_1, x_2, x_3)) + \frac{2}{\mathrm{Re}} \right. \right.$$

$$\left. \left. \times \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \right\} \frac{\partial n_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha} d\gamma dt \right]_{\alpha = 0}. \tag{85}$$

The second term in equation (84) is as follows:

$$\left[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left(\frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_2} \right) \right.$$

$$\left. \times \left(1 + \alpha \frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1} \right) n_2(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \right]_{\alpha = 0}$$

$$= \left[-\int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left\{ \frac{\partial}{\partial X_1} \left(\frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \frac{\partial X_1(X_1(\alpha, x_1, x_2, x_3))}{\partial \alpha} \right) \right. \right.$$

$$+\frac{\partial}{\partial X_2}\left(\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}\frac{\partial X_1(X_1(\alpha, x_1, x_2, x_3))}{\partial \alpha}\right)\Bigg\}\left(1+\alpha\frac{\partial \eta_{24}}{\partial x_1}\right)n_2 d\gamma dt\Bigg]_{\alpha=0}$$

$$+\Bigg[-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}+\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_2}\right)$$

$$\times\left(1+\alpha\frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1}\right)\frac{\partial n_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}\frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha}d\gamma dt\Bigg]_{\alpha=0}$$

$$+\Bigg[-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}+\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_2}\right)$$

$$\times\frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1}n_2(X_1(\alpha, x_1, x_2, x_3))d\gamma dt\Bigg]_{\alpha=0}. \tag{86}$$

The third term in equation (84) is as follows:

$$\Bigg[-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}+\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_3}\right)$$

$$\times\left(1+\alpha\frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1}\right)n_3(X_1(\alpha, x_1, x_2, x_3))d\gamma dt\Bigg]_{\alpha=0}$$

$$=\Bigg[-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\Bigg\{\frac{\partial}{\partial X_1}\left(\frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}\frac{\partial X_1(X_1(\alpha, x_1, x_2, x_3))}{\partial \alpha}\right)$$

$$+\frac{\partial}{\partial X_3}\left(\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}\frac{\partial X_1(X_1(\alpha, x_1, x_2, x_3))}{\partial \alpha}\right)\Bigg\}\left(1+\alpha\frac{\partial \eta_{24}}{\partial x_1}\right)n_3 d\gamma dt\Bigg]_{\alpha=0}$$

$$+\Bigg[-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}+\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_3}\right)$$

$$\times\left(1+\alpha\frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1}\right)\times\frac{\partial n_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}$$

$$\times\frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha}d\gamma dt\Bigg]_{\alpha=0}$$

$$+\Bigg[-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}+\frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_3}\right)$$

$$\times\frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1}n_3(X_1(\alpha, x_1, x_2, x_3))d\gamma dt\Bigg]_{\alpha=0}. \tag{87}$$

Using equation (76), it is derived as follows:

$$\frac{\partial X_1(\alpha, x_1, x_2, x_3)}{\partial \alpha} = \frac{\partial}{\partial \alpha}(x_1 + \alpha \eta_{24}(x_1, x_2, x_3)) = \eta_{24}(x_1, x_2, x_3) \text{ in } \Omega. \quad (88)$$

Equation (76) at $\alpha = 0$ is as follows:

$$X_l(0, x_1, x_2, x_3) = x_l, \quad l = 1, 2, 3 \text{ in } \Omega. \quad (89)$$

Equation (85) is as follows:

$$\Bigg[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \Bigg\{ -p(X_1(\alpha, x_1, x_2, x_3)) + \frac{2}{\mathrm{Re}} \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} \Bigg\} n_1(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \Bigg]_{\alpha=0}$$

$$= -\int_{t_s}^{t_e} \int_{\gamma} \left\{ -\frac{\partial p}{\partial x_1} \eta_{24} + \frac{2}{\mathrm{Re}} \frac{\partial}{\partial x_1} \left(\frac{\partial u_1}{\partial x_1} \eta_{24} \right) \right\} n_1 d\gamma dt$$

$$- \int_{t_s}^{t_e} \int_{\gamma} \left\{ -p + \frac{2}{\mathrm{Re}} \frac{\partial u_1}{\partial x_1} \right\} \frac{\partial n_1}{\partial x_1} \eta_{24} d\gamma dt. \quad (90)$$

Equation (86) is as follows:

$$\Bigg[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left(\frac{\partial u_2(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_2} \right) \times \left(1 + \alpha \frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1} \right) n_2(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \Bigg]_{\alpha=0}$$

$$= -\int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left\{ \frac{\partial}{\partial x_1} \left(\frac{\partial u_2}{\partial x_1} \eta_{24} \right) + \frac{\partial}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} \eta_{24} \right) \right\} n_2 d\gamma dt$$

$$- \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \frac{\partial n_2}{\partial x_1} \eta_{24} d\gamma dt - \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \frac{\partial \eta_{24}}{\partial x_1} n_2 d\gamma dt. \quad (91)$$

Equation (87) is as follows:

$$\Bigg[-\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e} \int_{\gamma} \frac{1}{\mathrm{Re}} \left(\frac{\partial u_3(X_1(\alpha, x_1, x_2, x_3))}{\partial X_1} + \frac{\partial u_1(X_1(\alpha, x_1, x_2, x_3))}{\partial X_3} \right) \times \left(1 + \alpha \frac{\partial \eta_{24}(X_1(\alpha, x_1, x_2, x_3))}{\partial x_1} \right) n_3(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt \Bigg]_{\alpha=0}$$

$$= -\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial x_1}\eta_{24}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}\eta_{24}\right)\right\} n_3 d\gamma dt$$

$$-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)\frac{\partial n_3}{\partial x_1}\eta_{24} d\gamma dt - \int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)\frac{\partial \eta_{24}}{\partial x_1} n_3 d\gamma dt. \quad (92)$$

Therefore, the first term in equation (75) is as follows:

$$\left[-\frac{\partial}{\partial \alpha}\int_{t_s}^{t_e}\int_{\gamma^*} T_1(X_1(\alpha, x_1, x_2, x_3)) d\gamma dt\right]_{\alpha=0}$$

$$= -\int_{t_s}^{t_e}\int_\gamma \left\{-\frac{\partial p}{\partial x_1} n_1 + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_1}\left(\frac{\partial u_i}{\partial x_1} + \frac{\partial u_1}{\partial x_i}\right) n_i\right\}\eta_{24} d\gamma dt$$

$$-\int_{t_s}^{t_e}\int_\gamma \left\{-p\frac{\partial n_1}{\partial x_1} + \frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_1} + \frac{\partial u_1}{\partial x_i}\right)\frac{\partial n_i}{\partial x_1}\right\}\eta_{24} d\gamma dt$$

$$-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_i}\right) n_i d\gamma dt$$

$$-\int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) n_2 \frac{\partial \eta_{24}}{\partial x_1} d\gamma dt - \int_{t_s}^{t_e}\int_\gamma \frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) n_3 \frac{\partial \eta_{24}}{\partial x_1} d\gamma dt. \quad (93)$$

The second term in equation (75) is expanded. The domain Ω is as follows:

$$\frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \\ \frac{\partial X_1}{\partial x_3} & \frac{\partial X_2}{\partial x_3} & \frac{\partial X_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial x_1}(x_1 + \alpha\eta_{24}(x_1, x_2, x_3)) & \frac{\partial x_2}{\partial x_1} & \frac{\partial x_3}{\partial x_1} \\ \frac{\partial}{\partial x_2}(x_1 + \alpha\eta_{24}(x_1, x_2, x_3)) & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_3}{\partial x_2} \\ \frac{\partial}{\partial x_3}(x_1 + \alpha\eta_{24}(x_1, x_2, x_3)) & \frac{\partial x_2}{\partial x_3} & \frac{\partial x_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \alpha\frac{\partial \eta_{24}(x_1, x_2, x_3)}{\partial x_1} & 0 & 0 \\ \alpha\frac{\partial \eta_{24}(x_1, x_2, x_3)}{\partial x_2} & 1 & 0 \\ \alpha\frac{\partial \eta_{24}(x_1, x_2, x_3)}{\partial x_3} & 0 & 1 \end{vmatrix} = 1 + \alpha\frac{\partial \eta_{24}(x_1, x_2, x_3)}{\partial x_1}. \quad (94)$$

The second term in equation (75) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} p(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial X_j}d\Omega dt\right]_{\alpha=0}$$

$$=\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} p(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial X_j}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$=\left[\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial p(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\frac{\partial\lambda_{j+1}(X_1)}{\partial X_j}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$+\left[\int_{t_s}^{t_e}\int_{\Omega} p(X_1)\frac{\partial}{\partial X_j}\left(\frac{\partial\lambda_{j+1}(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\right)\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$+\left[\int_{t_s}^{t_e}\int_{\Omega} p(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial X_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial p}{\partial x_1}\frac{\partial\lambda_{j+1}}{\partial X_j}\eta_{24}d\Omega dt+\int_{t_s}^{t_e}\int_{\Omega} p\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}\right)d\Omega dt$$

$$+\int_{t_s}^{t_e}\int_{\Omega} p\frac{\partial\lambda_{j+1}}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt$$

$$=\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial p}{\partial x_1}\frac{\partial\lambda_{j+1}}{\partial x_j}\eta_{24}d\Omega dt+\int_{t_s}^{t_e}\int_{\psi} p\frac{\partial\lambda_{j+1}}{\partial x_1}n_j\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial p}{\partial x_j}\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}d\Omega dt+\int_{t_s}^{t_e}\int_{\Omega} p\frac{\partial\lambda_{j+1}}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt. \tag{95}$$

Using equation (20), equation (95) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} p(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial x_j}d\Omega dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\psi} p\frac{\partial\lambda_{j+1}}{\partial x_1}n_j\eta_{24}d\psi dt-\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial p}{\partial x_j}\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}d\Omega dt. \tag{96}$$

The domain $X_1(\alpha, x_1, x_2, x_3)$ is abbreviated to X_1. This abbreviation is same in other equations. The time derivation in the third term is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} u_j(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial t}d\Omega dt\right]_{\alpha=0}$$

$$
\begin{aligned}
&= \left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial t}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&= \left[\int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\frac{\partial\lambda_{j+1}(X_1)}{\partial t}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&\quad + \left[\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial}{\partial t}\left(\frac{\partial\lambda_{j+1}(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\right)\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&\quad + \left[\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial\lambda_{j+1}(X_1)}{\partial t}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt\right]_{\alpha=0} \\
&= \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_1}\frac{\partial\lambda_{j+1}}{\partial t}\eta_{24}d\Omega dt + \int_{t_s}^{t_e}\int_{\Omega} u_j\frac{\partial}{\partial t}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}\right)d\Omega dt \\
&\quad + \int_{t_s}^{t_e}\int_{\Omega} u_j\frac{\partial\lambda_{j+1}}{\partial t}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt \\
&= \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_1}\frac{\partial\lambda_{j+1}}{\partial t}\eta_{24}d\Omega dt + \int_{\Omega}\left[u_j\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}\right]_{t_s}^{t_e}d\Omega \\
&\quad - \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial t}\frac{\partial\lambda_{j+1}}{\partial x_1}\eta_{24}d\Omega dt + \int_{t_s}^{t_e}\int_{\Omega} u_j\frac{\partial\lambda_{j+1}}{\partial t}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt. \qquad (97)
\end{aligned}
$$

The pressure term in the third term is as follows:

$$
\begin{aligned}
&\left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} u_i(X_1)\frac{\partial\lambda_1(X_1)}{\partial X_j}d\Omega dt\right]_{\alpha=0} \\
&= \left[-\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial\lambda_1(X_1)}{\partial X_j}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&= -\left[\int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\frac{\partial\lambda_1(X_1)}{\partial X_i}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&\quad -\left[\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial}{\partial X_j}\left(\frac{\partial\lambda_1(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\right)\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0} \\
&\quad -\left[\int_{t_s}^{t_e}\int_{\Omega} u_j(X_1)\frac{\partial\lambda_1(X_1)}{\partial X_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt\right]_{\alpha=0} \\
&= -\int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_1}\frac{\partial\lambda_1}{\partial x_j}\eta_{24}d\Omega dt - \int_{t_s}^{t_e}\int_{\Omega} u_j\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_1}{\partial x_1}\eta_{24}\right)d\Omega dt
\end{aligned}
$$

$$-\int_{t_s}^{t_e}\int_{\Omega} u_j \frac{\partial\lambda_1}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt$$

$$= \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_1}\frac{\partial\lambda_1}{\partial x_j}\eta_{24}d\Omega dt - \int_{t_s}^{t_e}\int_{\psi} \frac{\partial\lambda_1}{\partial x_1}u_j n_j \eta_{24}d\psi dt$$

$$+ \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_j}\frac{\partial\lambda_1}{\partial x_1}\eta_{24}d\Omega dt - \int_{t_s}^{t_e}\int_{\Omega} u_j \frac{\partial\lambda_1}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt. \tag{98}$$

The convection term in the third term is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} u_i(X_1)U_j \frac{\partial\lambda_{i+1}(X_1)}{\partial X_j}d\Omega dt\right]_{\alpha=0}$$

$$= \left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} u_i(X_1)U_j \frac{\partial\lambda_{i+1}(X_1)}{\partial X_j}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$= \left[\int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_i(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}U_i \frac{\partial\lambda_{i+1}(X_1)}{\partial X_j}\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$+ \left[\int_{t_s}^{t_e}\int_{\Omega} u_i(X_1)U_j \frac{\partial\lambda_{i+1}(X_1)}{\partial X_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt\right]_{\alpha=0}$$

$$+ \left[\int_{t_s}^{t_e}\int_{\Omega} u_i(X_1)U_j \frac{\partial}{\partial X_j}\left(\frac{\partial\lambda_{i+1}(X_1)}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\right)\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$= \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_i}{\partial x_1}U_j \frac{\partial\lambda_{i+1}}{\partial x_j}\eta_{24}d\Omega dt + \int_{t_s}^{t_e}\int_{\Omega} u_i U_j \frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{i+1}}{\partial x_1}\eta_{24}\right)d\Omega dt$$

$$+ \int_{t_s}^{t_e}\int_{\Omega} u_i U_j \frac{\partial\lambda_{i+1}}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt$$

$$= \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_i}{\partial x_1}U_j \frac{\partial\lambda_{i+1}}{\partial x_j}\eta_{24}d\Omega dt + \int_{t_s}^{t_e}\int_{\psi} u_i U_j \frac{\partial\lambda_{i+1}}{\partial x_1}n_j\eta_{24}d\psi dt$$

$$- \int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_i}{\partial x_j}U_j \frac{\partial\lambda_{i+1}}{\partial x_1}\eta_{24}d\Omega dt + \int_{t_s}^{t_e}\int_{\Omega} u_i U_j \frac{\partial\lambda_{i+1}}{\partial x_j}\frac{\partial\eta_{24}}{\partial x_1}d\Omega dt. \tag{99}$$

The viscous term in the third term is as follows:

$$\frac{1}{\mathrm{Re}}\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega^*} u_i(X_1)\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}(X_1)}{\partial x_i}+\frac{\partial\lambda_{i+1}(X_1)}{\partial x_j}\right)d\Omega dt\right]_{\alpha=0}$$

$$= \frac{1}{\mathrm{Re}}\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Omega} u_i(X_1)\frac{\partial}{\partial x_j}\left(\frac{\partial\lambda_{j+1}(X_1)}{\partial x_i}+\frac{\partial\lambda_{i+1}(X_1)}{\partial x_j}\right)\left(1+\alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$= \frac{1}{\mathrm{Re}}\left[\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial u_i(X_1)}{\partial X_1}\frac{\partial X_1}{\partial \alpha}\frac{\partial}{\partial x_j}\left(\frac{\partial \lambda_{i+1}(X_1)}{\partial x_i}+\frac{\partial \lambda_{i+1}(X_1)}{\partial x_j}\right)\left(1+\alpha\frac{\partial \eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$+\frac{1}{\mathrm{Re}}\left[\int_{t_s}^{t_e}\int_{\Omega}u_i(X_1)\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial \lambda_{j+1}(X_1)}{\partial x_i}+\frac{\partial \lambda_{i+1}(X_1)}{\partial x_j}\right)\right\}\frac{\partial \eta_{24}}{\partial x_1}d\Omega dt\right]_{\alpha=0}$$

$$+\frac{1}{\mathrm{Re}}\left[\int_{t_s}^{t_e}\int_{\Omega}u_i(X_1)\frac{\partial}{\partial x_i}\left\{\frac{\partial}{\partial x_i}\left(\frac{\partial \lambda_{j+1}(X_1)}{\partial X_1}\frac{\partial X_1}{\partial \alpha}\right)+\frac{\partial}{\partial x_j}\left(\frac{\partial \lambda_{i+1}(X_1)}{\partial X_1}\frac{\partial X_1}{\partial \alpha}\right)\right\}\right.$$

$$\left.\times\left(1+\alpha\frac{\partial \eta_{24}}{\partial x_1}\right)d\Omega dt\right]_{\alpha=0}$$

$$= \frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial u_i}{\partial x_1}\frac{\partial}{\partial x_i}\left(\frac{\partial \lambda_{j+1}}{\partial x_i}+\frac{\partial \lambda_{i+1}}{\partial x_j}\right)\eta_{24}d\Omega dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}u_i\left(\frac{\partial}{\partial x_i}\frac{\partial \lambda_{j+1}}{\partial x_1}+\frac{\partial}{\partial x_j}\frac{\partial \lambda_{i+1}}{\partial x_1}\right)n_j\eta_{24}d\Gamma dt$$

$$-\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}\frac{\partial u_i}{\partial x_j}\left(\frac{\partial \lambda_{j+1}}{\partial x_1}n_i+\frac{\partial \lambda_{i+1}}{\partial x_1}n_j\right)\eta_{24}d\psi dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\Omega}\left(\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial \lambda_{j+1}}{\partial x_1}+\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial \lambda_{i+1}}{\partial x_1}\right)\eta_{24}d\Omega dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\Omega}u_i\frac{\partial}{\partial x_j}\left(\frac{\partial \lambda_{j+1}}{\partial x_i}+\frac{\partial \lambda_{i+1}}{\partial x_j}\right)\frac{\partial \eta_{24}}{\partial x_1}d\Omega dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi}u_i\left(\frac{\partial \lambda_{j+1}}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_i}+\frac{\partial \lambda_{i+1}}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_j}\right)n_jd\Gamma dt. \tag{100}$$

Equations (96)-(99) are summarized as follows:

$$\int_{t_s}^{t_e}\int_{\Omega}u_i\left\{\frac{\partial \lambda_{i+1}}{\partial t}-\frac{\partial \lambda_1}{\partial x_i}+U_j\frac{\partial \lambda_{i+1}}{\partial x_j}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial \lambda_{j+1}}{\partial x_i}+\frac{\partial \lambda_{i+1}}{\partial x_j}\right)\right\}d\Omega dt$$

$$=\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial u_j}{\partial x_i}\left\{\frac{\partial \lambda_{j+1}}{\partial t}-\frac{\partial \lambda_1}{\partial x_j}+U_i\frac{\partial \lambda_{j+1}}{\partial x_i}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_i}\left(\frac{\partial \lambda_{i+1}}{\partial x_j}+\frac{\partial \lambda_{j+1}}{\partial x_i}\right)\right\}\eta_{24}d\Omega dt$$

$$+\int_{t_s}^{t_e}\int_{\Omega}\frac{\partial \lambda_{j+1}}{\partial x_1}\left\{-\frac{\partial u_j}{\partial t}-\frac{\partial p}{\partial x_j}-\frac{\partial u_i}{\partial x_j}U_j+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\}\eta_{24}d\Omega dt$$

$$+\int_{t_s}^{t_e}\int_{\Omega} u_j \left\{ \frac{\partial \lambda_{j+1}}{\partial t} - \frac{\partial \lambda_1}{\partial x_j} + U_i \frac{\partial \lambda_{j+1}}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial}{\partial x_{ij}} \left(\frac{\partial \lambda_{i+1}}{\partial x_j} + \frac{\partial \lambda_{j+1}}{\partial x_i} \right) \right\} \frac{\partial \eta_{24}}{\partial x_1} d\Omega dt$$

$$+\int_{t_s}^{t_e}\int_{\Omega} \frac{\partial u_j}{\partial x_j} \frac{\partial \lambda_1}{\partial x_1} \eta_{24} d\Omega dt + \int_{\Omega} \left[u_j \frac{\partial \lambda_{j+1}}{\partial x_1} \eta_{24} \right]_{t_s}^{t_e} d\Omega$$

$$+\int_{t_s}^{t_e}\int_{\psi} u_i \left\{ U_j \frac{\partial \lambda_{i+1}}{\partial x_1} n_j - \frac{\partial \lambda_1}{\partial x_1} n_i + \frac{1}{\mathrm{Re}} \left(\frac{\partial}{\partial x_i} \frac{\partial \lambda_{j+1}}{\partial x_1} + \frac{\partial}{\partial x_j} \frac{\partial \lambda_{i+1}}{\partial x_1} \right) n_j \right\} \eta_{24} d\Gamma dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} \left\{ -pn_i + \frac{1}{\mathrm{Re}} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) n_j \right\} \frac{\partial \lambda_{i+1}}{\partial x_1} \eta_{24} d\psi dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi} u_i \left(\frac{\partial \lambda_{j+1}}{\partial x_1} \frac{\partial \eta_{24}}{\partial x_i} + \frac{\partial \lambda_{i+1}}{\partial x_1} \frac{\partial \eta_{24}}{\partial x_j} \right) n_j d\Gamma dt. \tag{101}$$

Using equation (20) ~ equation (23), equation (14) ~ equation (17), equation (72), Table 1, equation (101) becomes as follows:

$$\int_{t_s}^{t_e}\int_{\Omega} u_i \left\{ \frac{\partial \lambda_{i+1}}{\partial t} - \frac{\partial \lambda_1}{\partial x_i} + U_j \frac{\partial \lambda_{i+1}}{\partial x_j} + \frac{1}{\mathrm{Re}} \frac{\partial}{\partial x_j} \left(\frac{\partial \lambda_{j+1}}{\partial x_i} + \frac{\partial \lambda_{i+1}}{\partial x_j} \right) \right\} d\Omega dt$$

$$= \int_{t_s}^{t_e}\int_{\psi} u_i \left\{ U_j \frac{\partial \lambda_{i+1}}{\partial x_1} n_j - \frac{\partial \lambda_1}{\partial x_1} n_i + \frac{1}{\mathrm{Re}} \left(\frac{\partial}{\partial x_i} \frac{\partial \lambda_{j+1}}{\partial x_1} + \frac{\partial}{\partial x_j} \frac{\partial \lambda_{i+1}}{\partial x_1} \right) n_j \right\} \eta_{24} d\Gamma dt$$

$$+\int_{\Omega} \left[u_j \frac{\partial \lambda_{j+1}}{\partial x_1} \eta_{24} \right]_{t_s}^{t_e} d\Omega - \int_{t_s}^{t_e}\int_{\psi} \left\{ -pn_i + \frac{1}{\mathrm{Re}} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) n_j \right\} \frac{\partial \lambda_{i+1}}{\partial x_1} \eta_{24} d\psi dt$$

$$+\frac{1}{\mathrm{Re}}\int_{t_s}^{t_e}\int_{\psi} u_i \left(\frac{\partial \lambda_{j+1}}{\partial x_1} \frac{\partial \eta_{24}}{\partial x_i} + \frac{\partial \lambda_{i+1}}{\partial x_1} \frac{\partial \eta_{24}}{\partial x_j} \right) n_j d\Gamma dt. \tag{102}$$

The fourth term in equations (75) is as follows:

$$-\left[\frac{\partial}{\partial \alpha} \int_{t_s}^{t_e}\int_{\psi^*} u_i(X_1) S_i(X_1) d\psi^* dt \right]_{\alpha=0}$$

$$= -\int_{t_s}^{t_e}\int_{\psi} \frac{\partial u_j}{\partial x_1} \left\{ \lambda_{j+1} U_i n_i - \lambda_1 n_j + \frac{1}{\mathrm{Re}} \left(\frac{\partial \lambda_{j+1}}{\partial x_i} + \frac{\partial \lambda_{i+1}}{\partial x_{j+1}} \right) n_i \right\} \eta_{24} d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j \left\{ \frac{\partial \lambda_{j+1}}{\partial x_1} U_i n_i - \frac{\partial \lambda_1}{\partial x_1} n_j + \frac{1}{\mathrm{Re}} \left(\frac{\partial}{\partial x_i} \frac{\partial \lambda_{j+1}}{\partial x_1} + \frac{\partial}{\partial x_{j+1}} \frac{\partial \lambda_{i+1}}{\partial x_1} \right) n_i \right\} \eta_{24} d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j\left\{\lambda_{j+1}U_i\frac{\partial n_i}{\partial x_1}-\lambda_1\frac{\partial n_j}{\partial x_1}+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_i}+\frac{\partial\lambda_{i+1}}{\partial x_j}\right)\frac{\partial n_i}{\partial x_i}\right\}\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_i}+\frac{\partial\lambda_{i+1}}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_j}\right)n_i\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_i\left\{\lambda_{i+1}U_2n_2+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{i+1}}{\partial x_2}+\frac{\partial\lambda_3}{\partial x_i}\right)n_2\right\}\frac{\partial\eta_{24}}{\partial x_1}d\psi dt$$

$$+\int_{t_s}^{t_e}\int_{\psi} u_2\lambda_1 n_2\frac{\partial\eta_{24}}{\partial x_1}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_i\left\{\lambda_{i+1}U_3n_3+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{i+1}}{\partial x_3}+\frac{\partial\lambda_4}{\partial x_i}\right)n_3\right\}\frac{\partial\eta_{24}}{\partial x_1}d\psi dt$$

$$+\int_{t_s}^{t_e}\int_{\psi^*} u_3\lambda_1 n_3\frac{\partial\eta_{24}}{\partial x_1}d\psi dt. \tag{103}$$

The calculation condition becomes $u_i = 0$ on the boundary γ. It becomes $n_2 = n_3 = 0$ on the boundary Γ_W, Γ_E. It becomes $n_1 = n_3 = 0$, $u_2 = 0$, $S_1 = 0$ and $S_3 = 0$ on the boundary Γ_N, Γ_S. It becomes $n_1 = n_2 = 0$, $u_3 = 0$, $S_1 = 0$ and $S_2 = 0$ on the boundary Γ_U, Γ_L. Therefore, the fifth-eighth terms become zero. Equation (103) is as follows:

$$-\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\psi^*} u_i(X_1)S_i(X_1)d\psi^* dt\right]_{\alpha=0}$$

$$=-\int_{t_s}^{t_e}\int_{\psi}\frac{\partial u_j}{\partial x_1}\left\{\lambda_{j+1}U_in_i-\lambda_1n_j+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_i}+\frac{\partial\lambda_{i+1}}{\partial x_{j+1}}\right)n_i\right\}\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j\left\{\frac{\partial\lambda_{j+1}}{\partial x_1}U_in_i-\frac{\partial\lambda_1}{\partial x_1}n_j+\frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial x_i}\frac{\partial\lambda_{j+1}}{\partial x_1}+\frac{\partial}{\partial x_{j+1}}\frac{\partial\lambda_{i+1}}{\partial x_1}\right)n_i\right\}\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j\left\{\lambda_{j+1}U_i\frac{\partial n_i}{\partial x_1}-\lambda_1\frac{\partial n_j}{\partial x_1}+\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_i}+\frac{\partial\lambda_{i+1}}{\partial x_j}\right)\frac{\partial n_i}{\partial x_1}\right\}\eta_{24}d\psi dt$$

$$-\int_{t_s}^{t_e}\int_{\psi} u_j\frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{j+1}}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_i}+\frac{\partial\lambda_{i+1}}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_j}\right)n_i\eta_{24}d\psi dt. \tag{104}$$

The fifth term in equation (75) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\gamma^*}\lambda_2(X_1)T_1(X_1)d(\Gamma_W^*+\gamma^*)dt\right]_{\alpha=0}$$

$$= \int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\frac{\partial\lambda_2}{\partial x_1}\left\{-pn_1+\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}+\frac{\partial u_1}{\partial x_i}\right)n_i\right\}\eta_{24}d(\Gamma_W+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\lambda_2\left\{-\frac{\partial p}{\partial x_1}n_1+\frac{1}{\text{Re}}\left(\frac{\partial}{\partial x_1}\frac{\partial u_i}{\partial x_1}+\frac{\partial}{\partial x_i}\frac{\partial u_1}{\partial x_1}\right)n_i\right\}\eta_{24}d(\Gamma_W+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\lambda_2\left\{-p\frac{\partial n_1}{\partial x_1}+\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}+\frac{\partial u_1}{\partial x_i}\right)\frac{\partial n_i}{\partial x_1}\right\}\eta_{24}d(\Gamma_W+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\lambda_2\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_i}\right)n_i d(\Gamma_W+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\lambda_2\frac{1}{\text{Re}}\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\frac{\partial\eta_{24}}{\partial x_1}n_2 d(\Gamma_W+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\lambda_2\frac{1}{\text{Re}}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)\frac{\partial\eta_{24}}{\partial x_1}n_3 d(\Gamma_W+\gamma)dt. \tag{105}$$

Using the boundary condition $\lambda_2 = 0$ on Γ_W and $\lambda_2 = 1$ on γ, equation (105) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\gamma^*}\lambda_2(X_1)T_1(X_1)d(\Gamma_W^*+\gamma^*)dt\right]_{\alpha=0}$$

$$= \int_{t_s}^{t_e}\int_{\Gamma_W+\gamma}\frac{\partial\lambda_2}{\partial x_1}\left\{-pn_1+\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}+\frac{\partial u_1}{\partial x_i}\right)n_i\right\}\eta_{24}d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\left\{-\frac{\partial p}{\partial x_1}n_1+\frac{1}{\text{Re}}\left(\frac{\partial}{\partial x_1}\frac{\partial u_i}{\partial x_1}+\frac{\partial}{\partial x_i}\frac{\partial u_1}{\partial x_1}\right)n_i\right\}\eta_{24}d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\left\{-p\frac{\partial n_1}{\partial x_1}+\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}+\frac{\partial u_1}{\partial x_i}\right)\frac{\partial n_i}{\partial x_1}\right\}\eta_{24}d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\frac{1}{\text{Re}}\left(\frac{\partial u_i}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_i}\right)n_i d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\frac{1}{\mathrm{Re}}\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\frac{\partial \eta_{24}}{\partial x_1}n_2 d\gamma dt$$

$$+\int_{t_s}^{t_e}\int_{\gamma}\frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)\frac{\partial \eta_{24}}{\partial x_1}n_3 d\gamma dt. \tag{106}$$

The sixth term in equation (75) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\Gamma_N^*+\Gamma_S^*+\gamma^*}\lambda_3(X_1)T_2(X_1)d(\Gamma_W^*+\Gamma_N^*+\Gamma_S^*+\gamma^*)dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\frac{\partial\lambda_3}{\partial x_1}\left\{-pn_2+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_2}+\frac{\partial u_2}{\partial x_i}\right)n_i\right\}$$

$$\times\eta_{24}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\lambda_3\left\{-\frac{\partial p}{\partial x_1}n_2+\frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial x_2}\frac{\partial u_i}{\partial x_1}+\frac{\partial}{\partial x_i}\frac{\partial u_2}{\partial x_1}\right)n_i\right\}$$

$$\times\eta_{24}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\frac{1}{\mathrm{Re}}\lambda_3\left(\frac{\partial u_i}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial \eta_{24}}{\partial x_i}\right)n_i d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\lambda_3\left\{-p+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_2}+\frac{\partial u_2}{\partial x_i}\right)\right\}\frac{\partial n_i}{\partial x_1}\eta_{24}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\lambda_3\left\{-p+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\right)\right\}n_2\frac{\partial \eta_{24}}{\partial x_1}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\lambda_3\frac{1}{\mathrm{Re}}\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)n_3\frac{\partial \eta_{24}}{\partial x_1}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt. \tag{107}$$

Using the boundary condition $\lambda_3 = 0$ on γ, Γ_W, Γ_N and Γ_S, equation (107) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\Gamma_N^*+\Gamma_S^*+\gamma^*}\lambda_3(X_1)T_2(X_1)d(\Gamma_W^*+\Gamma_N^*+\Gamma_S^*+\gamma^*)dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_N+\Gamma_S+\gamma}\frac{\partial\lambda_3}{\partial x_1}\left\{-pn_2+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_2}+\frac{\partial u_2}{\partial x_i}\right)n_i\right\}$$

$$\times\eta_{24}d(\Gamma_W+\Gamma_N+\Gamma_S+\gamma)dt. \tag{108}$$

The seventh term in equation (75) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\Gamma_U^*+\Gamma_L^*+\gamma^*}\lambda_4(X_1)T_3(X_1)d(\Gamma_W^*+\Gamma_U^*+\Gamma_L^*+\gamma^*)dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Gamma_M+\Gamma_U+\Gamma_L+\gamma}\frac{\partial\lambda_4}{\partial x_1}\left\{-pn_3+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_3}+\frac{\partial u_3}{\partial x_i}\right)n_i\right\}$$

$$\times\,\eta_{24}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\lambda_4\left\{-\frac{\partial p}{\partial x_1}n_3+\frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial x_3}\frac{\partial u_i}{\partial x_1}+\frac{\partial}{\partial x_i}\frac{\partial u_3}{\partial x_1}\right)n_i\right\}$$

$$\times\,\eta_{24}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\lambda_4\left\{-p\frac{\partial n_3}{\partial x_1}+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_3}+\frac{\partial u_3}{\partial x_i}\right)\right\}\frac{\partial n_i}{\partial x_1}\eta_{24}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\frac{1}{\mathrm{Re}}\lambda_4\left(\frac{\partial u_i}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial\eta_{24}}{\partial x_i}\right)n_id(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\lambda_4\frac{1}{\mathrm{Re}}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)n_2\frac{\partial\eta_{24}}{\partial x_1}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt$$

$$+\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\lambda_4\left\{-p+\frac{2}{\mathrm{Re}}\frac{\partial u_3}{\partial x_3}\right\}n_3\frac{\partial\eta_{24}}{\partial x_1}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt. \tag{109}$$

Using the boundary condition $\lambda_4=0$ on γ, Γ_U, Γ_L and Γ_W, equation (109) is as follows:

$$\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Gamma_W^*+\Gamma_U^*+\Gamma_L^*+\gamma^*}\lambda_4(X_1)T_3(X_1)d(\Gamma_W^*+\Gamma_U^*+\Gamma_L^*+\gamma^*)dt\right]_{\alpha=0}$$

$$=\int_{t_s}^{t_e}\int_{\Gamma_W+\Gamma_U+\Gamma_L+\gamma}\frac{\partial\lambda_4}{\partial x_1}\left\{-pn_3+\frac{1}{\mathrm{Re}}\left(\frac{\partial u_i}{\partial x_3}+\frac{\partial u_3}{\partial x_i}\right)n_i\right\}$$

$$\times\,\eta_{24}d(\Gamma_W+\Gamma_U+\Gamma_L+\gamma)dt. \tag{110}$$

The eighth term in equation (75) is as follows:

$$-\left[\frac{\partial}{\partial\alpha}\int_{\Omega^*}[\lambda_{i+1}u_i]_{t_s}^{t_e}d\Omega\right]_{\alpha=0}$$

$$= -\left[\frac{\partial}{\partial\alpha}\int_{\Omega}[\lambda_{i+1}u_i]_{t_s}^{t_e}\left(1 + \alpha\frac{\partial\eta_{24}(x_1, x_2, x_3)}{\partial x_1}\right)d\Omega\right]_{\alpha=0}$$

$$= -\left[\int_{\Omega}\left[\frac{\partial\lambda_{i+1}}{\partial X_1}\frac{\partial X_1}{\partial\alpha}u_i + \lambda_{i+1}\frac{\partial u_i}{\partial X_1}\frac{\partial X_1}{\partial\alpha}\right]_{t_s}^{t_e}d\Omega\right]_{\alpha=0}$$

$$-\left[\int_{\Omega}[\lambda_{i+1}u_i]_{t_s}^{t_e}\frac{\partial\eta_{24}(x_1, x_2, x_3)}{\partial x_1}d\Omega\right]_{\alpha=0}$$

$$= -\int_{\Omega}\left[\frac{\partial\lambda_{i+1}}{\partial x_1}u_i + \lambda_{i+1}\frac{\partial u_i}{\partial x_1}\right]_{t_s}^{t_e}\eta_{24}(x_1, x_2, x_3)d\Omega$$

$$-\int_{\Omega}[\lambda_{i+1}u_i]_{t_s}^{t_e}\frac{\partial\eta_{24}(x_1, x_2, x_3)}{\partial x_1}d\Omega. \tag{111}$$

The ninth term in equation (75) is as follows:

$$\kappa\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Pi^*}d\Pi dt\right]_{\alpha=0} = \kappa\left[\frac{\partial}{\partial\alpha}\int_{t_s}^{t_e}\int_{\Pi}\left(1 + \alpha\frac{\partial\eta_{24}}{\partial x_1}\right)d\Pi dt\right]_{\alpha=0}$$

$$= \kappa\int_{t_s}^{t_e}\int_{\Pi}\frac{\partial\eta_{24}}{\partial x_1}d\Pi dt = \kappa\int_{t_s}^{t_e}\int_{\gamma}n_1\eta_{24}d\gamma dt - \kappa\int_{t_s}^{t_e}\int_{\Pi}0\cdot\eta_{24}d\Pi dt$$

$$= \kappa\int_{t_s}^{t_e}\int_{\gamma}n_1\eta_{24}d\gamma dt. \tag{112}$$

Using equations (93)-(112), the first variation with respect to x_1 is summarized. Using equations (14)-(17), (20)-(23), (72) and Table 1 (the boundary condition) is as follows:

$$\left[\frac{\partial L(x_1 + \alpha\eta_{24})}{\partial\alpha}\right]_{\alpha=0} = -\int_{t_s}^{t_e}\int_{\gamma}\frac{\partial u_i}{\partial x_1}\left\{\lambda_{i+1}U_j n_j - \lambda_1 n_i + \frac{1}{\mathrm{Re}}\left(\frac{\partial\lambda_{i+1}}{\partial x_j} + \frac{\partial\lambda_{j+1}}{\partial x_i}\right)n_j\right\}$$

$$\times\,\eta_{24}d\Gamma dt + \kappa\int_{t_s}^{t_e}\int_{\gamma}n_1\eta_{24}d\gamma dt = 0. \tag{113}$$

The sensitivity equation with respect to x_1 is derived as shown in equation (26). The sensitivity equations with respect to x_2 and x_3 are also derived in the same operation.