Far East Journal of Mathematical Sciences (FJMS)
Volume 34, Issue 2, 2009, Pages 257-270
Published Online: September 7, 2009
This paper is available online at http://www.pphmj.com

ON THE SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS AND DOUBLE LAPLACE TRANSFORM

ADEM KILIÇMAN and HASSAN ELTAYEB

Department of Mathematics and Institute for Mathematical Research
University Putra Malaysia
43400 UPM Serdang, Selangor, Malaysia
e-mail: adem@math.upm.edu.my
eltayeb@putra.upm.edu.my

Abstract

In this study, we consider to solve the general linear second order partial differential equations with non-constant coefficients by using the double Laplace transform. In a special case, we provide solutions for the wave equation where the non-constant coefficients are polynomials.

The partial differential equations (PDEs) are very important in mathematical physics and occur in several places. In general, PDEs are two types: the homogenous equations with constant coefficients where they might accept many classical solutions such as: separation of variables, see [9]; the methods of characteristics, see [10] and [3]; integral transform methods, see [5]; and the non-homogenous equations with constant coefficients also might be solved by integral transform methods as well as operation calculus, see [11], [4], [2] and [6].

2000 Mathematics Subject Classification: Primary 35Qxx, 44A12; Secondary 44A35.

Keywords and phrases: polynomial, double convolution, double Laplace transform.

Received April 15, 2009

Definition 1. Let $F_1(x, y)$ and $F_2(x, y)$ be integrable functions. Then the *convolution of* $F_1(x, y)$ and $F_2(x, y)$ is defined as

$$F_1(x, y) * *F_2(x, y) = \int_0^y \int_0^x F_1(x - \zeta, y - \eta) F_2(\zeta, \eta) d\zeta d\eta$$

and known as double convolution with respect to *x* and *y*, see [11].

We apply double Laplace transform to solve the linear second order partial differential equations with non-constant coefficients, and discuss the non-homogenous wave and heat equations with non-constant coefficient by using the same techniques. For example, it was proved that if *F* and *G* are solutions for the wave equation with constant coefficients and non-constant coefficients, respectively, then the double convolution $F * * G$ is a solution for the following type of equations:

$$u_{tt}(x, t) - u_{xx}(x, t) - h(x, t) = \sum_{i=1}^{n} f(x, t) * * g_i(x, t) \quad (t, x) \in \mathbb{R}_+^2,$$

where $h(x, t)$ is called *remainder function*.

In [7], the *double Laplace transform* is defined as

$$L_x L_t [f(x, s)] = F(p, s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x, t) dt dx, \tag{1.1}$$

where $x, t > 0$, and *p* and *s* are complex values and further the first order partial derivatives are given by

$$L_x L_t \left[\frac{\partial f(x, t)}{\partial x} \right] = pF(p, s) - F(0, s).$$

Similarly the double Laplace transform for second partial derivative with respect to *x* is given by

$$L_{xx} \left[\frac{\partial^2 f(x, t)}{\partial^2 x} \right] = p^2 F(p, s) - pF(0, s) - \frac{\partial F(0, s)}{\partial x}$$

and with respect to *t* is given by

$$L_{tt} \left[\frac{\partial^2 f(x, t)}{\partial^2 t} \right] = s^2 F(p, s) - sF(p, 0) - \frac{\partial F(p, 0)}{\partial t},$$

see [11].

Next we study the uniqueness and existence of double Laplace transform. First of all, let $f(x, t)$ be a continuous function on the interval $[0, \infty)$ which is of exponential order, that is, for some $a, b \in \mathbb{R}$,

$$\sup_{\substack{t>0\\x>0}} \frac{|f(x,t)|}{e^{ax+bt}} < \infty. \tag{1.2}$$

In this case, the double Laplace transform of

$$F(p, s) = \int_0^\infty \int_0^\infty e^{-st-px} f(x, t)\,dxdt$$

exists for all $p > a$ and $s > b$ and is in fact infinitely differentiable with respect to $p > a$ and $s > b.$ All functions in this study are assumed to be of exponential order. The following theorem shows that $f(x, t)$ can uniquely be recovered from $F(p, s)$.

Theorem 1. *Let $f(x, t)$ and $g(x, t)$ be continuous functions defined for $x, t \geq 0$ have Laplace transforms $F(p, s)$ and $G(p, s)$, respectively. If $F(p, s) = G(p, s)$, then $f(x, t) = g(x, t)$.*

Proof. If α and β are sufficiently large, then the integral representation by

$$f(x, t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(p, s)\,ds\right)dp$$

for the double inverse Laplace transform can be used to obtain

$$\begin{aligned} f(x, t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(p, s)\,ds\right)dp \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}G(p, s)\,ds\right)dp \\ &= g(x, t) \end{aligned}$$

and the theorem is proved. □

In the next theorem, we study the existence of double Laplace transform as follows:

Theorem 2 (Existence of the Laplace transform). *If f is of exponential order, then its Laplace transform $L_x L_t[f(x, t)] = F(p, s)$ is given by*

$$F(p, s) = \int_0^\infty \int_0^\infty e^{-px-st} f(x, t)dtdx,$$

where $s = \eta + i\omega$ and $p = \zeta + i\mu$. The defining integral for F exists at points $s = \eta + i\tau$ and $p = \zeta + i\phi$ in the right half plane $\eta > K$ and $\zeta > L$.

Proof. Using $s = \eta + i\tau$ and $p = \zeta + i\phi$, we can express $F(p, s)$ as

$$F(p, s) = \int_0^\infty \int_0^\infty f(x, t)\cos(\phi x + \tau t)e^{-\zeta x - \eta t}dxdt$$

$$- i\int_0^\infty \int_0^\infty f(x, t)\sin(\phi x + \tau t)e^{-\zeta x - \eta t}dxdt.$$

Then, for values of $\eta > K$ and $\zeta > L$, we have

$$\int_0^\infty \int_0^\infty | f(x, t)||\cos(\phi x + \tau t)|e^{-\zeta x - \eta t}dxdt \le MN\int_0^\infty e^{(K-\eta)t}\left(\int_0^\infty e^{(L-\zeta)x}dx\right)dt$$

$$\le \left(\frac{M}{\zeta - L}\right)\left(\frac{N}{\eta - K}\right)$$

and

$$\int_0^\infty \int_0^\infty | f(x, t)||\sin(\phi x + \tau t)|e^{-\zeta x - \eta t}dxdt \le MN\int_0^\infty e^{(K-\eta)t}\left(\int_0^\infty e^{(L-\zeta)x}dx\right)dt$$

$$\le \left(\frac{M}{\zeta - L}\right)\left(\frac{N}{\eta - K}\right)$$

which imply that the integrals defining the real and imaginary parts of F exist for values of $\mathrm{Re}(p) > L$ and $\mathrm{Re}(s) > K$, completing the proof. □

Theorem 3 (Inversion formula). *A function $f(x, t)$ which is continuous on $[0, \infty)$ and satisfies the growth condition* (1.2) *can be recovered from $F(p, s)$ as*

$$f(x, t) = \lim_{\substack{m\to\infty \\ n\to\infty}} \frac{(-1)^{m+n}}{m!n!}\left(\frac{m}{x}\right)^{m+1}\left(\frac{n}{t}\right)^{n+1}\Psi^{m+n}\left(\frac{m}{x}, \frac{n}{t}\right),$$

where Ψ^{m+n} *denotes* $(m+n)$*th mixed partial derivatives of* $F(p, s)$ *defined by* $\Psi^{m+n} = \dfrac{\partial^{m+n}F(p, s)}{\partial p^m \partial s^n}$ *for* $x, t \geq 0$, *since the above theorem obtains* $f(x, t)$ *in term of* $F(p, s)$.

Of course, the main difficulty in using Theorem 3 for computing the inverse Laplace transform is the repeated symbolic differentiation of $F(p, s)$. However, we apply Theorem 3 in the next example.

Example 1. Let $f(x, t) = e^{-ax-bt}$. Then the Laplace transform is easily found to be

$$F(p, s) = \frac{1}{(p+a)(s+b)}.$$

It is also simple to verify that

$$\frac{\partial^{m+n}F(p, s)}{\partial p^m \partial s^n} = m!\,n!(-1)^{m+n}(p+a)^{-m-1}(s+b)^{-n-1}.$$

Putting this expression for $\dfrac{\partial^{m+n}F(p, s)}{\partial p^m \partial s^n}$ into Theorem 3 gives

$$f(x, t) = \lim_{\substack{m\to\infty \\ n\to\infty}} \frac{m^{m+1}n^{n+1}}{x^{m+1}t^{n+1}}\left(a + \frac{m}{x}\right)^{-m-1}\left(b + \frac{n}{t}\right)^{-n-1}$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} \left(1 + \frac{ax}{m}\right)^{-m-1}\left(1 + \frac{bt}{n}\right)^{-n-1}.$$

The last limit is easy to evaluate, take the natural log of both sides and write the result in the form of $-\dfrac{\ln\left(1+\frac{ax}{m}\right)}{1/(m+1)} - \dfrac{\ln\left(1+\frac{bt}{n}\right)}{1/(n+1)}$. L'Hopital's rule reveals that the indeterminate from approaches $-ax - bt$. The continuity of the natural logarithm shows that $\ln(f(x, t)) = -ax - bt$, then $f(x, t) = e^{-ax-bt}$.

Properties of the double Laplace transform

In this part, we consider some of the properties of the double Laplace transform that will enable us to find further transform pairs $\{f(x, t), F(p, s)\}$ without having to compute.

(I) $F(p+d,\ s+c) = L_x L_t[e^{-dx-ct} f(x,\ t)](p,\ s)$.

(II) $\frac{1}{k} F\left(\frac{p}{\alpha},\ \frac{s}{\beta}\right) = L_x L_t[f(\alpha x,\ \beta t)](p,\ s)$, where $k = \alpha\beta$.

(III) $\frac{\partial^{m+n} F(p, s)}{\partial p^m \partial s^n} = L_x L_t[(-1)^{m+n} x^m t^n f(x,\ t)](p,\ s)$.

The three properties above are very useful for the proof of Theorem 3.

The proof of Theorem 3. Let us define the set of functions depending on parameters m and n,

$$g_{m,n}(x,\ t) = \frac{m^{m+1} n^{n+1}}{m!\, n!} x^m t^n e^{-mx-nt} \quad \text{so} \quad \int_0^\infty \int_0^\infty g_{m,n}(x,\ t)\, dxdt = 1, \tag{1.3}$$

and

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_0^\infty \int_0^\infty g_{m,n}(x,\ t) \varphi(x,\ t)\, dxdt = \varphi(1,\ 1), \tag{1.4}$$

where $\varphi(x,\ t)$ is any continuous function, let us denote its Laplace transform as a function of p and s by $L_x L_t[\varphi(x,\ t)](p,\ s)$. Now we define the function $\Psi(x,\ t) = f(xx_0,\ tt_0)$ and using the property (II), we have

$$L_x L_t[\Psi(x,\ t)](p,\ s) = L_x L_t[f(xx_0,\ tt_0)](p,\ s) = \frac{1}{x_0 t_0} F\left(\frac{p}{x_0},\ \frac{s}{t_0}\right). \tag{1.5}$$

We apply the property (III), (we must evaluate the $m+n$ mixed partial derivatives of $F(p,\ s)$ at the points $p = \frac{m}{x}$ and $s = \frac{n}{t}$)

$$\frac{\partial^{m+n}}{\partial p^m \partial s^n} (L_x L_t[\Psi(x,\ t)])(p,\ s) = \frac{1}{x_0^{m+1} t_0^{n+1}} \frac{\partial^{m+n}}{\partial p^m \partial s^n} F\left(\frac{p}{x_0},\ \frac{s}{t_0}\right). \tag{1.6}$$

Let $\varphi(x,\ t) = e^{-px-st}\Psi(x,\ t)$. By using equations (1.4) and (1.3), we have

$$\begin{aligned} \varphi(1,\ 1) &= e^{-p-s}\Psi(1,\ 1) = e^{-p-s} f(x_0,\ t_0) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{m^{m+1} n^{n+1}}{m!\, n!} \int_0^\infty \int_0^\infty x^m t^n e^{-px-st} e^{-mx-nt} \Psi(x,\ t)\, dxdt \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{m^{m+1} n^{n+1}}{m!\, n!} L_x L_t[x^m t^n e^{-mx-nt} \Psi(x,\ t)](p,\ s). \end{aligned} \tag{1.7}$$

Using the above properties (I) and (II) of double Laplace transform, equation (1.6), and the definition of $\Psi(x, t)$, we have

$$\begin{aligned} & L_x L_t [x^m t^n e^{-mx-nt} \Psi(x, t)](p, s) \\ &= (-1)^{m+n} \frac{\partial^{m+n}}{\partial p^m \partial s^n} (L_x L_t (e^{-mx-nt} \Psi(x, t)))(p, s) \\ &= (-1)^{m+n} \frac{\partial^{m+n}}{\partial p^m \partial s^n} (L_x L_t (\Psi(x, t)))(p+m, s+n) \\ &= (-1)^{m+n} \frac{1}{z} \frac{\partial^{m+n}}{\partial p^m \partial s^n} (L_x L_t (f(xx_0, tt_0))) \left(\frac{p+m}{x_0}, \frac{s+n}{t_0} \right) \\ &= (-1)^{m+n} \frac{1}{z} \frac{\partial^{m+n}}{\partial p^m \partial s^n} \left(F\left(\frac{p+m}{x_0}, \frac{s+n}{t_0} \right) \right), \end{aligned} \tag{1.8}$$

where $\frac{1}{z} = \frac{1}{x_0^{m+1} t_0^{n+1}}$, from equations (1.7) and (1.8), with $f(x_0, t_0) = e^{p+s} \varphi(1, 1)$, we have yield

$$\begin{aligned} f(x_0, t_0) = e^{p+s} \lim_{m \to \infty} \lim_{n \to \infty} \frac{(-1)^{m+n}}{m!\, n!} \left(\frac{m}{x_0} \right)^{m+1} \left(\frac{n}{t_0} \right)^{n+1} \\ \times \frac{\partial^{m+n}}{\partial p^m \partial s^n} \left(F\left(\frac{p+m}{x_0}, \frac{s+n}{t_0} \right) \right) \end{aligned}$$

for any p and s. The statement in Theorem 3 is actually just the special cases $p = 0$ and $s = 0$.

Now consider a linear second order partial differential equation with constant coefficients

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = \sum_{i=1}^{n} f_i(x, y) ** g_i(x, y), \tag{1.9}$$

where a_i's are constants and further assume that equation (1.9) has a solution which can be obtained by using the Laplace transform, then in order to produce an equation with non-constant coefficients we may use a convolution method and multiply the left hand side of equation as follows:

$$p(x, y) ** (a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u)$$

$$= \sum_{i=1}^{n} f_i(x, y) ** g_i(x, y) \tag{1.10}$$

under the boundary conditions

$$u(x, 0) = f_1(x), \quad u(0, y) = f_2(y),$$

$$u_y(x, 0) = h_1(x), \quad u_x(0, y) = h_2(y) \quad \text{and} \quad u(0, 0) = 0.$$

Thus equation (1.10) has also a solution found by using the double Laplace transform, see [8].

Now consider a linear second order partial differential equation with non-constant coefficients in the form of

$$p(x, y) ** [u_{xx} + u_{xy} + u_{yy} + u_x + u_y + u] = f(x, y), \tag{1.11}$$

where $p(x, y) = \sum_{j=1}^{m} \sum_{i=1}^{n} x^i y^j$ is a polynomial and the boundary conditions are given by

$$u(x, 0) = h_1(x), \quad u(0, y) = g_1(y),$$

$$u_y(x, 0) = \frac{\partial}{\partial x} h_1(x), \quad u_x(0, y) = \frac{\partial}{\partial y} g_1(y) \quad \text{and} \quad u(0, 0) = 0, \tag{1.12}$$

then by taking double Laplace transform and using single Laplace transform for equation (1.12), we obtain

$$u(x, y) = L_p^{-1} L_q^{-1} \left[\frac{F(p, q)}{P(p, q)(p^2 + pq + q^2 + p + q + 1)} \right]$$

$$+ L_p^{-1} L_q^{-1} \left[\frac{(p + q + 1) G_1(q)}{(p^2 + pq + q^2 + p + q + 1)} \right]$$

$$+ L_p^{-1} L_q^{-1} \left[\frac{(p + q + 1) H_1(p)}{(p^2 + pq + q^2 + p + q + 1)} \right]$$

$$+ L_p^{-1} L_q^{-1}\left[\frac{qG_1(q) - G_1(0)}{(p^2 + pq + q^2 + p + q + 1)}\right]$$

$$+ L_p^{-1} L_q^{-1}\left[\frac{pH_1(p) - H_1(0)}{(p^2 + pq + q^2 + p + q + 1)}\right]$$

provided that the double inverse Laplace transform exists.

In particular, consider a non-homogenous one dimensional wave equation

$$u_{tt} - u_{xx} = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.13}$$

and the conditions

$$u(0, t) = g_1(t), \quad u(x, 0) = h_1(x),$$

$$u_x(0, t) = \frac{\partial}{\partial t} g_1(t), \quad u_t(x, 0) = \frac{\partial}{\partial t} h_1(x),$$

then by using the double convolution, we can obtain a wave equation with non-constant coefficient in the form

$$p(x, t) ** [u_{tt} - u_{xx}] = \sum_{i=1}^{n} f(x, t) ** g_i(x, t), \tag{1.14}$$

where $p(x, t)$ and $g_i(x, t)$ are polynomials such that the degree $p(x, t)$ is greater than the degree of $g_i(x, t)$.

Now we let $F(x, t)$ be a solution of

$$u_{tt}(x, t) - u_{xx}(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \quad (x, t) \in \mathbb{R}_+^2 \tag{1.15}$$

and further consider $K(x, t)$ is a solution of

$$p(x, t) ** (u_{tt}(x, t) - u_{xx}(x, t)) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \quad (t, x) \in \mathbb{R}_+^2. \tag{1.16}$$

Thus $F(x, t)$ satisfies equation (1.15),

$$F_{tt}(x, t) - F_{xx}(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.17}$$

and similarly, $K(x, t)$ satisfies equation (1.16),

$$K_{tt}(x, t) - K_{xx}(x, t) = \frac{1}{i!\, j!} f(x, t). \tag{1.18}$$

Now we can easily check whether the convolution $F(x, t) ** K(x, t)$ is a solution or not for equation (1.15). By substitution, we obtain

$$(F(x, t) ** K(x, t))_{tt} - (F(x, t) ** K(x, t))_{xx} \stackrel{?}{=} \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.19}$$

by using the partial derivative of the convolution; the left hand side of equation (1.19) follows:

$$F_{tt}(x, t) ** K(x, t) - F_{xx}(x, t) ** K(x, t)$$

$$= F(x, t) ** K_{tt}(x, t) - F(x, t) ** K_{xx}(x, t)$$

and then equation (1.19) can be written in the form

$$F(x, t) ** [K_{tt}(x, t) - K_{xx}(x, t)] \stackrel{?}{=} \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.20}$$

and

$$[F_{tt}(x, t) - F_{xx}(x, t)] ** K(x, t) \stackrel{?}{=} \sum_{i=1}^{n} f(x, t) ** g_i(x, t). \tag{1.21}$$

By substituting equation (1.18) into (1.20) and equation (1.17) into (1.21), we have

$$F(x, t) ** \frac{1}{i!\, j!} f(x, t) \neq \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.22}$$

and

$$f(x, t) ** K(x, t) \neq \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.23}$$

and thus we can easily see from equations (1.22) and (1.23) that the convolution $F(x, t) ** K(x, t)$ is not a solution for equation (1.15), however it is a solution for another type of equation as in the following theorem.

Theorem 4. *If* $F(x, t)$ *is a solution of*

$$u_{tt} - u_{xx} = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.24}$$

under the initial conditions

$$u(0, t) = g_1(t), \quad u(x, 0) = h_1(x),$$

$$u_x(0, t) = \frac{\partial}{\partial t} g_1(t), \quad u_t(x, 0) = \frac{\partial}{\partial t} h_1(x)$$

and $K(x, t)$ *is a solution of*

$$p(x, t) ** (u_{tt} - u_{xx}) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \quad (t, x) \in \mathbb{R}_+^2 \tag{1.25}$$

under the same conditions, then $F(x, t) ** K(x, t)$ *is a solution for the following equation*:

$$u_{tt}(x, t) - u_{xx}(x, t) - h(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \quad (t, x) \in \mathbb{R}_+^2, \tag{1.26}$$

where $f(x, t)$ *is an exponential function and* $p(x, t) = \sum_{j=1}^{m} \sum_{i=1}^{n} x^i t^j$.

Proof. Since $F(x, t)$ is a solution of equation (1.24),

$$F_{tt}(x, t) - F_{xx}(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t) \tag{1.27}$$

holds and $K(x, t)$ is a solution of equation (1.25),

$$K_{tt}(x, t) - K_{xx}(x, t) = \frac{1}{i!\, j!} f(x, t) \tag{1.28}$$

is also true and by substitution, we have

$$(F(x, t) ** K(x, t))_{tt} - (F(x, t) ** K(x, t))_{xx} - h(x, t)$$

$$= \sum_{i=1}^{n} f(x, t) ** g_i(x, t). \tag{1.29}$$

By using the partial derivative of convolution, we obtain

$$F_{tt}(x, t) ** K(x, t) - F_{xx}(x, t) ** K(x, t)$$
$$= F(x, t) ** K_{tt}(x, t) - F(x, t) ** K_{xx}(x, t) \tag{1.30}$$

and then equation (1.29) is followed by

$$[F_{tt}(x, t) - F_{xx}(x, t)] ** K(x, t) - h(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t). \tag{1.31}$$

By substituting equation (1.27) in (1.31), we have

$$f(x, t) ** K(x, t) - h(x, t) = \sum_{i=1}^{n} f(x, t) ** g_i(x, t). \tag{1.32}$$

This shows that the convolution $F(x, t) ** K(x, t)$ is a solution of equation (1.26). □

In the next example, we apply double Laplace transform technique and Theorem 4 in order to solve one dimensional wave equation as follows.

Example 2. Consider the one dimensional wave equation in the forms

$$u_{tt} - u_{xx} = e^{x+t} ** x^2t^3 + e^{x+t} ** x^3t \quad (t, x) \in \mathbb{R}_+^2, \tag{1.33}$$

$$u(x, 0) = xe^x, \quad u_t(x, 0) = xe^x + e^x, \tag{1.34}$$

$$u(0, t) = te^t, \quad u_x(0, t) = te^t + e^t \tag{1.35}$$

by taking double Laplace transform for equation (1.33) and single Laplace transform for equations (1.34) and (1.35) and taking the double inverse Laplace transform, we obtain the solution of equation (1.33) in the form

$$\begin{aligned} u(x, t) = {} & 36 + \frac{3}{20}t^5x - 24e^tx + \frac{1}{2}t^2x^3 + 6t^3 + \frac{1}{60}t^6 + 12xt^2 + 9x^2t \\ & + 6e^xt^2 + 2e^xt^3 + \frac{1}{4}t^4x^2 + \frac{9}{2}x^2t^2 + 30e^xt + \frac{5}{4}t^4 + 9x^2 \\ & + 24xt + \frac{9}{2}e^{-t+x} - 36e^t + 30e^x + \frac{1}{420}t^7 + \frac{1}{4}t^5 - \frac{69}{2}e^{t+x} \\ & + \frac{3}{2}x^2t^3 + 4xt^3 + 18t^2 + x^3 + \frac{1}{20}t^5x^2 + \frac{3}{4}t^4x - 9e^tx^2 \\ & + x^3t + 9e^{t+x}t + 36t + 24x - e^tx^3 + \frac{1}{6}t^3x^3. \end{aligned}$$

Now if we consider to multiply the left hand side of equation (1.33) with a non-constant coefficient x^3t^4 by using the double convolution and use the same technique that was applied above we get the solution in the form of

$$v(x,\ t) = \frac{7}{96}e^{t+x} + \frac{1}{16}e^{t+x}t - \frac{1}{32}e^{-t+x}. \tag{1.36}$$

If we take second derivatives of equation (1.36), and taking the difference we obtain a nonhomogenous term and plus a function $h(x,\ t)$, that is,

$$(x^3t^4) ** (v_{tt} - v_{xx}) = (u_{tt} - u_{xx}) + h(x,\ t).$$

We can also apply same method to solve non-homogenous one dimensional heat as well as Laplace's equations in two dimensions.

Acknowledgement

The authors gratefully acknowledge that this research paper partially supported by Ministry of Science, Technology and Innovations (MOS Ti), Malaysia under the e-Science Fund Grant 06-01-04-SF0115.

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