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AN EXPRESSION FOR THE BAKER-FORRESTER'S CONSTANT TERM BY q-GAMMA FUNCTIONS

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Abstract

The Baker-Forrester's constant term conjecture is an extension of the q-Morris constant term identity [2]. In this conjecture, the cases from $N_1 = 1$ to $N_1 = 6$ are proved [6, 10]. Here we consider that the case $N_1 = N_2 = 2$ of the multi-component q-generalization of Baker-Forrester's constant term conjecture. The purpose of this paper is to express the constant term for the special cases $N_0 = 0$, $N_1 = N_2 = 2$ and $a - b = 3$ (general k) by the product of q-gamma functions.

1. Introduction and Preliminary

1.1. Baker-Forrester's conjecture

Fix q with $0 < q < 1$ and set $(x)_\infty = (x;\, q)_\infty = \prod_{i=0}^{\infty}(1 - xq^i)$ and $(x)_a = (x;\, q)_a = (x)_\infty/(xq^a)_\infty$. Define

$$\Gamma_q(x) := \frac{(q;\, q)_\infty}{(q^x;\, q)_\infty}(1-q)^{1-x} \quad \text{and} \quad [x]_q := \frac{1-q^x}{1-q} = \frac{\Gamma_q(x+1)}{\Gamma_q(x)}.$$

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For complex Q, we denote by ${}_Q\Delta_N(x)$ the product $\prod_{1\le i<j\le N}(x_i - Qx_j)$ and by $\Delta_N(x)$ the product $\prod_{1\le i<j\le N}(x_i - x_j)$.

Suppose $a, b, k \in \mathbb{Z}_{\ge 0}$ and let

$$H_q(x_1, \ldots, x_N; a, b, k) := \prod_{l=1}^{N}(x_l)_a\left(\frac{q}{x_l}\right)_b \prod_{1\le i<j\le N}\left(q\frac{x_j}{x_i}\right)_k\left(\frac{x_i}{x_j}\right)_k.$$

The q-Morris constant term identity [5, 7, 12] states that

$$CT_{\{x\}}H_q(x_1, \ldots, x_N; a, b, k) = M_N(a, b, k; q),$$

where $CT_{\{x\}}$ denotes the constant term in the Laurent polynomial expansion as a function of $\{x_1, \ldots, x_N\}$ and

$$M_N(a, b, k; q) := \prod_{l=0}^{N-1}\frac{\Gamma_q(a+b+1+kl)\Gamma_q(1+k(l+1))}{\Gamma_q(a+1+kl)\Gamma_q(b+1+kl)\Gamma_q(1+k)}.$$

In [2], Baker and Forrester have given a conjecture extending this identity to read as follows.

Conjecture 1.1. For $a, b, k \in \mathbb{Z}_{\ge 0}$, we have

$$CT_{\{x\}}\prod_{N_0+1\le i<j\le N}\left(1-q^k\frac{x_i}{x_j}\right)\left(1-q^{k+1}\frac{x_j}{x_i}\right)H_q(x_1, \ldots, x_N; a, b, k)$$

$$= M_{N_0}(a, b, k; q)$$

$$\times\prod_{j=0}^{N_1-1}\frac{[(1+k)(j+1)]_q\Gamma_q((k+1)j+a+b+kN_0+1)\Gamma_q((k+1)(j+1)+kN_0)}{\Gamma_q(k+2)\Gamma_q((k+1)j+a+kN_0+1)\Gamma_q((k+1)j+b+kN_0+1)}, \quad (1.1)$$

where

$$N = N_0 + N_1.$$

Conjecture 1.2. Let

$$D_p(N_1, \ldots, N_p; N_0, a, b, k; q)$$

$$= CT_{\{x\}} \prod_{N_0+1 \le i < j \le N_0+N_1} \left(1 - q^k \frac{x_i}{x_j}\right)\left(1 - q^{k+1} \frac{x_j}{x_i}\right)$$

$$\times \prod_{N_0+N_1+1 \le i < j \le N_0+N_1+N_2} \left(1 - q^k \frac{x_i}{x_j}\right)\left(1 - q^{k+1} \frac{x_j}{x_i}\right)$$

$$\times \cdots \times \prod_{\sum_{q=0}^{p-1} N_q + 1 \le i < j \le \sum_{q=0}^{p} N_p} \left(1 - q^k \frac{x_i}{x_j}\right)\left(1 - q^{k+1} \frac{x_j}{x_i}\right) H_q(x_1, \ldots, x_N; a, b, k), \tag{1.2}$$

where $N = \sum_{q=0}^{p} N_q$, and suppose $N_j \le N_p + 1$ $(j = 1, \ldots, p-1)$. As a function of N_p the function D_p satisfies the recurrence

$$\frac{D_p(N_1, \ldots, N_{p-1}, N_p + 1; N_0, a, b, k; q)}{D_p(N_1, \ldots, N_{p-1}, N_p; N_0, a, b, k; q)}$$

$$= [(k+1)(N_p + 1)]_q$$

$$\times \frac{\Gamma_q\left((k+1)N_p + a + b + k\sum_{j=1}^{p-1} N_j + 1\right)\Gamma_q\left((k+1)(N_p+1) + k\sum_{j=1}^{p-1} N_j\right)}{\Gamma_q(k+2)\Gamma_q((k+1)N_p + a + kN_0 + 1)\Gamma_q\left((k+1)N_p + b + k\sum_{j=0}^{p-1} N_j + 1\right)}.$$

In the limit $q \to 1$, the formula in above conjecture is equivalent to the recurrence equation conjectured in [4].

In Conjecture 1.1, they proved this conjecture in the special cases $a = k$ and $a = b = 0$ by employing a theorem of Bressoud and Goulden [3]. Also the special case $N_1 = 2$ is proved by adapting the method of Zeilberger [12]. In [6], the cases $N_1 = 2, 3$ are proved by making use of the q-integration formula of Macdonald polynomial $P_\lambda(x_1, \ldots, x_N; q, t)$ [8]. Moreover, in [10], it is proved that the cases $N_1 = 4, 5, 6$. While the proof given in [2], is of strongly combinatorial nature, our proof was computational and straightforward with some tedious calculation.

Now, we consider Conjecture 1.2.

1.2. The recurrence equation in Conjecture 1.2

We gave a proof of Baker-Forrester's constant term conjecture for the cases $N_1 = 2, 3$ [6]. In these cases, the recurrence equation in Conjecture 1.2 is satisfied.

Proposition 1.3. *The case* $N_1 = 2$ *satisfies the recurrence*

$$\frac{D_1(N_1+1=3;\ N_0)}{D_1(N_1=2;\ N_0)}$$

$$= \frac{[(k+1)(N_1+1)]_q \Gamma_q((k+1)N_1+a+b+1)\Gamma_q((k+1)(N_1+1))}{\Gamma_q(k+2)\Gamma_q((k+1)N_1+a+kN_0+1)\Gamma_q((k+1)N_1+b+kN_0+1)},$$

where $N = N_0 + N_1$.

In the limit $q \to 1$, *the formula is equivalent to the recurrence equation conjectured in* [4].

Proof. It is easy to see by the result in [6]. □

1.3. The q-integration formula of Macdonald polynomials

Let $\lambda = (\lambda_1, ..., \lambda_N)$ be a partition. The integer $|\lambda| = \sum_i \lambda_i$ is called the *weight* of λ. The number of nonzero λ_i is called the *length* of λ. For partitions λ and μ with the same weight $|\lambda| = |\mu|$, we define dominance (partial) ordering $\geq$ by

$$\lambda \geq \mu \Leftrightarrow \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \text{for all } i \geq 1.$$

Given a partition $\lambda = (\lambda_1, ..., \lambda_N)$ of length $\leq N$, the monomial symmetric polynomial $m_\lambda = m_\lambda(x_1, ..., x_N)$ is defined by

$$m_\lambda(x_1, ..., x_N) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N},$$

where the sum is over all distinct monomials obtainable from $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$ by permutations of the x's. Let q and t be independent indeterminates. We have the following theorem and definition of Macdonald polynomials ([11, Chapter 6, (4.7)]):

Theorem 1.4. *Let* $\mathbb{Q}(q, t)$ *be the field of rational functions in q and t. For each partition* λ *of length* $\leq N$, *there exists a unique symmetric polynomial* $P_\lambda = P_\lambda(x_1, \ldots, x_N; q, t)$ *with coefficients in* $\mathbb{Q}(q, t)$ *satisfying*

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu,$$

where $u_{\lambda\mu} \in \mathbb{Q}(q, t)$ *and* $u_{\lambda\lambda} = 1$; *and*

$$D_N^1 P_\lambda = e_\lambda P_\lambda,$$

where D_N^1 *is defined by*

$$D_N^1 = \sum_{i=1}^{N} A_i T_{q, x_i}, \quad A_i := \prod_{j=1,\, j \neq i}^{N} \frac{tx_i - x_j}{x_i - x_j},$$

$$T_{q, x_i} f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, qx_i, \ldots, x_N)$$

and $e_\lambda = e_\lambda(q, t)$ *is*

$$e_\lambda(q, t) := \sum_{i=1}^{N} q^{\lambda_i} t^{N-i}.$$

Moreover, the Macdonald polynomial P_λ is the joint eigenpolynomial for the q-difference operators

$$D_N^r = \sum_{I \subset \{1, 2, \ldots, N\}, |I| = r} A_I(x, t) \prod_{i \in I} T_{q, x_i} \quad (r = 0, 1, \ldots, N), \tag{1.3}$$

where

$$A_I(x, t) = t^{\frac{r(r-1)}{2}} \prod_{i \in I,\, j \notin I} \frac{tx_i - x_j}{x_i - x_j}$$

and the eigenvalue for D_N^r is the coefficient of X^r in the polynomial $\prod_{i=1}^{N}(1 + Xt^{N-i} q^{\lambda_i})$ ([11, pp. 315-317]).

Next we recall the q-integration formula of Macdonald polynomials ([8, Proposition 5.2]). Let λ' be the conjugate partition of a partition λ. For each square $s = (i, j)$ in the diagram of λ, let

$$a(s) = \lambda_i - j, \quad a'(s) = j - 1,$$
$$l(s) = \lambda'_j - i, \quad l'(s) = i - 1$$

and put

$$h_\lambda(q, t) = \prod_{s\in\lambda}(1 - q^{a(s)}t^{l(s)+1}) \quad \text{and} \quad (a)_\lambda^{(q,t)} = \prod_{s\in\lambda}(t^{l'(s)} - q^{a'(s)}a). \tag{1.4}$$

It holds [11] that

$$P_\lambda(1, t, t^2, \dots, t^{N-1}) = \frac{(t^N)_\lambda^{(q,t)}}{h_\lambda(q, t)}. \tag{1.5}$$

We write $H_\lambda(t) := \dfrac{(t^N)_\lambda^{(q,t)}}{h_\lambda(q, t)}$.

The q-integral of a function $f(x_1, \dots, x_N)$ over $[0, 1]^N$ is defined by

$$\int_{[0,1]^N} f(x_1, \dots, x_N)d_qx_1 \cdots d_qx_N = (1 - q)^N \sum_{s_i\in\mathbb{Z}_{\geq 0}} f(q^{s_1}, \dots, q^{s_N})q^{s_1} \cdots q^{s_N}.$$

For a Macdonald polynomial $P_\lambda(x_1, \dots, x_N; q, t)$, we have

$$\int_{[0,1]^N} P_\lambda(x; q, q^k)\prod_{j=1}^{N} x_j^{x-1} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \prod_{1\leq i<j\leq N} x_i^{2k}\left(q^{1-k}\frac{x_j}{x_i}\right)_{2k} d_qx_1 \cdots d_qx_N$$

$$= q^{kx\binom{N}{2}+2k^2\binom{N}{3}} \frac{(q^{kN})_\lambda^{(q,q^k)}}{h_\lambda(q, q^k)}$$

$$\times \prod_{i=1}^{N} \frac{\Gamma_q(ik + 1)\Gamma_q(x + (N - i)k + \lambda_i)\Gamma_q(y + (N - i)k)}{\Gamma_q(k + 1)\Gamma_q(x + y + (2N - i - 1)k + \lambda_i)}$$

$$= {}_qS_N(x, y; k)\frac{(q^{kN})_\lambda^{(q,q^k)}}{h_\lambda(q, q^k)} \prod_{i=1}^{N} \frac{(q^{x+(N-i)k})_{\lambda_i}}{(q^{x+y+(2N-i-1)k})_{\lambda_i}}, \tag{1.6}$$

where

$$ {}_qS_N(x,\ y;\ k) := \int_{[0,1]^N} \prod_{j=1}^{N} x_j^{x-1} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \prod_{1\le i<j\le N} x_i^{2k} \left(q^{1-k}\frac{x_j}{x_i}\right)_{2k} d_qx_1 \cdots d_qx_N $$

$$ = q^{kx\binom{N}{2}+2k^2\binom{N}{3}} \prod_{i=1}^{N} \frac{\Gamma_q(ik+1)\Gamma_q(x+(N-i)k)\Gamma_q(y+(N-i)k)}{\Gamma_q(k+1)\Gamma_q(x+y+(2N-i-1)k)}. $$

We shall use the following q-Selberg type integral evaluation; Proposition 4.1 in [2] that for a general Laurent polynomial $f(x_1, \ldots, x_N)$,

$$ \left(\frac{\Gamma_q(x+y)}{\Gamma_q(x)\Gamma_q(y)}\right)^N \int_{[0,1]^N} \prod_{j=1}^{N} x_j^{x-1} \frac{(qx_j;\ q)_\infty}{(q^y x_j;\ q)_\infty} f(x_1, \ldots, x_N) d_qx_1 \cdots d_qx_N $$

$$ = \left(\frac{(q)_a(q)_b}{(q)_{a+b}}\right)^N CT\{x\} \prod_{i=1}^{N} (x_j;\ q)_a \left(\frac{q}{x_j};\ q\right)_b f(q^{-(b+1)}x_1, \ldots, q^{-(b+1)}x_N) \tag{1.7} $$

provided $x = -b,\ \ y = a+b-1$ and

$$ f_{N_0, N_1, N}(x) := \prod_{N_0+1\le i<j\le N_0+N_1} \left(1-q^k\frac{x_i}{x_j}\right)\left(1-q^{k+1}\frac{x_j}{x_i}\right) $$

$$ \times \prod_{N_0+N_1+1\le i<j\le N} \left(1-q^k\frac{x_i}{x_j}\right)\left(1-q^{k+1}\frac{x_j}{x_i}\right) f_N, $$

$$ f_N(x) := \prod_{1\le i<j\le N} \left(\frac{x_i}{x_j};\ q\right)_k \left(q\frac{x_j}{x_i};\ q\right)_k. $$

Let $\mathcal{A}$ denote the antisymmetrizer

$$ \mathcal{A}f(x_1, \ldots, x_N) = \sum_{\sigma\in S_N} \varepsilon(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}), $$

where S_N is the symmetric group of degree N and $\varepsilon(\sigma)$ is the sign of the permutation σ. Since we have

$$ f_N = (-1)^{\frac{kN(N-1)}{2}} q^{\frac{k(k-1)N(N-1)}{4}} \prod_{j=1}^{N} x_j^{-k(N-1)} \prod_{1\le i<j\le N} x_i^{2k} \left(q^{1-k}\frac{x_j}{x_i};\ q\right)_{2k}, \tag{1.8} $$

the integral evaluation is equivalent to

$$\int_{[0,1]^N} \prod_{j=1}^{N} x_j^{x-k(N-1)-N_1-N_2+1} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \prod_{1\le i<j\le N} x_i^{2k} \left(q^{1-k}\frac{x_j}{x_i}\right)_{2k}$$

$$\times \prod_{i=1}^{N_0} x_i^{N_1-1} \prod_{N_0+1\le i<j\le N_0+N_1} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)$$

$$\times \prod_{i=1}^{N_0+N_1} x_i^{N_2-1} \prod_{N_0+N_1+1\le i<j\le N} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)\, d_qx_1\cdots d_qx_N$$

$$= B_N \left(\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\right)^N \left(\frac{(q)_a(q)_b}{(q)_{a+b}}\right)^N \ \ \text{((1.2) of conjecture)}, \tag{1.9}$$

where

$$B_N := (-1)^{\frac{kN(N-1)}{2}+\frac{N_1(N_1-1)}{2}+\frac{N_2(N_2-1)}{2}} q^{-\frac{k(k-1)N(N-1)}{4}-\frac{kN_1(N_1-1)}{2}-\frac{kN_2(N_2-1)}{2}}.$$

The symmetrized form of this evaluation is

$$\int_{[0,1]^N} \prod_{j=1}^{N} x_j^{x-k(N-1)-N_1-N_2+1} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \prod_{1\le i<j\le N} x_i^{2k-1} \left(q^{1-k}\frac{x_j}{x_i}\right)_{2k-1}$$

$$\times \mathcal{A}\Bigg({}_{q^k}\Delta_N(x) \prod_{i=1}^{N_0} x_i^{N_1-1} \prod_{N_0+1\le i<j\le N_0+N_1} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)$$

$$\times \prod_{i=1}^{N_0+N_1} x_i^{N_2-1} \prod_{N_0+N_1+1\le i<j\le N} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)\Bigg) d_qx_1\cdots d_qx_N$$

$$= N!\,B_N \left(\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\right)^N \left(\frac{(q)_a(q)_b}{(q)_{a+b}}\right)^N \ \ \text{((1.2) of conjecture)}. \tag{1.10}$$

Note that

$$\mathcal{A}({}_{q^k}\Delta_N(x)) = \frac{(q^k;q^k)_N}{(1-q^k)^N}\Delta_N(x),$$

which can easily be verified and is also the simplest case of the following Kadell's lemma ([7, (4.10), p. 976], [11, Chapter 3, (1.3)]):

Lemma 1.5. *Let* $M \subset \{1, \ldots, N\}$. *Then*

$$\mathcal{A}\left(\prod_{j \in M} x_{jQ} \Delta_N(x)\right) = Q^{e(M)} \frac{(Q;\, Q)_{|M|}(Q;\, Q)_{N-|M|}}{(1-Q)^N} e_{|M|}(x) \Delta_N(x),$$

where $e(M) = |\{(i,\ j) | 1 \le i < j \le N,\ i \notin M,\ j \in M\}|$ *and* $e_r(x)$ *denotes the elementary symmetric polynomial of degree* r.

Hence symmetrizing the integrand of the left-hand side of (1.6) gives

$$\int_{[0,1]^N} P_\lambda(x;\, q,\, q^k) \prod_{j=1}^{N} x_j^{x-1} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \Delta_N(x)$$

$$\times \prod_{1 \le i < j \le N} x_i^{2k-1} \left(q^{1-k} \frac{x_j}{x_i}\right)_{2k-1} d_q x_1 \cdots d_q x_N$$

$$= N! \frac{(1-q^k)^N}{(q^k;\, q^k)_N} {}_q S_N(x,\ y;\ k) \frac{(q^{kN})_\lambda^{(q,\, q^k)}}{h_\lambda(q, q^k)} \prod_{i=1}^{N} \frac{(q^{x+(N-i)k})_{\lambda_i}}{(q^{x+y+(2N-i-1)k})_{\lambda_i}}. \tag{1.11}$$

We express the symmetric homogeneous polynomial

$$F_{N, N_1, N_2}(x_1, \ldots, x_N)$$

$$:= \Delta_N(x)^{-1} \mathcal{A}\Bigg({}_{q^k}\Delta_N(x) \prod_{i=1}^{N_0} x_i^{N_1 - 1} \prod_{N_0+1 \le i < j \le N_0 + N_1} (x_i - q^{-k} x_j)(x_i - q^{k+1} x_j)$$

$$\times \prod_{i=1}^{N_0+N_1} x_i^{N_2 - 1} \prod_{N_0 + N_1 + 1 \le i < j \le N} (x_i - q^{-k} x_j)(x_i - q^{k+1} x_j) \Bigg) \tag{1.12}$$

in terms of Macdonald polynomials $P_\lambda(x_1, \ldots, x_N;\, q,\, q^k)$.

We consider that the cases $N_1 = N_2 = 2$ and $N_0 = N - 4$. Let $\lambda^{(1)} = (3,\, 2^{N-3},\, 1)$, $\lambda^{(2)} = (3,\, 2^{N-4},\, 1^3)$, $\lambda^{(3)} = (2^{N-1})$ and $\lambda^{(4)} = (2^{N-2},\, 1^2)$. We have

$$\int_{[0,1]^N} \prod_{j=1}^{N} x_j^{x-k(N-1)-3} \frac{(qx_j)_\infty}{(q^y x_j)_\infty} \prod_{1 \le i < j \le N} x_i^{2k-1} \left(q^{1-k} \frac{x_j}{x_i}\right)_{2k-1}$$

$$\times \mathcal{A}\Bigg({}_{q^k}\Delta_N(x) \prod_{i=1}^{N_0} x_i \prod_{N_0+1 \le i < j \le N_0 + 2} (x_i - q^{-k} x_j)(x_i - q^{k+1} x_j)$$

$$\times \prod_{i=1}^{N_0+2} x_i \prod_{N_0+3\le i<j\le N} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)\Bigg) d_q x_1 \cdots d_q x_N$$

$$= N!\, B_N \left(\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\right)^N \left(\frac{(q)_a(q)_b}{(q)_{a+b}}\right)^N \quad \text{((1.2) of conjecture)}, \tag{1.13}$$

$$F_{N,2,2}(x_1, \ldots, x_N)$$

$$:= \Delta_N(x)^{-1}\mathcal{A}\Big(_{q^k}\Delta_N(x)\prod_{i=1}^{N-4} x_i \prod_{N-3\le i<j\le N-2} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)$$

$$\times \prod_{i=1}^{N-2} x_i \prod_{N-1\le i<j\le N} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)\Big).$$

We put

$$F_{N,2,2} := \sum_{i=1}^{4} b_{\lambda^{(i)}} P_{\lambda^{(i)}}. \tag{1.14}$$

To calculate the coefficient $b_{\lambda^{(i)}}$, we shall make use of extending version of the following Kaneko's lemma ([6, Lemma 2.2]) for the case $N_1 = N_2 = 2$:

Lemma 1.6. *We assume* $t = q^k$ *in the definition of* D_N^r, (1.3). *Then*

$$D_N^r F_{N,N_1}(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = 0 \quad \begin{cases} r = 0, \ldots, N, & \text{if } N_1 \ge 3, \\ r = 0, & \text{if } N_1 = 2. \end{cases}$$

Proof. We have

$$A_I(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = 0$$

unless $I = \{N-r+1, N-r+2, \ldots, N\}$. Hence it suffices to show

$$\prod_{i\in\{N-r+1, N-r+2, \ldots, N\}} T_{q,x_i}\Bigg\{\mathcal{A}\Big(_{q^k}\Delta_N(x)\prod_{i=1}^{N_0} x_i^{N_1-1}$$

$$\times \prod_{N_0+1\le i<j\le N} (x_i - q^{-k}x_j)(x_i - q^{k+1}x_j)\Big)\Bigg\} = 0,$$

where $x_i = q^{k(i-1)}$, $i = 1, \ldots, N$. For any element $\sigma \in S_N$, put

$$\sigma(p_1) = N - r + 1, \sigma(p_2) = N - r + 2, \ldots, \sigma(p_r) = N,$$

$$\sigma(q_1) = 1, \sigma(q_2) = 2, \ldots, \sigma(q_{N-r}) = N - r.$$

We may assume that $1 \le p_1 < \cdots < p_r \le N$ and $1 \le q_1 < \cdots < q_{N-r} \le N$. In fact, if this is not the case, then it is clear that there exists a pair $p_i > p_{i+1}$, $1 \le i \le r - 1$, or a pair $q_j > q_{j+1}$, $1 \le j \le N - r - 1$. In both cases the term corresponding to σ is zero because of the factor ${}_{q^k}\Delta_N(x)$. Since $N_1 \ge 3$ or $r = 0$ if $N_1 = 2$, it holds that $N_0 + 1 \le p_{r-1} < p_r \le N$ or $N_0 + 1 \le q_{N-r-1} < q_{N-r} \le N$. Hence the term corresponding to σ is zero because of the factor $\prod_{N_0+1 \le i < j \le N} (x_i - q^{-k} x_j)$. □

We have the following extension of Kaneko’s lemma:

Corollary 1.7. *We assume* $t = q^k$ *in the definition of* D_N^r, (1.3). *Then for the cases* $N_1 = N_2 = 2$ *and* $r = 0, 1, 2,$ *we have*

$$D_N^r F_{N,2,2}(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = 0.$$

2. The Case $N_1 = N_2 = 2$ of Conjecture

2.1. Application of the q-integration formula of Macdonald polynomials

To rewrite in terms of Macdonald polynomials, we set

1. $\Delta_N(x)^{-1} \mathcal{A}\left({}_{q^k}\Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^2 x_N^0 \right) = \sum_{i=1}^{4} w_{\lambda^{(i)}}^1 P_{\lambda^{(i)}},$

2. $\Delta_N(x)^{-1} \mathcal{A}\left({}_{q^k}\Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^0 x_N^2 \right) = \sum_{i=1}^{4} w_{\lambda^{(i)}}^2 P_{\lambda^{(i)}},$

3. $\Delta_N(x)^{-1} \mathcal{A}\left({}_{q^k}\Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3} x_{N-2}^3 x_{N-1}^2 x_N^0 \right) = \sum_{i=1}^{4} w_{\lambda^{(i)}}^3 P_{\lambda^{(i)}},$

4. $\Delta_N(x)^{-1} \mathcal{A}\left({}_{q^k}\Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3} x_{N-2}^3 x_{N-1}^0 x_N^2 \right) = \sum_{i=1}^{4} w_{\lambda^{(i)}}^4 P_{\lambda^{(i)}},$ (2.1)

where let

$$\lambda^{(1)} = (3,\ 2^{N-3},\ 1),\quad \lambda^{(2)} = (3,\ 2^{N-4},\ 1^3),\quad \lambda^{(3)} = (2^{N-1}),\quad \lambda^{(4)} = (2^{N-2}, 1^2).$$

For each integer $m \geq 0$, put

$$v_m(t) = \prod_{i=1}^{m} \frac{1-t^i}{1-t}$$

and for a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ of length $\leq N$, define

$$v_\lambda = \prod_{i \geq 0} v_{m_i},$$

where m_i is the number of λ_j equal to i.

Lemma 2.1. *We have*

$$w^1_{\lambda^{(1)}} = Q^{N-3} v_{\lambda^{(1)}},\quad w^2_{\lambda^{(1)}} = Q^{N-2} v_{\lambda^{(1)}},$$

$$w^3_{\lambda^{(1)}} = Q^{N-2} v_{\lambda^{(1)}},\quad w^4_{\lambda^{(1)}} = Q^{N-1} v_{\lambda^{(1)}}.$$

Proof. We give only the proof of the formula of $w^1_{\lambda^{(1)}}$. Proofs of other formulas are similar. We may put

$$\mathcal{A}\left(q^k \Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^2 x_N^0 \right) = \Delta_N(x) \sum_{i=1}^{4} c_i m_{\lambda^{(i)}}.$$

We have the coefficient of $x_1^{N+2} x_2^N x_3^{N-1} \cdots x_{N-2}^4 x_{N-1}^2$ in the right-hand side is c_1.

To calculate the antisymmetrizer $\mathcal{A}$, we first note that the symmetric group S_N, has a right coset decomposition with respect to the symmetric group S_{N-1} which fix the letter N : $S_N = \bigcup_{i=1}^{N-1} (i,\ N) S_{N-1} \bigcup S_{N-1}$, where $(i,\ N)$ denotes the transposition of i and N. Accordingly the antisymmetrizer $\mathcal{A} = \mathcal{A}_N$ decomposes in the following way:

$$\mathcal{A}_N = \mathcal{A}_{N-1} - \sum_{i=1}^{N-1} (i,\ N) \mathcal{A}_{N-1},$$

where $\mathcal{A}_{N-1}$ is the antisymmetrizer with respect to $1, \ldots, N-1$. For $1 \leq i \leq N-1$, let $\mathcal{A}^{(i)}_{N-1}$ denote the antisymmetrizer with respect to $\{1, \ldots, i-1, N, i+1, \ldots, N-1\}$.

Clearly it suffices to consider the term

$$x_N^0 (x_1 - q^k x_N)(x_2 - q^k x_N)\cdots(x_{N-1} - q^k x_N)$$
$$\times \mathcal{A}_{N-1}\left({}_{q^k}\Delta_{N-1}(x_1, x_2, \ldots, x_{N-1})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^2\right).$$

Moreover, we have

$$\mathcal{A}_{N-1}\left({}_{q^k}\Delta_{N-1}(x_1, x_2, \ldots, x_{N-1})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^2\right)$$
$$= x_{N-1}^2 (x_1 - q^k x_{N-1})(x_2 - q^k x_{N-1})\cdots(x_{N-2} - q^k x_{N-1})$$
$$\times \mathcal{A}_{N-2}\left({}_{q^k}\Delta_{N-2}(x_1, x_2, \ldots, x_{N-2})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2}\right)$$
$$+\cdots$$
$$(-1)^{N-1-i} x_i^2 (x_1 - q^k x_i)(x_2 - q^k x_i)\cdots(x_{N-1} - q^k x_i)$$
$$\times \mathcal{A}_{N-2}^{(i)}({}_{q^k}\Delta_{N-2}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-1}) x_1^2 \cdots x_{i-1}^2 x_{N-1}^2 x_{i+1}^2 \cdots x_{N-3}^3 x_{N-2})$$
$$+\cdots.$$

Here, for example, we have

$$\mathcal{A}_{N-2}\left({}_{q^k}\Delta_{N-2}(x_1, x_2, \ldots, x_{N-2})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2}\right)$$
$$= x_{N-2}(x_1 - q^k x_{N-2})(x_2 - q^k x_{N-2})\cdots(x_{N-3} - q^k x_{N-2})$$
$$\times \mathcal{A}_{N-3}\left({}_{q^k}\Delta_{N-3}(x_1, x_2, \ldots, x_{N-3})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3\right),$$

$$A_{N-3}\left({}_{q^k}\Delta_{N-3}(x_1, x_2, \ldots, x_{N-3})\prod_{i=1}^{N-4} x_i^2 x_{N-3}^3\right)$$
$$= (x_1^2 x_2^2 \cdots x_{N-3}^2)$$
$$\times [x_{N-3}(x_1 - q^k x_{N-3})(x_2 - q^k x_{N-3})\cdots(x_{N-4} - q^k x_{N-3})$$
$$\times \mathcal{A}_{N-4}({}_{q^k}\Delta_{N-4}(x_1, x_2, \ldots, x_{N-4}))$$
$$+\cdots + (-1)^{N-3-i} x_i (x_1 - q^k x_i)\cdots(x_{i-1} - q^k x_i)(x_{i+1} - q^k x_i)\cdots(x_{N-3} - q^k x_i)$$
$$\times \mathcal{A}_{N-4}^{(i)}({}_{q^k}\Delta_{N-4}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-3})) + \cdots],$$

where

$$\mathcal{A}_{N-4}^{(i)}\left({}_{q^k}\Delta_{N-4}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-3})\right)$$

$$= \prod_{i=1}^{N-4} \frac{1-q^{ki}}{1-q^k} \Delta_{N-4}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-3})$$

by Lemma 1.5.

The coefficient in the right-hand side is easy to compute and the result is $c_1 = q^{k(N-3)} v_{\lambda(1)}$. Since we have $P_{\lambda(1)} = m_{\lambda(1)} +$ (the other terms) by Theorem 1.4, we obtain

$$\Delta_N(x)^{-1} \mathcal{A}\left({}_{q^k}\Delta_N(x) \prod_{i=1}^{N-4} x_i^2 x_{N-3}^3 x_{N-2} x_{N-1}^2 x_N^0 \right)$$

$$= \sum_{i=1}^{4} w_{\lambda(i)}^1 P_{\lambda(i)} = w_{\lambda(1)}^1 m_{\lambda(1)} + (\text{the other terms}).$$

Therefore, we get

$$w_{\lambda(1)}^1 = c_1 = q^{k(N-3)} v_{\lambda(1)}. \qquad \square$$

Proposition 2.2. *We have* $b_{\lambda(1)} = Q^{N-3}(1+qQ)^2 v_{\lambda(1)}$.

Proof. Since the coefficients of the left-hand sides of 1-4 of (2.1) in $F_{N,2,2}$ are $1, q, q, q^2$, respectively, it follows that $b_{\lambda(1)} = Q^{N-3}(1+qQ)^2 v_{\lambda(1)}$. $\square$

By Corollary 1.7, we have the following.

Proposition 2.3. *We have*

(1) $D_N^0 F_{N,2,2}(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = \sum_{i=1}^4 b_{\lambda(i)} P_{\lambda(i)} = 0,$

(2) $D_N^1 F_{N,2,2}(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = \sum_{i=1}^4 b_{\lambda(i)} e_{\lambda(i)}^1 P_{\lambda(i)} = 0,$

(3) $D_N^2 F_{N,2,2}(1, q^k, q^{2k}, \ldots, q^{k(N-1)}) = \sum_{i=1}^4 b_{\lambda(i)} e_{\lambda(i)}^2 P_{\lambda(i)} = 0,$

where $e_\lambda^r(q, t)$ *is the eigenvalue of* P_λ *for* D_N^r.

We set $A_i := e^1_{\lambda^{(i)}} - e^1_{\lambda^{(4)}}$ and $B_i := e^2_{\lambda^{(i)}} - e^2_{\lambda^{(4)}}$.

By Proposition 2.2 and solving the simultaneous system of equations in Proposition 2.3, we obtain

$$b_{\lambda^{(2)}} = \frac{A_3B_1 - A_1B_2}{A_2B_3 - A_3B_2}\frac{H_{\lambda^{(1)}}}{H_{\lambda^{(2)}}} b_{\lambda^{(1)}}, \quad b_{\lambda^{(3)}} = \frac{A_1B_2 - A_2B_1}{A_2B_3 - A_3B_2}\frac{H_{\lambda^{(1)}}}{H_{\lambda^{(3)}}} b_{\lambda^{(1)}},$$

$$b_{\lambda^{(4)}} = -\frac{H_{\lambda^{(1)}}}{H_{\lambda^{(4)}}} b_{\lambda^{(1)}}(1 + (i) + (ii)),$$

where

$$\frac{A_3B_1 - A_1B_2}{A_2B_3 - A_3B_2} := (i) \text{ and } \frac{A_1B_2 - A_2B_1}{A_2B_3 - A_3B_2} := (ii).$$

Replacing x with $x - k(N-1) - 2$ in (1.11), we have

$$\sum_{j=1}^{4}\prod_{i=1}^{N}\frac{(q^{x-k(i-1)-2})_{\lambda^{(j)}_{(i)}}}{(q^{x+y+k(N-i)-2})_{\lambda^{(j)}_{(i)}}} = \prod_{i=1}^{N-1} L_i \prod_{i=1}^{N-3} M_i \sum_{i=1}^{4} K_i,$$

where

$$L_i = \frac{1-q^{x-k(i-1)-2}}{1-q^{x+y+k(N-i)-2}}, \quad M_i = \frac{1-q^{x-k(i-1)-1}}{1-q^{x+y+k(N-i)-1}},$$

$$K_1 = \frac{(1-q^x)(1-q^{x-k(N-3)-1})}{(1-q^{x+y+k(N-1)})(1-q^{x+y+2k-1})},$$

$$K_2 = \frac{(1-q^x)(1-q^{x-k(N-1)-2})}{(1-q^{x+y+k(N-1)})(1-q^{x+y-2})},$$

$$K_3 = \frac{(1-q^{x-k(N-2)-1})(1-q^{x-k(N-3)-1})}{(1-q^{x+y+k-1})(1-q^{x+y+2k-1})},$$

$$K_4 = \frac{(1-q^{x-k(N-1)-2})(1-q^{x-k(N-3)-1})}{(1-q^{x+y-2})(1-q^{x+y+2k-1})},$$

$$\begin{aligned} &{}_qS_N(x-k(N-1)-2,\ y,\ k) \\ &= q^{6kx-10k^2-12k} \\ &\quad\times\prod_{i=1}^{N}\frac{\Gamma_q(ik+1)\Gamma_q(x-(i-1)k-2)\Gamma_q(y+(N-i)k)}{\Gamma_q(k+1)\Gamma_q(x+y+(N-i)k-2)}. \end{aligned}$$

2.2. Expression by the product of q-gamma functions

Making use of the q-integration formula of Macdonald polynomials (1.11), we obtain the following in the left-hand side of (1.13):

$$\prod_{i=1}^{N-1} L_i \prod_{i=1}^{N-3} M_i \sum_{i=1}^{4} b_{\lambda^{(i)}} H_{\lambda^{(i)}} K_i = \prod_{i=1}^{N-1} L_i \prod_{i=1}^{N-3} M_i b_{\lambda^{(1)}} H_{\lambda^{(1)}} \widetilde{R}, \tag{2.2}$$

where $\widetilde{R}$ is some polynomial. We are unable to give a form of q-gamma functions' product at this stage. However, if $N_0 = 0$ and $a - b = 3$, then we get the result without loss of generality. Then the partitions of these cases are $\lambda^{(1)} = (3, 2, 1)$, $\lambda^{(2)} = (3, 1^3)$, $\lambda^{(3)} = (2^3)$ and $\lambda^{(4)} = (2^2, 1^2)$, and $\widetilde{R}$ is the following:

$$\widetilde{R} = \frac{R_0 R_1 R_2 R_3 R_4 R_5 q^{2x-4} t^{3-2N} (1-q)^2 (1-q^{a-2})(1+q^{a-1}t^2)}{R_0 R_2 R_6 R_7 R_8 R_9 (1-q)^2 (1+t)(1-qt^2)}$$

$$= \frac{R_1 R_3 R_4 R_5 q^{2x-4} t^{3-2N} (1-q^{a-2})(1+q^{a-1}t^2)}{R_6 R_7 R_8 R_9 (1+t)(1-qt^2)}$$

$$= q^{k(3-2N)+2x-4} \frac{\begin{array}{l}[k(N-2)+1]_q [k(N-1)+2]_q [k(N-1)+y]_q \\ \times [a-2]_q [k]_q [2(2k+a-1)]_q\end{array}}{\begin{array}{l}[x+y-2]_q [k+x+y-1]_q [2k+x+y-1]_q \\ \times [k(N-1)+x+y]_q [2k]_q [2k+a-1]_q\end{array}}, \tag{2.3}$$

where $R_0 = 1 - qt$, $R_1 = 1 - qt^2$, $R_2 = 1 - qt^{N-3}$, $R_3 = 1 - qt^{N-2}$, $R_4 = 1 - q^2 t^{N-1}$, $R_5 = 1 - q^y t^{N-1}$, $R_6 = 1 - q^{x+y-2}$, $R_7 = 1 - q^{x+y-1}t$, $R_8 = 1 - q^{x+y-1}t^2$ and $R_9 = 1 - q^{x+y} t^{N-1}$.

Proposition 2.4. *For the cases* $N = 4$ $(N_0 = 0,\ N_1 = N_2 = 2,\ \text{general } k)$ *and* $a - b = 3$, *we have*

$$(2.2) = q^{k(N^2-4N+5)} \widetilde{R} \frac{(1-q^{x-2})(1-q^{x-k-2})(1-q^{x-2k-2})}{(1-q^{x+y+k(N-1)-2})(1-q^{x+y+k(N-2)-2})(1-q^{x+y+k(N-3)-2})}$$

$$\times \frac{(1-q^{x-1})}{(1-q^{x+y+k(N-1)-1})} \frac{(1-q^{2(k+1)})^2}{(1-q^{k+1})^2}$$

$$\times \frac{(1-q^{kN})(1-q^{kN+1})(1-q^{kN+2})(1-q^{k(N-2)})(1-q^{k(N-1)})(1-q^{k(N-1)+1})}{(1-q^k)^3(1-q^{2k+1})(1-q^{k(N-1)+2})(1-q^{k(N-2)+1})}$$

$$= q^{k(N^2-4N+5)}\tilde{R}$$

$$\times \frac{[x-2]_q[x-k-2]_q[x-2k-2]_q}{[x+y+k(N-1)-2]_q[x+y+k(N-2)-2]_q[x+y+k(N-3)-2]_q}$$

$$\times \frac{[x-1]_q[2(k+1)]_q^2}{[x+y+k(N-1)-1]_q[k+1]_q^2}$$

$$\times \frac{[kN]_q[kN+1]_q[kN+2]_q[k(N-2)]_q[k(N-1)]_q[k(N-1)+1]_q}{[k]_q^3[2k+1]_q[k(N-1)+2]_q[k(N-2)+1]_q}, \tag{2.4}$$

where $b = a-3$, $x = -b = 3-a$ *and* $y = 2a-2$ *so* $x+y = a+1$ *and* $a+b = 2a-3$.

Therefore, we have

$$\text{(LHS of (1.13))} = N!\frac{(1-q^k)^N}{(q^k;q^k)_N}\ {}_qS_N(x-k(N-1)-2, y, k)$$

$$\times \text{ (formula of (2.4))}. \tag{2.5}$$

This is the product of q-gamma functions.

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