Far East Journal of Mathematical Sciences (FJMS)
Volume 34, Issue 2, 2009, Pages 207-218
Published Online: September 7, 2009
This paper is available online at http://www.pphmj.com

FIXED POINT THEOREMS OF CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

R. KRISHNAKUMAR and M. MARUDAI

Department of Mathematics
Urumu Dhanalakshmi College
Tiruchirappalli-620019
Tamil Nadu, India
e-mail: srksacet@yahoo.co.in

Department of Mathematics
Bharathidasan University
Tiruchirappalli-620024, India
e-mail: marudaim@hotmail.com

Abstract

Let P be a subset of a Banach space E, where P is normal and regular cone on E. We prove several fixed point theorems on cone metric spaces and these theorems generalize the recent results of various authors.

1. Introduction and Preliminaries

In recent years, several authors (see [1-4]) have studied the strong convergence to a fixed point with contractive constant in cone metric spaces. Rezapour and Hamlbarani [4] have proved certain fixed point theorems by using self mapping in the setting of contractive constant in cone metric spaces. We first recall definitions and known results that are needed in the sequel.

2000 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.

Keywords and phrases: cone metric space, self mappings, fixed point.

Submitted by E. Thandapani

Received May 8, 2009

Let E be a Banach space, and a subset P of E is said to be a *cone* if it satisfies the following conditions:

(i) $P \neq \emptyset$ and P is closed;

(ii) $ax + by \in P$ for all $x, y \in P$, and a and b are non-negative real numbers;

(iii) $P \cap (-P) = \emptyset$.

The partial ordering $\leq$ with respect to the cone P by $x \leq y$ if and only if $y - x \in P$. If $y - x \in$ interior of P, then it is denoted by $x \ll y$. The cone P is said to be a *normal* if a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The cone P is called *regular* if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.1. Let X be a non-empty set, and suppose the mapping $d : X \times X \to E$ is said to be a *cone metric space* if it satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 1.2. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a *cone metric space* [1].

Definition 1.3. Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

Definition 1.4. Let (X, d) be a *complete cone metric space*, if every Cauchy sequence is convergent in X.

Theorem 1.5 [4]. *Let* (X, d) *be a complete cone metric space and the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfy the contractive condition*

$$d(Tx, Ty) \leq kd(Tx, y) + ld(x, Ty)$$

for all $x, y \in X$, *where* $k, l \in [0, 1)$ *are constants. Then T has a unique fixed point in X. Also the fixed point of T is unique whenever* $k + l < 1$.

In this paper, we prove the theorems which are the generalization of the theorems of Rezapour and Hamlbarani [4].

2. Main Results

Theorem 2.1. *Let* (X, d) *be a complete cone metric space and the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfy the contractive condition*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]$$

for all $x, y \in X$, *and* $a + b < \frac{1}{2}$, $a, b \in [0, 1/2)$. *Then T has a unique fixed point in X.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_n, x_n) + d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\ &\leq (a + b)[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)], \end{aligned}$$

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}), \quad \text{where} \quad L = \frac{(a + b)}{(1 - (a + b))},$$

$$d(x_{n+1}, x_n) \leq L^n d(x_1, x_0).$$

For $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [L^{n-1} + L^{n-2} + \cdots + L^m] d(x_1, x_0) \\ &\leq \frac{L^m}{(1-L)} d(x_1, x_0). \end{aligned}$$

For given $0 \ll c$, choose a natural number N_1 such that $\frac{L^m}{(1-L)} d(x_1, x_0) \ll c$ for all $m \geq N_1$. This implies $d(x_n, x_m) \ll c$. For $n > m$, $\{x_n\}$ is a Cauchy sequence in (X, d) which is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$. Choose a natural number N_2 such that $d(x_n, p) \ll \frac{c(1-L)}{3}$, for all $n \geq N_2$. Hence for $n \geq N_2$, we have $d(x_n, P) \ll \frac{c(1-k)}{3}$, where $k = a + b$,

$$\begin{aligned} d(Tp, p) &\leq d(Tx_n, Tp) + d(Tx_n, p) \\ &\leq a[d(x_n, Tx_n) + d(p, Tp)] + b[d(x_n, Tp) + d(Tx_n, p)] + d(x_{n+1}, p) \\ &\leq a[d(x_n, x_{n+1}) + d(p, Tp)] + b[d(x_n, Tp) + d(x_{n+1}, p)] + d(x_{n+1}, p) \\ &\leq a[d(x_n, x_{n+1}) + d(p, Tp)] + b[d(x_n, p) + d(p, Tp) + d(x_{n+1}, p)] \\ &\quad + d(x_{n+1}, p), \end{aligned}$$

$$\begin{aligned} (1-k)d(Tp, p) &\leq kd(x_n, p) + kd(x_{n+1}, p) + d(x_{n+1}, p) \\ &\leq d(x_n, p) + d(x_{n+1}, p) + d(x_{n+1}, p), \end{aligned}$$

$$d(Tp, p) \leq \frac{[d(x_n, p) + d(x_{n+1}, p) + d(x_{n+1}, p)]}{(1-k)},$$

$$d(Tp, p) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3},$$

$$d(Tp, p) \ll c,$$

for all $n \geq N_2$, $d(Tp, p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(Tp, p) \in P$, and as $m \to \infty$, we get $\frac{c}{m} \to 0$ and P is closed $-d(Tp, p) \in P$, but $d(Tp, p) \in P$, hence $d(Tp, p) = 0$ and so $Tp = p$. □

The following result of Rezapour and Hamlbarani [4] is a special case of the previous theorem.

Corollary 2.1. *Let* (X, d) *be a complete cone metric space and the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfy the contractive condition*

$$d(Tx, Ty) \leq a(d(Tx, y), d(x, Ty))$$

for all $x, y \in X$, *where* $a \in [0, 1/2)$ *is a constant. Then T has a unique fixed point in X. For each* $x \in X$, *the iterative sequence* $\{T^n x\}_{n \geq 1}$ *converges to the fixed point.*

Proof. The proof of the corollary immediately follows by putting $b = 0$ in the previous theorem. □

Theorem 2.2. *Let* (X, d) *be a complete cone metric space and the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfy the contractive condition*

$$d(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$ *and* $r \in [0, 1)$. *Then T has a unique fixed point in X.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq r \max[d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})] \\ &\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)] \\ &\leq rd(x_{n-1}, x_n) \\ &\leq r^n d(x_1, x_0). \end{aligned}$$

For $n > m,$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [r^{n-1} + r^{n-2} + \cdots + r^m] d(x_1, x_0) \\ &\leq \frac{r^m}{(1-r)} d(x_1, x_0). \end{aligned}$$

For given $0 \ll c$, choose a natural number N_1 such that $\frac{r^m}{(1-r)} d(x_1, x_0) \ll c$ for all $m \geq N_1$. This implies $d(x_n, x_m) \ll c.$ For $n > m,$ $\{x_n\}$ is a Cauchy sequence in (X, d) which is a complete cone metric space and hence there exists $p \in X$ such that $x_n \to p.$ Choose a natural number N_2 such that $d(x_n, p) \ll \frac{c}{3},$ for all $n \geq N_2.$ Then for $n \geq N_2,$ we have $d(x_n, p) \ll \frac{c}{3},$

$$\begin{aligned} d(Tp, p) &\leq d(Tx_n, Tp) + d(Tx_n, p) \\ &\leq r \max[d(x_n, p), d(x_n, Tx_n), d(p, Tp)] + d(x_{n+1}, p) \\ &\leq r \max[d(x_n, p), d(x_n, x_{n+1}), d(p, Tp)] + d(x_{n+1}, p) \\ &\leq r \max[d(x_n, p), d(x_n, p) + d(p, x_{n+1}), d(p, Tp)] + d(x_{n+1}, p) \end{aligned}$$

$$d(Tp, p) \ll c,$$

for all $n \geq N_2,$ $d(Tp, p) \ll \frac{c}{m}$ for all $m \geq 1,$ we get $\frac{c}{m} - d(Tp, p) \in P,$ and as $m \to \infty,$ we get $\frac{c}{m} \to 0$ and P is closed $-d(Tp, p) \in P,$ but $d(Tp, p) \in P \Rightarrow$ $d(Tp, p) = 0,$ and so $Tp = p.$ □

Corollary 2.2. *Let* (X, d) *be a complete cone metric space and the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfy the contractive condition* $d(Tx, Ty) \leq kd(x, y)$ *for all* $x, y \in X$, *where* $k \in [0, 1)$ *is a constant. Then* T *has a unique fixed point in* X. *For each* $x \in X$, *the iterative sequence* $\{T^n x\}_{n \geq 1}$ *converges to the fixed point.*

Proof. The proof of the corollary immediately follows since

$$d(x, y) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \qquad \square$$

Note 2.3. We prove the above theorems in the setting of normal cone with normal constant *K*.

Theorem 2.4. *Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose the mapping $T : \mathbf{X} \to \mathbf{X}$ satisfies the contractive condition*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]$$

for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, 1/2)$. Then T has a unique fixed point in X. Also, for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_n, x_n) + d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\ &\leq (a + b)[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)], \end{aligned}$$

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}), \quad \text{where} \quad L = \frac{(a+b)}{(1-(a+b))} < 1,$$

$$d(x_{n+1}, x_n) \leq L^n d(x_1, x_0).$$

For $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [L^{n-1} + L^{n-2} + \cdots + L^m] d(x_1, x_0) \\ &\leq \frac{L^m}{(1-L)} d(x_1, x_0). \end{aligned}$$

We get $\| d(x_n, x_m) \| \leq K \frac{L^m}{(1-L)} \| d(x_1, x_0) \|,$ and hence $d(x_n, x_m) \to 0$ as $n, m \to \infty.$ Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of *X*, there is $p \in X$ such that $x_n \to p$ as $n \to \infty,$

$$\begin{aligned} d(Tp,\ p) &\leq d(Tx_n,\ Tp) + d(Tx_n,\ p) \\ &\leq a[d(x_n,\ Tx_n) + d(p,\ Tp)] + b[d(x_n,\ Tp) + d(Tx_n,\ p)] + d(x_{n+1},\ p) \\ &\leq a[d(x_n,\ x_{n+1}) + d(p,\ Tp)] + b[d(x_n,\ Tp) + d(x_{n+1},\ p)] + d(x_{n+1},\ p) \\ &\leq a[d(x_n,\ x_{n+1}) + d(p,\ Tp)] + b[d(x_n,\ p) + d(p,\ Tp) + d(x_{n+1},\ p)] \\ &\quad + d(x_{n+1},\ p), \end{aligned}$$

$$\begin{aligned} (1-k)d(Tp,\ p) &\leq kd(x_n,\ p) + kd(x_{n+1},\ p) + d(x_{n+1},\ p), \quad (\text{where } k = a + b) \\ &\leq k[d(x_n,\ p) + d(x_{n+1},\ p)], \end{aligned}$$

$$d(Tp,\ p) \leq \frac{k}{1-k}[d(x_n,\ p) + d(x_{n+1},\ p)],$$

$$\| d(Tp,\ p) \| \leq K \frac{k}{1-k}[\| d(x_n,\ p) \| + \| d(x_{n+1},\ p) \|].$$

Hence $\| d(Tp,\ p) \| = 0 \Rightarrow Tp = p.$ If *q* is another fixed point of *T* in *X*, then

$$\begin{aligned} d(p,\ q) &= d(Tp,\ Tq) \\ &\leq a[d(Tp,\ p) + d(q,\ Tq)] + b[d(p,\ Tq) + d(Tp,\ q)] \\ &\leq a[d(p,\ p) + d(q,\ q)] + b[d(p,\ q) + d(p,\ q)] \\ &\leq 2b[d(p,\ q)]. \end{aligned}$$

This is contradiction, and hence *T* has a unique fixed point in *X*. □

Theorem 2.5. *Let* $(X,\ d)$ *be a complete cone metric space, and P be a normal cone with normal constant K. Suppose the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfies the contractive condition*

$$d(Tx,\ Ty) \leq r \max[d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty),\ d(x,\ Ty),\ d(Tx,\ y)]$$

for all $x,\ y \in X$, *and* $r \in [0,\ 1).$ *Then T has a unique fixed point in X. Also, for any* $x \in X$, *iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$,

$$d(x_{n+1}, x_n)$$
$$= d(Tx_n, Tx_{n-1})$$
$$\leq r \max[d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(Tx_n, x_{n-1})]$$
$$\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n+1}, x_{n-1})]$$
$$\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n-1})]$$
$$\leq r \max[d(x_n, x_{n-1}), d(x_{n+1}, x_{n-1})].$$

Case (i). If $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$, then we get $d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$.

For $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [r^{n-1} + r^{n-2} + \cdots + r^m] d(x_1, x_0) \\ &\leq \frac{r^m}{(1-r)} d(x_1, x_0). \end{aligned}$$

We get $\| d(x_n, x_m) \| \leq K \frac{r^m}{(1-r)} \| d(x_1, x_0) \|$, and hence $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$,

$$\begin{aligned} d(Tp, p) &\leq d(Tx_n, Tp) + d(Tx_n, p) \\ &\leq r \max[d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)] \\ &\quad + d(x_{n+1}, p) \\ &\leq r \max[d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)] \\ &\quad + d(x_{n+1}, p) \\ &\leq rd(p, Tp), \end{aligned}$$

$d(Tp, p) = 0$ hence $Tp = p$.

Case (ii). If $d(x_{n+1}, x_n) \leq rd(x_{n+1}, x_{n-1})$, then

$$d(x_{n+1}, x_n) \leq r[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]$$

$$\leq \frac{r}{1-r}[d(x_n, x_{n-1})]$$

$$\leq h[d(x_n, x_{n-1})], \quad \text{where} \quad h = \frac{r}{1-r} < 1.$$

For $n > m$,

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$

$$\leq [h^{n-1} + h^{n-2} + \cdots + h^m]d(x_1, x_0)$$

$$\leq \frac{h^m}{(1-h)} d(x_1, x_0).$$

We get $\|d(x_n, x_m)\| \leq K\frac{h^m}{(1-h)}\|d(x_1, x_0)\|$, and hence $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$,

$$d(Tp, p) \leq d(Tx_n, Tp) + d(Tx_n, p)$$

$$\leq r \max[d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)]$$
$$+ d(x_{n+1}, p)$$

$$\leq r \max[d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)]$$
$$+ d(x_{n+1}, p)$$

$$\leq rd(p, Tp),$$

$d(Tp, p) = 0.$ Hence $Tp = p$,

$$d(p, q) = d(Tp, Tq)$$
$$\leq r \max[d(p, q), d(p, Tp), d(q, Tq), d(p, Tq), d(Tp, q)]$$
$$\leq r \max[d(p, q), d(p, p), d(q, q), d(p, q), d(p, q)]$$
$$\leq r[d(p, q)].$$

This is contradiction and hence T has a unique fixed point in X. □

Corollary 2.3. *Let* (X, d) *be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping* $T : \mathbf{X} \to \mathbf{X}$ *satisfies the contractive condition*

$$d(Tx, Ty) \leq r \max[d(x, y), d(x, Tx), d(y, Ty)]$$

for all $x, y \in X$, *and* $r \in [0, 1)$. *Then T has a unique fixed point in X. Also, for any* $x \in X$, *iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Proof. The proof of the corollary immediately follows since

$$\max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

$$\leq \max[d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)]. \qquad \square$$

Theorem 2.6. *Let* (X, d) *be a cone metric space, and* $P \subset X$ *and P be a regular cone, and let S be the class of functions* $\alpha : \mathbf{R}^+ \to [\mathbf{0}, \mathbf{1})$ *satisfying* $\alpha(t_n) \to 1 \Rightarrow t_n \to 0$ *and* $T : \mathbf{X} \to \mathbf{X}$ *satisfying the contractive condition*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$. *Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$$

$$\leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}).$$

If $d(x_n, x_{n-1}) = 0$, then $\{d(x_{n+1}, x_n)\}$ is a monotonically decreasing and bounded below, as P is regular we have $\{d(x_{n+1}, x_n)\}$ is convergent. Also if $d(x_n, x_{n-1}) > 0$, then $\dfrac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \alpha(d(x_n, x_{n-1})) \to 1$ as $n \to \infty$ and $\alpha \in S$, then we get $d(x_{n+1}, x_n)$ is a monotonically decreasing and bounded below, as P is regular we have $\{d(x_{n+1}, x_n)\}$ is convergent.

$$d(Tp, p) \leq d(Tx_n, Tp) + d(Tx_n, p)$$

$$\leq \alpha(d(x_n, p))d(x_n, p) + d(x_{n+1}, p) \to 0.$$

This proves the theorem. $\square$

References

[1] Huang Gaung and Zhang Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[2] H. Mohebi, Topical functions and their properties in a class of ordered Banach spaces, Continuous Optimization, Current Trends and Modern Applications, Part II, Springer, 2005, pp. 343-361.

[3] H. Mohebi, H. Sadeghi and A. M. Rubinov, Best approximation in a class of normed spaces with star-shaped cone, Numer. Funct. Anal. Optim. 27(3-4) (2006), 411-436.

[4] Sh. Rezapour and R. Hamlbarani, Some notes on the paper: cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345 (2008), 719-724.