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FINITE DIMENSIONAL CONNECTED AND GEOMETRIC-LIKE HOPF ALGEBRAS

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Abstract

This paper presents a description of some connected finite dimensional Hopf algebras over a field in terms of its primitive elements. If the field has characteristic a prime, then those descriptions generalize to include induced primitives. We also deal with geometric-like Hopf algebras, which expand the connected case and have their own descriptions in terms of primitives and induced primitives.

1. Introduction

Any map $\varphi : A_1 \to A_2$ between algebras over a ring R can be completely determined by the values it takes on the generators of A_1, when such a set exists. If we look at maps between *Hopf algebras*, then we run into additional restrictions and problems because of having to deal with the co-product.

Typically, a Hopf algebra H may or may not be generated as an algebra by its primitive elements. In this paper, we look at some cases where we can guarantee this, which then implies that a map between such Hopf algebras will be determined

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by the values it takes on its primitive elements. The proof of this fact implies analyzing all possible combinations $\sum x \otimes y$ in $H \otimes H$, satisfying convenient properties of co-associativity and co-commutativity, and finding an element a in H that has as co-product that exact combination. When the Hopf algebras are *connected* and R is a field of characteristic zero, then we can use induction on the primitive filtration and thus obtain an a for any possible combination in $H \otimes H$ as above.

Over fields of characteristic a prime, however, we may run into trouble even in the connected case. For example, if the characteristic is 2, there may happen that no a will satisfy $\Psi(a) = 1 \otimes a + a \otimes 1 + x \otimes x$, for some x, even primitive, and thus we cannot guarantee any element is a sum of products of primitives (If the characteristic of the field is different from 2, though, we would always have that $a = x^2/2$ satisfies $\Psi(a) = 1 \otimes a + a \otimes 1 + x \otimes x$ when x is a primitive).

For these cases, we define induced primitives, each of which is always associated with a given primitive in the Hopf algebra and may or may not be non-zero. Finite dimensional connected Hopf algebras over a field will then be generated by its primitive and induced primitive elements.

Section 2 elaborates on these definitions, and Section 3 presents the results for finite dimensional connected Hopf algebras. Section 4 then deals with a generalization of connected Hopf algebras, those called *geometric-like* and defined as having a "primitive" filtration for each generalized unit. Connected Hopf algebras have just one such generalized unit, namely its *unit* 1, and so the methods in Section 3 are expanded in Section 4 in order to accommodate this generalization.

The main results in this paper are Theorems 3.6, 3.8 and 3.10 for finite dimensional connected Hopf algebras over fields of characteristic respectively zero, 2 and an odd prime; and Theorem 4.3 for finite dimensional geometric-like Hopf algebras over fields of zero and prime characteristic.

These descriptions of connected and geometric-like Hopf algebras are also used to elucidate on maps $f : H_1 \to H_2$ between Hopf algebras in those categories.

2. Definitions and First Properties

All Hopf algebras in this paper will be finite dimensional, co-commutative, co-associative, with co-unit, over a commutative ring *R*. The unit 1 on any such Hopf algebra gives a co-augmentation $R \to H$ and we can construct from it a *co-augmentation filtration* $\{F_q H\}_{q \geq 0}$ by taking the short exact sequence

$$0 \to R \to H \to J(H) \to 0$$

and, using the iterated co-product on *H*, defining

$$F_q H = \ker\left(H \to J(H)^{\otimes(q+1)}\right)$$

for $q \geq 0$. In particular, $F_0 H \simeq R$ and $F_1 H / F_0 H \simeq P(H)$, the primitive elements of *H*. A Hopf algebra will be called *connected* if its coaugmentation filtration is exhaustive; that is, if any $x \in A$ is in some $F_q A$ for some $q \geq 0$.

Given one such Hopf algebra *H*, if we write

$$\Psi(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$$

for each $x \in F_q H$, with $q > 0$, then all the x' and x'' that appear in the expression are in those $F_{q'} H$ that have $q' < q$. This suggests an inner grading of *H* based on the "degree of primitiveness" of its elements.

Non-connected Hopf algebras can be further explored as follows.

A *generalized unit* in *H* will be an element *a* with coproduct $\Psi(a) = a \otimes a$. The unit on *H* is a generalized unit. Moreover, we will call a *generalized primitive* (with respect to *a*), or simply an *a-primitive*, any element $x \in H$ that does $\Psi(x) = a \otimes x + x \otimes a$. A regular primitive element is 1-primitive.

Any generalized unit *a* gives naturally rise to a coaugmentation $R \to H$, and we can thus construct an exact sequence

$$0 \to R \to H \to J^a(H) \to 0$$

and a corresponding *a*-primitive filtration:

$$F_q^a H = \ker\left(H \to J^a(H)^{\otimes(q+1)}\right)$$

for $q \geq 0$. Here, $F_1 H / F_0 H \simeq P^a(H)$, the *a*-primitives of *H*.

A Hopf algebra will be called *geometric-like* if it is exhausted by all its a-primitive filtrations. Such Hopf algebras do depend directly on their generalized units and, as will be seen in the following sections, are completely determined by their collections of a-primitives (for all generalized units a).

Connected (and geometric-like) Hopf algebras appear for example in Homotopy theory, as homotopy classes of classifying spaces.

3. A Description of Connected Hopf Algebras over a Field

We will see that a connected Hopf algebra over R is generated by its primitives if R is a field of characteristic zero. This comes from understanding elements in each $F_q(H)$ in the exhaustive primitive filtration as sums of products of primitives.

In this section, all Hopf algebras H are connected. We start with some lemmas needed for the proof of the main result.

Lemma 3.1. *If* $x \in H$ *is such that* $\Psi(x)$ *contains a term of the form* $p \otimes q_1 q_2 \cdots q_n$, *where* p *and all* q_i *are primitives with* $p \neq q_1$, *then* $\Psi(x)$ *also contains* $pq_1 \otimes q_2 \cdots q_n$.

Proof. We use co-associativity.

$(1 \otimes \Psi)(p \otimes q_1 q_2 \cdots q_n)$ contains a term of the form $p \otimes q_1 \otimes q_2 \cdots q_n$ and, since $1 \otimes \Psi = \Psi \otimes 1$, $\Psi(x)$ has a term of the form $b \otimes q_2 \cdots q_n$, where b is such that $\Psi(b)$ contains $p \otimes q_1$.

By co-commutativity, $\Psi(b)$ also contains $q_1 \otimes p$. Since p and q_1 are primitive, we have $\Psi(pq_1) = pq_1 \otimes 1 + 1 \otimes pq_1 + p \otimes q_1 + q_1 \otimes p$, and so b must contain a term pq_1.

Thus $\Psi(x)$ contains a $pq_1 \otimes q_2 \cdots q_n$. □

Lemma 3.2. *If* $x \in H$ *is such that* $\Psi(x)$ *contains a term of the form* $p_1 \cdots p_m \otimes q_1 q_2 \cdots q_n$, *where the* p_i *and* q_j *are all primitives and* $q_1 \neq p_i$ *for all* i, *then* $\Psi(x)$ *also contains* $p_1 \cdots p_m q_1 \otimes q_2 \cdots q_n$.

Proof. We use induction on m.

$m = 1$ is the previous lemma. If $p_1 \cdots p_{m-1}p_m \otimes q_1q_2 \cdots q_n$ is in $\Psi(x)$ (with $p_m \neq q_1$), then $p_1 \cdots p_{m-1}p_m \otimes q_1 \otimes q_2 \cdots q_n$ is in $(1 \otimes \Psi)(\Psi(x))$, thus in $(\Psi \otimes 1)(\Psi(x))$; so $\Psi(x)$ has a term $b \otimes q_2 \cdots q_n$, where $\Psi(b)$ contains $p_1 \cdots p_{m-1}p_m \otimes q_1$.

By the previous lemma (and co-commutativity), $\Psi(b)$ contains $p_1 \cdots p_{m-1} \otimes p_mq_1$.

Consider all terms $\alpha \otimes \beta$, where α and β are products of the $p_1, ..., p_m, q$ such that $\alpha\beta = p_1 \cdots p_mq$ (with $\alpha \neq 1$ and $\beta \neq 1$). In these terms appear exactly all of these $m + 1$ primitives, taken in any possible permutation that forms two products α and β (except those with either being 1). By induction, any such term must be in $\Psi(b)$, and so, since $\Psi(p_1 \cdots p_mq_1) - 1 \otimes p_1 \cdots p_mq_1 - p_1 \cdots p_mq_1 \otimes 1$ consists of precisely those elements, we have that b contains $p_1 \cdots p_mq_1$ and so $\Psi(x)$ contains $p_1 \cdots p_mq_1 \otimes q_2 \cdots q_n$. □

The lemma above deals with elements in a $\Psi(x)$ made of products of different primitives. The following show what must happen when those terms contain repetitions.

Lemma 3.3. *For R a field of characteristic zero, if* $\Psi(x)$ *contains a term* $p^k \otimes p^m$, *with p primitive, k and* $m \geq 1$, *then it also contains* $\alpha p^{k+1} \otimes p^{m-1}$, *where* $\alpha = \dfrac{m}{k+1}$.

Proof. First consider $k = 1$. In this case, $\alpha = m/2$.

We prove the result by induction on m. $m = 1$ is trivial, since for any primitive p we have $\Psi(p^2/2) = 1 \otimes p^2/2 + p^2/2 \otimes 1 + p \otimes p$.

For any primitive p, we also get $\Psi(p^m) = 1 \otimes p^m + p^m \otimes 1 + \sum_{i=1}^{m-1} \binom{m}{i} p^i \otimes p^{m-i}$ so, if $\Psi(x)$ contains $p \otimes p^m$, $(1 \otimes \Psi)(\Psi(x))$ contains $m\ p \otimes p \otimes p^{m-1}$, thus $(\Psi \otimes 1)(\Psi(x))$ contains the same, and $\Psi(x)$ contains a $b \otimes p^{m-1}$ with $\Psi(b)$ containing $mp \otimes p$. This means $b = \dfrac{m}{2} p^2$ mod primitives, and $\Psi(x)$ contains $\dfrac{m}{2} p^2 \otimes p^{m-1}$.

Now suppose $\Psi(x)$ contains $p^k \otimes p^m$, with $k > 1$.

Then $(1 \otimes \Psi)(\Psi(x)) = (\Psi \otimes 1)(\Psi(x))$ contains $mp^k \otimes p \otimes p^{m-1}$, and so $\Psi(x)$ contains $b \otimes p^{m-1}$ with $\Psi(b)$ containing $m\ p^k \otimes p$ and also (by co-commutativity) $m\ p \otimes p^k$. By hypothesis, it also contains $m \frac{k}{2} p^2 \otimes p^{k-1}$, and in general $m \frac{k(k-1)\cdots(k-i+2)}{i!} p^i \otimes p^{k+1-i}$ for $i = 1, \ldots, k$.

Thus $\Psi(b)$ contains

$$\sum_{i=1}^{k} m \frac{k(k-1)\cdots(k-i+2)}{i!} p^i \otimes p^{k+1-i} = \frac{m}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} p^i \otimes p^{k+1-i}$$

and we conclude that b contains $\frac{m}{k+1} p^{k+1}$.

Thus $\Psi(x)$ contains $\frac{m}{k+1} p^{k+1} \otimes p^{m-1}$. □

This gives directly the following.

Proposition 3.4. *For R a field of characteristic zero, if $\Psi(x)$ contains a term $p^k \otimes p^m$, with k and $m \geq 1$, then x contains $\frac{m!\,k!}{(m+k)!} p^{m+k}$.*

Proof. As in the proof of the previous lemma, if $\Psi(x)$ contains $p^k \otimes p^m$, then it contains $\frac{m}{k+1} p^{k+1} \otimes p^{m-1}$, and in general

$$\frac{m(m-1)\cdots(m-j+1)}{(k+1)(k+2)\cdots(k+j)} p^{k+j} \otimes p^{m-j} = \frac{m!\,k!}{(k+j)!(m-j)!} p^{k+j} \otimes p^{m-j}$$

for $j = 0, \ldots, m$.

It contains thus

$$\sum_{j=0}^{m} \frac{m!\,k!}{(k+j)!(m-j)!} p^{k+j} \otimes p^{m-j} = \sum_{i=k}^{k+m} \frac{m!\,k!}{i!(m+k-i)!} p^i \otimes p^{m+k-i}.$$

By co-commutativity, it also has

$$\sum_{i=k}^{k+m} \frac{m!\,k!}{i!(m+k-i)!} p^{m+k-i} \otimes p^{i} = \sum_{j=0}^{m} \frac{m!\,k!}{(m+k-j)!\,j!} p^{j} \otimes p^{m+k-j}.$$

Since by co-commutativity k and m are interchangeable in the statement of the proposition, we finally can conclude that $\Psi(x)$ has

$$\sum_{i=0}^{k+m} \frac{m!\,k!}{i!(m+k-i)!} p^{i} \otimes p^{m+k-i} = \frac{m!\,k!}{(m+k)!} \Psi(p^{m+k})$$

and the result follows. □

Lemma 3.5. *For R a field of characteristic zero, suppose $\Psi(x)$ contains a term of the form $p_1 \cdots p_m p^k \otimes p^r q_1 \cdots q_n$, where all p_i, q_j and p are primitives with $p_i \neq p$ and $q_j \neq p$ for all i and j.*

Then x contains $\frac{k!\,r!}{(k+r)!} p^{k+r} y$, where y is such that $\Psi(y)$ contains $p_1 \cdots p_m \otimes q_1 \cdots q_n$.

Proof. $\Psi(x)$ contains

$$p_1 \cdots p_m p^k \otimes p^r q_1 \cdots q_n = (p_1 \cdots p_m \otimes q_1 \cdots q_n)(p^k \otimes p^r).$$

By the previous lemma, if an element has coproduct containing $p^k \otimes p^r$, then that element must contain $a = \frac{k!\,r!}{(k+r)!} p^{k+r}$.

Put $y = a^{-1}x$.

Then, since p does not appear on both products $p_1 \cdots p_m$ and $q_1 \cdots q_n$, from the relation $\Psi(x) = \Psi(a)\Psi(y)$, we get that $\Psi(y)$ must have a term of the form $p_1 \cdots p_m \otimes q_1 \cdots q_n$. □

Theorem 3.6. *Any finite dimensional connected Hopf algebra over a field of characteristic zero is polynomial on its primitive elements.*

Proof. We use induction on the exhaustive primitive filtration.

Consider a generic element x in F_q of the filtration.

By induction, we know that

$$\Psi(x) = 1 \otimes x + x \otimes 1 + \sum a \otimes b,$$

where both a and b are a product of primitives (suppose for now all these primitives are different). Say $a = p_1 \cdots p_k$ and $b = q_1 \cdots q_r$.

By Lemma 3.2, all elements of the form $\alpha \otimes \beta$, with $\alpha\beta = ab$ (and both α and β not equal to 1) must be in $\Psi(x)$.

Thus, the element $\sum(ab)$, whose coproduct is

$$\Psi(\sum(ab)) = 1 \otimes \sum(ab) + \sum(ab) \otimes 1 + \sum \alpha \otimes \beta$$

(where α and β are, as above, all possibilities of products of the p_i and q_j), differs from x by a primitive, and so x is a sum of products of primitives.

If there are repetitions among the p_i and q_j, we use Lemma 3.5. Suppose $\Psi(x)$ contains a term $cp_1 \cdots p_m p^k \otimes p^r q_1 \cdots q_n$, with $p \neq p_i$ and $p \neq q_j$ for all i and j (c is a constant). Note that the p_i and q_j can still harbour further repetitions.

Then, since p does not occur among the p_i and q_j, $x = \left[\frac{k!r!}{(k+r)!} p^{k+r} + s\right] y$, with s a primitive and $\Psi(y)$ containing $p_1 \cdots p_m \otimes q_1 \cdots q_n$.

Applying similarly and in succession Lemma 3.5 to the remaining products $p_1 \cdots p_m \otimes q_1 \cdots q_n$, we get that x is a sum of products of primitives, and the result follows. □

Corollary 3.7. *Any Hopf algebra map $f : H_1 \to H_2$ between two connected Hopf algebras over a field of characteristic zero is completely determined by the values it takes on primitives.*

If we deal with a field of characteristic different from zero, Lemma 3.3 and Proposition 3.4 are no longer true. For example, if the characteristic is 2, no element

$1/2$ exists. In this case, squares of primitives are still primitive, and so we cannot directly find a term x whose co-product is $1 \otimes x + x \otimes 1 + p \otimes p$ (when p is a primitive) as happened with $\frac{1}{2} p^2$ in the proofs of Lemma 3.3 and Proposition 3.4. We can however still get an easy description of connected Hopf algebras for characteristic 2 or an odd prime.

We begin with the case of characteristic 2. Define a *first-order induced primitive* element of H (relative to the primitive p) any $x \in H$ that does $\Psi(x) = 1 \otimes x + x \otimes 1 + p \otimes p.$ The difference between two induced primitives (relative to the same p) is always a primitive element. Any first order induced primitive relative to p will be denoted by $\hat{p}$ (or $p^{(1)}$).

Given a connected Hopf algebra H over a field of characteristic 2 and a primitive element p in the Hopf algebra, there may or may not exist a first-order induced primitive element on H relative to this p. However, if a $\Psi(x)$ features a term of the form $p \otimes p,$ then it is clear that x must have a $\hat{p}.$ Such an element behaves as $p^2/2$ would in a Hopf algebra over a field of characteristic zero.

A *second-order induced primitive* element of H (relative to the primitive p) is any x such that $\Psi(x)$ contains a term $\hat{p} \otimes \hat{p}$ and additionally just the extra terms that the definition of Hopf algebra imposes. Specifically, if $\Psi(x)$ contains $\hat{p} \otimes \hat{p},$ then $(1 \otimes \Psi)(x)$ contains $\hat{p} \otimes p \otimes p,$ so $(\Psi \otimes 1)(x)$ contains $a \otimes p$ with $\Psi(a)$ containing $\hat{p} \otimes p.$ Following this reasoning through, similarly to the kind of deductions one did for example in the proofs of Lemma 3.3 and Proposition 3.4, one gets that any second-order induced primitive x relative to p must actually make

$$\Psi(x) = 1 \otimes x + x \otimes 1 + p \otimes p\hat{p} + p\hat{p} \otimes p + p^2 \otimes \hat{p} + \hat{p} \otimes p^2 + \hat{p} \otimes \hat{p}.$$

This is the minimum number of terms necessary for $\Psi(x)$ to feature a $\hat{p} \otimes \hat{p}.$

We will denote by $p^{(2)}$ any second-order induced primitive relative to p. Any two such differ by a primitive.

As $\hat{p}$ in a sense generalized for characteristic 2 what $p^2/2$ did for characteristic zero, we can define from $p^{(2)}$ terms in characteristic 2 that generalize others that existed in the previous setting. Working in characteristic zero, $\hat{p} = p^2/2$, so from Proposition 3.4 we get that, if $x = p^{(2)}$, then $\Psi(x)$ contains $\hat{p} \otimes \hat{p} = \frac{1}{4}\, p^2 \otimes p^2$, and so

$$x = \frac{1}{4}\,\frac{2!\,2!}{4!}\,p^4 = \frac{1}{3}\,\hat{p}\,\frac{p^2}{2^2}.$$

This suggests, in characteristic 2, considering the element $[\hat{p}]^{-1} p^{(2)}$, which behaves as $p^2/2^2$ did in characteristic zero.

Next we define higher order induced primitives. Working inductively, and emulating the definition of second-order induced primitives, denote by $p^{(k)}$ an *induced primitive of order k* relative to the primitive *p*, which is defined as any *x* in *H* whose co-product contains a term $p^{(k-1)} \otimes p^{(k-1)}$ plus all the minimum required terms that the definition of Hopf algebra imposes. From these induced primitives of all orders we can define in characteristic 2 terms that behave as the $p^2/2^k$ did in characteristic zero. For example, in characteristic zero, $\Psi(p^{(3)})$ has $p^{(2)} \otimes p^{(2)} = \frac{1}{4!}\frac{1}{4!}\, p^4 \otimes p^4$, so (Proposition 3.4)

$$p^{(3)} = \frac{1}{8!}\, p^8 = \frac{1}{3^2 \cdot 5 \cdot 7}\,\frac{p^8}{2^6} = \frac{1}{3^2 \cdot 5 \cdot 7}\,[\hat{p}]^3\,\frac{p^2}{2^3}.$$

Thus, in characteristic 2, the term $[\hat{p}]^{-3} p^{(3)}$ behaves as a $p^2/2^3$.

We can moreover write $p^{(0)}$ when referring to a primitive *p*.

We are now ready for a result that generalizes Theorem 3.6.

Theorem 3.8. *Any finite dimensional connected Hopf algebra over a field of characteristic* 2 *is polynomial on its primitive and induced primitive elements.*

Proof. If the characteristic of the base field is zero, we know from Theorem 3.6 that every element in the Hopf algebra is a sum of products of primitives. This notion works in characteristic 2 except in those cases where the coefficients

$\frac{m!\,k!}{(m+k)!}$ one gets from Proposition 3.4 do not make sense, that is, if such coefficients are of the form $\frac{a}{2^r b}$, with a and b free of factors of 2. When such a case arises, however, we can always write $\frac{a}{2^r b}p^{m+k}$ as a product of induced primitives relative to p, as the above comments indicate, and so indeed any x in the Hopf algebra is a sum of products of primitives and induced primitives. □

The above result does not indicate that any connected Hopf algebra over a field of characteristic 2 must have all possible induced primitives for all its primitives p. What it says is that the Hopf algebra will be polynomial on those induced primitives that are indeed present.

We can in any case define all induced primitives for all primitives p as follows. If $p^{(i)} \otimes p^{(i)}$ is present as an element of a co-product in $H \otimes H$ for $i = 0, ..., k-1$ but no $p^{(k)} \otimes p^{(k)}$ appears, we define $p^{(m)} = 0$ for $m \geq k+1$. This notation permits the following corollary.

Corollary 3.9. *Any Hopf algebra map* $f : H_1 \to H_2$ *between two connected Hopf algebras over a field of characteristic* 2 *is completely determined by the values it takes on primitives and induced primitives.*

Proof. A Hopf algebra map must by definition make

$$f(p^{(n)}) = [f(p)]^{(n)}$$

for any primitive p and any $n \geq 0$. The term on the right makes sense from the commentary before the enunciation of this corollary. If, for example, there is in H_1 an induced primitive of order n relative to the primitive p but that does not happen to $f(p)$ in H_2, then $[f(p)]^n = 0$ and no contradiction arises from f being a Hopf algebra map. □

If the characteristic of the field is an odd prime, then the definitions of induced primitives must be done differently. In such a field, for instance, it is clear that elements like $p^2/2$ make sense, and so now there always exist elements x with coproduct $1 \otimes x + x \otimes 1 + p \otimes p$ for a given primitive p.

On the other hand, calling r the odd prime characteristic of the field, no term p^r/r makes sense now. Define in characteristic r a *first-order induced primitive* element $\hat{p}$ (relative to the primitive p) any $x \in H$ whose coproduct contains $p \otimes p^{r-1}$ and also just all the terms that the definition of Hopf algebra imposes.

Specifically, and from Lemma 3.3, $\Psi(x)$ also has $\frac{r-1}{2} p^2 \otimes p^{r-2}$, $\frac{(r-1)(r-2)}{3 \cdot 2} p^3 \otimes p^{r-3}$ and a generic

$$\frac{(r-1)!}{k!(r-k)!} p^k \otimes p^{r-k}$$

for $k = 1, \ldots, r-1$, so a first-order induced primitive does

$$\Psi(\hat{p}) = 1 \otimes \hat{p} + \hat{p} \otimes 1 + \sum_{k=1}^{r-1} \frac{(r-1)!}{k!(r-k)!} p^k \otimes p^{r-k}.$$

We can then consider any $\hat{p}$ as a generalization from characteristic zero of a term p^r/r, as this term does (in characteristic zero) exactly

$$\Psi(p^r/r) = 1 \otimes p^r/r + p^r/r \otimes 1 + \sum_{k=1}^{r-1} \frac{(r-1)!}{k!(r-k)!} p^k \otimes p^{r-k}.$$

A connected Hopf algebra over a field of characteristic r may or may not have a first-order induced primitive $\hat{p}$ for each of its primitives p.

Higher-order induced primitives are defined as follows. Write $p^{(k)}$ for an *induced primitive of order k* (relative to the primitive p), defined inductively as any $x \in H$ whose coproduct contains $p^{(k-1)} \otimes [p^{(k-1)}]^{r-1}$ and additionally just all the terms required by the definition of Hopf algebra.

As happened in characteristic 2, induced primitives of higher orders permit us to define in characteristic r terms that behave like p^r/r^k did $(\text{for } k \geq 0)$ in characteristic zero.

Consider for instance a $p^{(2)}$. In characteristic zero, $\Psi(p^{(2)})$ contains

$$\hat{p} \otimes [\hat{p}]^{r-1} = \frac{1}{r} \cdot \frac{1}{r^{r-1}} p^r \otimes p^{r(r-1)},$$

so

$$p^{(2)} = \frac{1}{r^r} \cdot \frac{r!(r(r-1))!}{(r^2)!} p^{r^2},$$

mod primitives.

Counting factors of *r*, we get one such factor in $r!$, $r-1$ in $(r(r-1))!$ and $r+1$ in $(r^2)!$. This gives

$$p^{(2)} = \alpha \frac{1}{r^{r+1}} p^{r^2},$$

where α is a fraction free of factors of *r*.

Thus $p^{(2)} = \alpha \left[\frac{p^r}{r}\right]^{r-1} \frac{p^r}{r^2}$, and in characteristic *r* the element $\alpha^{-1}[\hat{p}]^{1-r} p^{(2)}$ behaves as a p^r/r^2.

For $p^{(3)}$, we get (in characteristic zero), that $\Psi(p^{(3)})$ contains

$$p^{(2)} \otimes [p^{(2)}]^{r-1} = \alpha^r \frac{1}{r^{r(r+1)}} p^{r^2} \otimes p^{r^2(r-1)},$$

so

$$p^{(3)} = \alpha^r \frac{1}{r^{r(r+1)}} \frac{(r^2)!(r^2(r-1))!}{(r^3)!} p^{r^3},$$

mod primitives.

$(r^2)!$ has $r+1$ factors of *r*, $(r^2(r-1))!$ has $(r-1)(r+1)$ factors of *r* and $(r^3)!$ has $r(r+1)+1$ factors of *r*, so

$$p^{(3)} = \beta \frac{1}{r^{r(r+1)+1}} \cdot p^{r^3},$$

mod primitives (where β is free of factors of *r*).

Thus

$$p^{(3)} = \beta \frac{p^r}{r^3} \left[\frac{p^r}{r^2}\right]^{r-1} \left[\frac{p^r}{r}\right]^{r^2-r} = \alpha^{1-r} \beta \frac{p^r}{r^3} [p^{(2)}]^{r-1} [\hat{p}]^{r-1}$$

and

$$\frac{p^r}{r^3} = \beta^{-1}\alpha^{r-1}[\hat{p}]^{1-r}[p^{(2)}]^{1-r} p^{(3)}.$$

Analyzing induced primitives in higher orders permits us then similarly to get terms that do what the p^r/r^k did in characteristic zero.

Theorem 3.10. *Any finite dimensional connected Hopf algebra over a field of characteristic an odd prime r is polynomial on its primitive and induced primitive elements.*

Proof. As in the proof of Theorem 3.8, use Theorem 3.6 to write (in characteristic zero) any element in the Hopf algebra as a sum of products of primitives. The coefficients $\dfrac{m!\,k!}{(m+k)!}$ one gets from Proposition 3.4 can then be written in the form $\dfrac{a}{r^s b}$, with a and b free of powers of r, and we can describe $\dfrac{a}{r^s b} p^{m+k}$ as a product of induced primitives using the relations between the $p^{(i)}$ and the p^r/r^i previously presented. □

Just as for characteristic 2, we get

Corollary 3.11. *Any Hopf algebra map $f : H_1 \to H_2$ between two connected Hopf algebras over a field of characteristic an odd prime is completely determined by the values it takes on primitives and induced primitives.*

4. A Description of Geometric-like Hopf Algebras over a Field

This section deals with geometric-like Hopf algebras.

Let $a \in H$ be a generalized unit, i.e., an element a such that $\Psi(a) = a \otimes a$. An a-primitive x of H (relative to a) makes $\Psi(x) = a \otimes x + x \otimes a$, and any generalized unit a gives then rise to an a-primitive filtration

$$\{F_q^a H : q \geq 0\}$$

Similarly to what happened with 1-primitive filtrations, if an element $x \in F_q^a H$ has coproduct

$$\Psi(x) = a \otimes x + x \otimes a + \sum x' \otimes x'',$$

then all the x' and x'' that appear in the expression are in those $F^a_{q'}H$ that have $q' < q$.

We define H^a, the a-connected component of H, as the union $\bigcup_{q \geq 0} F^a_q H$.

Lemma 4.1. *If H is a geometric-like Hopf algebra and a is a generalized unit in H, we have*

$$F^a_q H = aF_q H$$

for all $q \geq 0$.

Proof. By induction on q we first have that $a \otimes a = a\Psi(1)$, so indeed $F^a_0 H = aF_0 H$. The same is true for a-primitives: If x is an a-primitive, then $\Psi(a^{-1}x) = 1 \otimes (a^{-1}x) + (a^{-1}x) \otimes 1$, and so $a^{-1}x$ is a primitive.

If the result is true up to a $q - 1$, then an $x \in F^a_q H$ has coproduct

$$a \otimes x + x \otimes a + \sum x' \otimes x'',$$

where each x' or x'' is in some $F^a_{q'}H$, and so each $a^{-1}x'$ or $a^{-1}x''$ is in some $F_{q'}H$.

Thus $a^{-1}x$ has coproduct

$$1 \otimes (a^{-1}x) + (a^{-1}x) \otimes 1 + \sum (a^{-1}x') \otimes (a^{-1}x'')$$

and so $a^{-1}x \in F_q H$ as desired. □

Considerations on the *exhaustive* part of the definition of connected and geometric-like Hopf algebras now gives:

Corollary 4.2. *For a generalized unit a in a geometric-like Hopf algebra H, $a^{-1}H^a$ is a connected Hopf algebra.*

From the previous section, we then know that, for any generalized unit *a*, $a^{-1}H^{a}$ will be generated by its primitive elements (and eventually also by its induced primitives, if the base field has characteristic different from zero).

Call *induced a-primitive* in a geometric-like Hopf algebra an element of the form *ax*, with *x* an induced primitive and *a* a generalized unit in the Hopf algebra.

The generalized units in a geometric-like Hopf algebra generate a sub-Hopf algebra, and the product of any two such generalized units is still a generalized unit. Moreover, the *a*-connected components H^{a} of a geometric-like Hopf algebra are disjoint.

We get a description for geometric-like Hopf algebras in the line with what was done in the previous section.

Theorem 4.3. *Any finite dimensional geometric-like Hopf algebra over a field of characteristic zero is generated by its generalized units and a-primitives*.

When the field is of prime characteristic, *it is generated by its generalized units*, *a-primitives and induced a-primitives*.

Assuming the notation mentioned before Corollary 3.9, we finally get

Corollary 4.4. *Any Hopf algebra map* $f : H_1 \to H_2$ *between two geometric-like Hopf algebras over a field of characteristic zero is completely determined by the values it takes on generalized units and a-primitives*.

If the field has prime characteristic, *such a map will be completely determined by the values it takes on generalized units, a-primitives and generalized a-primitives*.

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