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# RELATIVE DEFINABLE $C^rG$ TRIVIALITY OF $G$ INVARIANT PROPER DEFINABLE $C^r$ FUNCTIONS

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## Abstract

Let $G$ be a compact definable $C^r$ group and $1 \leq r < \infty$. Let $X$ be an affine definable $C^rG$ manifold and $X_1, \ldots, X_k$ definable $C^rG$ submanifolds of $X$ such that $X_1, \ldots, X_k$ are in general position in $X$. Suppose that $f : X \to \mathbb{R}$ is a $G$ invariant proper surjective submersive definable $C^r$ function such that for every $1 \leq i_1 < \cdots < i_s \leq k$, $f \mid X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. We prove that there exists a definable $C^rG$ diffeomorphism $h = (h', f) : (X; X_1, \ldots, X_k) \to (X^*; X_1^*, \ldots, X_k^*) \times \mathbb{R}$, where $Z^*$ denotes $Z \cap f^{-1}(0)$ for a subset $Z$ of $X$.

Moreover, we prove an equivariant definable $C^\infty$ version under some conditions and its application.

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## 1. Introduction

Coste and Shiota [1] proved that a proper Nash surjective submersion $f$ from an affine Nash manifold $X$ to $\mathbb{R}$ is Nash trivial, namely there exist a point $a \in \mathbb{R}$ and a Nash map $h : X \to f^{-1}(a)$ such that $(h, f) : X \to f^{-1}(a) \times \mathbb{R}$ is a Nash diffeomorphism.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [2], [5], see also [15]. The Nash category is a special case of the definable $C^\infty$ category and it coincides with the definable $C^\infty$ category based on $\mathcal{R}$ [16]. Further properties and constructions of them are studied in [3], [4], [6], [13] and there are uncountably many o-minimal expansions of $\mathcal{R}$ [14]. Equivariant definable $C^r$ categories are studied in [8-11]. Everything is considered in $\mathcal{M}$ and each manifold does not have boundary unless otherwise stated.

A map $\psi : M \to N$ between topological spaces is *proper* if for any compact set $C \subset N$, $\psi^{-1}(C)$ is compact.

Let $X$ be a $C^r$ manifold, $X_1, \dots, X_n$ $C^r$ submanifolds of $X$ and $r \geq 1$. We say that $\{X_i\}_{i=1}^n$ *are in general position in X* if for each $i \in I$ and $J \subset I - \{i\}$, $X_i$ intersects transverse to $\bigcap_{j \in J} X_j$.

The following is an equivariant relative definable $C^r$ version of [1].

**Theorem 1.1.** *Let G be a compact definable $C^r$ group and $1 \leq r < \infty$. Let X be an affine definable $C^r G$ manifold and $X_1, \dots, X_k$ definable $C^r G$ submanifolds of X such that $X_1, \dots, X_k$ are in general position in X. Suppose that $f : X \to \mathbb{R}$ is a G invariant proper surjective submersive definable $C^r$ function such that for every $1 \leq i_1 < \cdots < i_s \leq k$, $f | X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. Then there exists a definable $C^r G$ diffeomorphism $h = (h', f) : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*) \times \mathbb{R}$, where $Z^*$ denotes $Z \cap f^{-1}(0)$ for a subset Z of X.*

Let $X = \{(x, y) | y = 0\} \cup \{(x, y) | xy = 1\} \subset \mathbb{R}^2$ and $f : X \to \mathbb{R}$, $f(x, y) = x$. Then $f$ is a surjective submersive polynomial map and it is not definably trivial. Thus even in the non-equivariant category, the proper condition in Theorem 1.1 is necessary.

Let $1 \le r < \infty$ and let $F : \mathbb{R} \to (-1, 1)$ be a definable $C^r$ function such that $F(x) = x$ in a definable open neighborhood of 0, $F | (-\infty, -2] = -1/2$ and $F | [2, \infty) = 1/2$. Suppose that $X = S^1 \times \mathbb{R} \subset \mathbb{R}^3$, $f : S^1 \times \mathbb{R} \to \mathbb{R}$, $f(x, y, t) = t$, $X_1 = \{(0, 1)\} \times \mathbb{R}$ and $X_2 = \{(x, y, t) \in S^1 \times \mathbb{R} | x = F(t), y = \sqrt{1 - x^2}\}$. Then $X_1$, $X_2$ are in general position in $X$, $f$, $f | X_1$, $f | X_2$ are proper surjective submersions and $f | X_1 \cap X_2 : X_1 \cap X_2 \to \mathbb{R}$ is not surjective. Since there exists no definable $C^1$ diffeomorphism $h : (h', f) : (X; X_1, X_2) \to (X^*; X_1^*, X_2^*) \times \mathbb{R}$, even in the non-equivariant setting, the condition that every $f | X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion is necessary.

Let $f : U \to \mathbb{R}$ be a definable $C^\infty$ function on a definable open subset $U \subset \mathbb{R}^n$. We say that $f$ has *controlled derivatives* if there exist a definable continuous function $u : U \to \mathbb{R}$, real numbers $C_1, C_2, \ldots$ and positive integers $E_1, E_2, \ldots$ such that $| D^\alpha f(x) | \le C_{|\alpha|} u(x)^{E_{|\alpha|}}$ for all $x \in U$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We say that $\mathcal{M}$ has *piecewise controlled derivatives* if for every definable $C^\infty$ function $f : U \to \mathbb{R}$ defined in a definable open subset $U$ of $\mathbb{R}^n$, there exist definable open sets $U_1, \ldots, U_l \subset U$ such that $\dim(U - \cup_{i=1}^l U_i) < n$ and each $f | U_i$ has controlled derivatives.

The following is an equivariant definable $C^\infty$ version of Theorem 1.1.

**Theorem 1.2.** *Suppose that $\mathcal{M}$ is exponential, admits the $C^\infty$ cell decomposition and has piecewise controlled derivatives. Let $G$ be a compact definable $C^\infty$ group, $X$ be an affine definable $C^\infty G$ manifold and $X_1, \ldots, X_k$ be definable $C^\infty G$ submanifolds of $X$ such that $X_1, \ldots, X_k$ are in general position in $X$. Suppose that*

*$f : X \to \mathbb{R}$ is a G invariant proper surjective submersive definable $C^\infty$ function such that for every $1 \le i_1 < \cdots < i_s \le k$, $f \mid X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. Then there exists a definable $C^\infty G$ diffeomorphism $h = (h', f) : (X;\ X_1, ..., X_k) \to (X^*;\ X_1^*, ..., X_k^*) \times \mathbb{R}$, where $Z^*$ denotes $Z \cap f^{-1}(0)$ for a subset Z of X.*

Let $G$ be a compact definable $C^\infty$ group, $X$ be a noncompact definable $C^\infty G$ manifold and $X_1, ..., X_k$ be noncompact definable $C^\infty G$ submanifolds of $X$ in general position in $X$. If $\mathcal{M}$ is exponential, admits the $C^\infty$ cell decomposition and has piecewise controlled derivatives and $X$ is affine, then by Proposition 3.2, we may assume that $X$ is a bounded definable $C^\infty G$ submanifold of some representation $\Omega$ of $G$. We say that $(X;\ X_1, ..., X_k)$ satisfies *the frontier condition* if each $\overline{X}_i - X_i$ is contained in $\overline{X} - X$, where $\overline{X}_i$ (resp. $\overline{X}$) denotes the closure of $X_i$ (resp. $X$) in $\Omega$. We say that $(X;\ X_1, ..., X_k)$ is *simultaneously definably $C^\infty G$ compactifiable* if there exist a compact definable $C^\infty G$ manifold $Y$ with boundary $\partial Y$, compact definable $C^\infty G$ submanifolds $Y_1, ..., Y_k$ of $Y$ with boundary $\partial Y_1, ..., \partial Y_n$, respectively, and a definable $C^\infty G$ diffeomorphism $f : X \to \mathit{Int}\, Y$ such that for any $i$, $f(X_i) = \mathit{Int}\, Y_i$, each $\partial Y_i$ is contained in $\partial Y$, and $Y_1, ..., Y_k$ and $\partial Y$ are in general position in $Y$. Here $\mathit{Int}\, Y$ (resp. $\mathit{Int}\, Y_i$) denotes the interior of $Y$ (resp. $Y_i$).

As an application of Theorem 1.2, we have the following theorem.

**Theorem 1.3.** *Suppose that $\mathcal{M}$ is exponential, admits the $C^\infty$ cell decomposition and has piecewise controlled derivatives. Let G be a compact definable $C^\infty$ group, X be a noncompact affine definable $C^\infty G$ manifold and $X_1, ..., X_k$ be noncompact definable $C^\infty G$ submanifolds of X in general position in X such that $(X;\ X_1, ..., X_k)$ satisfies the frontier condition. Then $(X;\ X_1, ..., X_k)$ is simultaneously definably $C^\infty G$ compactifiable.*

Theorem 1.3 is an equivariant relative definable version of [1] and an equivariant definable $C^r$ version is proved in [12] when $r$ is a positive integer.

## 2. Proof of Theorem 1.1

Let $r$ be a non-negative integer, $\infty$ or $\omega$. A definable $C^r$ manifold $G$ is a *definable* $C^r$ *group* if the group operations $G \times G \to G$ and $G \to G$ are definable $C^r$ maps.

Let $G$ be a definable $C^r$ group. A *representation map* of $G$ is a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is a definable $C^r$ map. A *representation* means the representation space of a representation map of $G$. In this paper, we assume that every representation of $G$ is orthogonal. A *definable* $C^rG$ *submanifold* of a representation $\Omega$ of $G$ is a $G$ invariant definable $C^r$ submanifold of $\Omega$. A *definable* $C^rG$ *manifold* is a pair $(X, \phi)$ consisting of a definable $C^r$ manifold $X$ and a group action $\phi : G \times X \to X$ which is a definable $C^r$ map. We simply write $X$ instead of $(X, \phi)$. A definable $C^rG$ manifold is *affine* if it is definably $C^rG$ diffeomorphic to (definably $G$ homeomorphic to if $r = 0$) a definable $C^rG$ submanifold of some representation of $G$. Definable $C^rG$ manifolds and affine definable $C^rG$ manifolds are introduced in [10].

Let $G$ be a definable $C^r$ group, $X$ be definable $C^rG$ manifold and $Y$ be a definable $C^r$ manifold. A $G$ invariant definable $C^r$ map $f : X \to Y$ is *definably* $C^rG$ *trivial* if there exist a point $y \in Y$ and a definable $C^rG$ map $h : X \to f^{-1}(y)$ such that $H = (h, f) : X \to f^{-1}(y) \times Y$ is a definable $C^rG$ diffeomorphism.

The following is piecewise definable $C^rG$ triviality of $G$ invariant surjective submersive definable $C^r$ maps [10].

**Theorem 2.1** (1.1 [10]). *Let $r$ be a positive integer. Let $G$ be a compact definable $C^r$ group, $X$ be an affine definable $C^rG$ manifold and $Y$ be a definable $C^r$ manifold. Suppose that $f : X \to Y$ is a $G$ invariant surjective submersive definable $C^r$ map. Then there exists a finite decomposition $\{T_i\}$ of $Y$ into definable $C^r$ submanifolds of $Y$ such that each $f \mid f^{-1}(T_i) : f^{-1}(T_i) \to T_i$ is definably $C^rG$*

*trivial. If* $\mathcal{M}$ *admits the* $C^\infty$ $(resp.\ C^\omega)$ *cell decomposition, then we can take* $r = \infty$ $(resp.\ \omega)$.

The following provides the existence of a definable $C^rG$ tubular neighborhood of a definable $C^rG$ submanifold of a representation of $G$.

**Theorem 2.2** [11, 9]**.** *Let r be a non-negative integer,* $\infty$ *or* $\omega$. *Then every definable* $C^rG$ *submanifold X of a representation* $\Omega$ *of G has a definable* $C^rG$ *tubular neighborhood* $(U, \theta_X)$ *of X in* $\Omega$, *namely U is a G invariant definable open neighborhood of X in* $\Omega$ *and* $\theta_X : U \to X$ *is a definable* $C^rG$ *map with* $\theta_X \mid X = id_X$.

**Proposition 2.3** (P4 [12])**.** *Let r be a positive integer. Let Y, Z be affine definable* $C^rG$ *manifolds,* $Y_1, \ldots, Y_k$ $(resp.\ Z_1, \ldots, Z_k)$ *definable* $C^rG$ *submanifolds of Y (resp. Z) in general position in Y (resp. Z). Suppose that* $F : \left(\bigcup_{i=1}^k Y_i;\, Y_1, \ldots, Y_k\right) \to \left(\bigcup_{i=1}^k Z_i;\, Z_1, \ldots, Z_k\right)$ *is a definable continuous G map. If each* $F \mid Y_i$ *is a definable* $C^rG$ *map* $(Y_i;\, Y_i \cap Y_1, \ldots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \ldots, Y_i \cap Y_k) \to (Z_i;\, Z_i \cap Z_1, \ldots, Z_i \cap Z_{i-1}, Z_i \cap Z_{i+1}, \ldots, Z_i \cap Z_k)$, *then there exist a G invariant definable open neighborhood W of* $\bigcup_{i=1}^n Y_i$ *in Y and a definable* $C^rG$ *map* $H : (W;\, Y_1, \ldots, Y_k) \to (Z;\, Z_1, \ldots, Z_k)$ *such that* $H \mid \bigcup_{i=1}^k Y_i = F$.

Let $1 \le r < \infty$ and $Def^r(\mathbb{R}^n)$ denote the set of definable $C^r$ functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, the $\varepsilon$-neighborhood $N(f; \varepsilon)$ of $f$ in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) \mid |D^\alpha(h - f)| < \varepsilon,\ \forall \alpha \in (\mathbb{N} \cup \{0\})^n,\ |\alpha| \le r\}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We call the topology defined by these $\varepsilon$-neighborhoods the *definable* $C^r$ *topology*. By taking the relative topology of the definable $C^r$ topology of $\mathbb{R}^n$, we can define the *definable* $C^r$ *topology* of a definable $C^r$ submanifold $X$ of $\mathbb{R}^n$.

Let $X, Y \subset \mathbb{R}^n$ be definable $C^r$ submanifolds. Note that if $X$ is compact, then the definable $C^r$ topology of the set of definable $C^r$ maps from $X$ to $Y$ coincides the $C^r$ Whitney topology of it [15].

**Theorem 2.4** [15]**.** *Let $X$ and $Y$ be definable $C^s$ submanifolds of $\mathbb{R}^n$ and $0 < s < \infty$. Let $f : X \to Y$ be a definable $C^s$ map. If $f$ is an immersion* (*resp. a diffeomorphism, a diffeomorphism onto its image*), *then an approximation of $f$ in the definable $C^s$ topology is an immersion* (*resp. a diffeomorphism, a diffeomorphism onto its image*). *Moreover, if $f$ is a diffeomorphism, then $h^{-1} \to f^{-1}$ as $h \to f$.*

**Proof of Theorem 1.1.** Since $X$ is affine, we may assume that $X$ is a definable $C^rG$ submanifold of a representation $\Omega$ of $G$.

We first prove the case where $k = 0$. Applying Theorem 2.1, we have a partition $-\infty = a_0 < a_1 < a_2 < \cdots < a_j < a_{j+1} = \infty$ of $\mathbb{R}$ and definable $C^rG$ diffeomorphisms $w_i : f^{-1}((a_i, a_{i+1})) \to f^{-1}(y_i) \times (a_i, a_{i+1})$ with $f \mid f^{-1}((a_i, a_{i+1})) = p_i \circ w_i$, $0 \leq i \leq j$, where $p_i$ denotes the projection $f^{-1}(y_i) \times (a_i, a_{i+1}) \to (a_i, a_{i+1})$ and $y_i \in (a_i, a_{i+1})$.

Now we prove that for each $a_i$ with $1 \leq i \leq j$, there exist an open interval $I_i$ containing $a_i$ and a definable $C^rG$ map $\pi_i : f^{-1}(I_i) \to f^{-1}(a_i)$ such that $F_i = (\pi_i, f) : f^{-1}(I_i) \to f^{-1}(a_i) \times I_i$ is a definable $C^rG$ diffeomorphism. By Theorem 2.2, we have a definable $C^rG$ tubular neighborhood $(U_i, \theta_{f^{-1}(a_i)})$ of $f^{-1}(a_i)$ in $X$. Since $f$ is proper, there exists an open interval $I_i$ containing $a_i$ such that $f^{-1}(I_i) \subset U_i$. Note that if $f$ is not proper, then such an open interval does not always exist. Hence shrinking $I_i$, if necessary, $F_i = (\pi_i, f) : f^{-1}(I_i) \to f^{-1}(a_i) \times I_i$ is the required definable $C^rG$ diffeomorphism.

By the above argument, we have a finite family of $\{J_i\}_{i=1}^{l}$ of open intervals and definable $C^rG$ diffeomorphisms $\phi_i : f^{-1}(J_i) \to f^{-1}(y_i) \times J_i$, $1 \leq i \leq l$, such that

$y_i \in J_i$, $\bigcup_{i=1}^{l} J_i = \mathbb{R}$ and the composition of $\phi_i$ with the projection $f^{-1}(y_i) \times J_i$ onto $J_i$ is $f \mid f^{-1}(J_i)$.

Now we glue these trivializations to get a global one. We can suppose that $i \geq 2$, $U_{i-1} \cap J_i = (a, b)$ and $\psi_{i-1} : f^{-1}(U_{i-1}) \to f^{-1}(y_1) \times U_{i-1}$ is a definable $C^r G$ diffeomorphism with $f \mid f^{-1}(U_{i-1}) = proj_{i-1} \circ \psi_{i-1}$, where $U_{i-1} = \bigcup_{s=1}^{i-1} J_s$ and $proj_{i-1}$ denotes the projection $f^{-1}(y_1) \times U_{i-1} \to U_{i-1}$. Take $z \in (a, b) = U_{i-1} \cap J_i$. Then since $f^{-1}(y_1) \cong f^{-1}(z) \cong f^{-1}(y_i)$, $f^{-1}(y_1)$ is definably $C^r G$ diffeomorphic to $f^{-1}(y_i)$. Hence we may assume that $\phi_i'$ is a definable $C^r G$ diffeomorphism from $f^{-1}(J_i)$ to $f^{-1}(y_1) \times J_i$. Then we have a definable $C^r G$ diffeomorphism

$$\psi_{i-1} \circ (\phi_i')^{-1} : f^{-1}(y_1) \times (a, b) \to f^{-1}(y_1) \times (a, b),\ (x, t) \mapsto (q(x, t), t).$$

Take a $C^r$ Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \dfrac{a+b}{2}$ on $\left(-\infty, \dfrac{3}{4}a + \dfrac{1}{4}b\right]$ and $u = id$ on $\left[\dfrac{1}{4}a + \dfrac{3}{4}b, \infty\right)$. Let

$$\tau : f^{-1}(y_1) \times (a, b) \to f^{-1}((a, b)),\ \tau(x, t) = \psi_{i-1}^{-1}(q(x, u(t)), t).$$

Then $\tau$ is a definable $C^r G$ diffeomorphism such that $\tau = (\phi_i')^{-1}$ if $\dfrac{1}{4}a + \dfrac{3}{4}b \leq t \leq b$ and $\tau = \psi_{i-1}^{-1} \circ (P \times id)$ if $a \leq t \leq \dfrac{3}{4}a + \dfrac{1}{4}b$, where $P : f^{-1}(y_1) \to f^{-1}(y_1)$, $P(x) = q\left(x, \dfrac{a+b}{2}\right)$. Thus we can define

$$\tilde{\psi}_i : f^{-1}(U_i) \to f^{-1}(y_1) \times U_i,$$

$$\tilde{\psi}_i(x) = \begin{cases} (P \times id)^{-1} \circ \psi_{i-1}(x), & f(x) \leq \dfrac{3}{4}a + \dfrac{1}{4}b, \\ \tau^{-1}(x), & \dfrac{3}{4}a + \dfrac{1}{4}b \leq f(x) \leq b, \\ \phi_i(x), & f(x) > b. \end{cases}$$

Then $\tilde{\psi}_i$ is a definable $C^rG$ diffeomorphism. Thus $\tilde{\psi}_l : X \to f^{-1}(y_1) \times \mathbb{R}$ is a definable $C^rG$ diffeomorphism. Therefore, we have the required definable $C^rG$ diffeomorphism $(H, f) : X \to X^* \times \mathbb{R}$.

We now prove the general case by induction on $k$.

Let $k \geq 1$. By the inductive hypothesis, for any $i$, there exists a definable $C^rG$ diffeomorphism $h_i = (h_i', f) : (X_i; X_i \cap X_1, ..., X_i \cap X_{i-1}, X_i \cap X_{i+1}, ..., X_i \cap X_k) \to (X_1^*; X_i^* \cap X_1^*, ..., X_i^* \cap X_{i-1}^*, X_i^* \cap X_{i+1}^*, ..., X_i^* \cap X_k^*) \times \mathbb{R}$. In particular, $h_1' | X_2 \cap X_1 : (X_2 \cap X_1; X_2 \cap X_1 \cap X_3, ..., X_2 \cap X_1 \cap X_k) \to (X_2^* \cap X_1^*; X_2^* \cap X_1^* \cap X_3^*, ..., X_2^* \cap X_1^* \cap X_k^*)$ is a definable $C^rG$ map. By Theorem 2.2, we have a $G$ invariant definable open neighborhood $W_2$ of $X_1 \cap X_2$ in $X_2$ and a definable $C^rG$ map $\Phi_2 : (W_2; X_2 \cap X_1; X_2 \cap X_1 \cap X_3, ..., X_2 \cap X_1 \cap X_k) \to (X_2^*; X_2^* \cap X_1^*; X_2^* \cap X_1^* \cap X_3^*, ..., X_2^* \cap X_1^* \cap X_k^*)$ such that $\Phi_2 | X_2 \cap X_1 = h_1' | X_2 \cap X_1$. Take a $G$ invariant definable open neighborhood $W_2' \subset W_2$ of $X_1 \cap X_2$ in $X_2$ whose closure in $X_2$ is properly contained in $W_1$ and a $G$ invariant definable $C^r$ function $a : X_2 \to \mathbb{R}$ such that its support lies in $W_2$ and $a | W_2' = 1$. By Theorem 2.2, we have a $G$ invariant definable open neighborhood $O$ of $X_2^*$ in $\Omega$ and a definable $C^rG$ map $\theta_{X_2^*} : O \to X_2^*$ with $\theta | X_2^* = id_{X_2^*}$.

Define

$$\Psi_2'(x) = \begin{cases} \theta_{X_2^*}((1 - a(x))h_2'(x) + a(x)\Phi_2(x)), & x \in W_1, \\ h_2'(x), & x \in X_2 - W_1. \end{cases}$$

Then $\Psi_2' : (X_2; X_2 \cap X_1, ..., X_2 \cap X_k) \to (X_2^*; X_2^* \cap X_1^*, ..., X_2^* \cap X_k^*)$ is a definable $C^rG$ map which is an approximation of $h_2'$. Thus $h_1'$ is extensible to a definable continuous $G$ map $\tilde{\Psi}_2 : X_1 \cup X_2 \to X^*$ such that $\tilde{\Psi}_2 | X_1$ and $\tilde{\Psi}_2 | X_2$ are definable $C^rG$ maps.

Repeating this process, we have a definable continuous $G$ map $\Phi$ :

$\left(\bigcup_{i=1}^k X_i;\ X_1, ..., X_k\right) \to (X^*;\ X_1^*, ..., X_k^*)$ such that each $\Phi | X_i$ is a definable $C^r G$ map which is an approximation of $h_i'$.

By Proposition 2.3, we have a $G$ invariant definable open neighborhood $U$ of $\bigcup_{i=1}^k X_i$ and a definable $C^r G$ map $L : U \to X^*$ extending $\Phi$.

Take a $G$ invariant definable open neighborhood $U'$ of $\bigcup_{i=1}^k X_i$ in $X$ whose closure in $X$ is properly contained in $U$ and a $G$ invariant definable $C^r$ function $b : X \to \mathbb{R}$ such that its support lies in $U$ and $b|U' = 1$. By Theorem 2.2, we have a $G$ invariant definable open neighborhood $V$ of $X^*$ in $\Omega$ and a definable $C^r G$ map $\theta_{X^*} : V \to X^*$ with $\theta_{X^*} | X^* = id_{X^*}$.

Define

$$h'(x) = \begin{cases} \theta_{X^*}((1 - b(x))H(x) + b(x)L(x)), & x \in U, \\ H(x), & x \in X - U. \end{cases}$$

Then $h' : (X;\ X_1, ..., X_k) \to (X^*;\ X_1^*, ..., X_k^*)$ is a definable $C^r G$ map. Thus $h = (h',\ f) : (X;\ X_1, ..., X_k) \to (X^*;\ X_1^*, ..., X_k^*) \times \mathbb{R}$ is a definable $C^r G$ map which is an approximation of $(H,\ f)$. Therefore, by Theorem 2.4, $h$ is the required definable $C^r G$ diffeomorphism. □

## 3. Proofs of Theorems 1.2 and 1.3

From now on we assume that $\mathcal{M}$ is exponential, admits the $C^\infty$ cell decomposition and has piecewise controlled derivatives.

**Theorem 3.1** (1.2 [7])**.** *Every definable closed subset of* $\mathbb{R}^n$ *is the zero set of a definable* $C^\infty$ *function on* $\mathbb{R}^n$.

**Proposition 3.2.** *Let G be a compact definable* $C^\infty$ *group and X be a definable* $C^\infty G$ *manifold in a representation* $\Omega$ *of G. Then X is definably* $C^\infty G$ *imbeddable* *into* $\Omega \times \mathbb{R}^2$ *such that* $X$ *is bounded and* $\overline{X} - X$ *consists of at most one point, where* $\overline{X}$ *denotes the closure of* $X$.

**Proof.** We may assume that $X$ is noncompact. Then $\overline{X} - X$ is a $G$ invariant closed definable subset of $\Omega$. Let $\pi : \Omega \to \Omega/G \subset \mathbb{R}^s$ denote the orbit map. Then $i \circ \pi : \Omega \to \mathbb{R}^s$ is a proper polynomial map (see Section 4 [11]), where $i : \Omega/G \to \mathbb{R}^s$ denotes the inclusion. Hence $i \circ \pi | \overline{X} - X : \overline{X} - X \to \mathbb{R}^s$ is proper because $\overline{X} - X$ is closed in $\Omega$. Thus $i \circ \pi(\overline{X} - X)(= \pi(\overline{X} - X))$ is a definable closed subset of $\mathbb{R}^s$. Applying Theorem 3.1, there exists a definable $C^\infty$ function $f : \mathbb{R}^s \to \mathbb{R}$ with $\pi(\overline{X} - X) = f^{-1}(0)$. Hence $F := f \circ \pi : \Omega \to \mathbb{R}$ is a $G$ invariant definable $C^\infty$ function with $\overline{X} - X = F^{-1}(0)$. Therefore, replacing the graph of $1/F$ by $X$, we may assume that $X$ is closed in $\Omega \times \mathbb{R}$. Applying the stereographic projection $s : \Omega \times \mathbb{R} \to S(\Omega \times \mathbb{R}^2)$, $s(X)$ satisfies our requirements, where $S(\Omega \times \mathbb{R}^2)$ denotes the unit sphere of $\Omega \times \mathbb{R}^2$. □

The proof of Proposition 3.2 proves the following two theorems and proposition.

**Theorem 3.3.** *Let* $G$ *be a compact definable* $C^\infty$ *group and* $\Omega$ *be a representation of* $G$. *Every* $G$ *invariant definable closed subset of* $\Omega$ *is the zero set of a* $G$ *invariant definable* $C^\infty$ *function on* $\Omega$.

**Theorem 3.4.** *Let* $G$ *be a compact definable* $C^\infty$ *group and* $X$ *be an affine definable* $C^\infty G$ *manifold. Suppose that* $A$, $B$ *are* $G$ *invariant definable disjoint closed subsets of* $X$. *Then there exists a* $G$ *invariant definable* $C^\infty$ *function* $f : X \to \mathbb{R}$ *such that* $f | A = 1$ *and* $f | B = 0$.

**Proposition 3.5.** *Let* $G$ *be a compact definable* $C^\infty$ *group,* $X$ *be a noncompact affine definable* $C^\infty G$ *manifold and* $X_1, \ldots, X_n$ *be noncompact definable* $C^r G$ *submanifolds of* $X$ *in general position in* $X$ *such that* $(X; X_1, \ldots, X_n)$ *satisfies the frontier condition. Then we may assume that* $X$ *is a bounded definable* $C^\infty G$

*submanifold of some representation* $\Omega$ *of* $G$ *such that* $\overline{X}_1 - X_1 = \cdots = \overline{X}_n - X_n = \overline{X} - X = \{0\}$, *where* $\overline{X}$ (*resp.* $\overline{X}_i$) *denotes the closure of* $X$ (*resp.* $X_i$) *in* $\Omega$.

Using Theorem 3.4, a similar proof of P4 [12] proves the following proposition.

**Proposition 3.6.** *Let* $Y, Z$ *be affine definable* $C^\infty G$ *manifolds*, $Y_1, \ldots, Y_k$ (*resp*. $Z_1, \ldots, Z_k$) *definable* $C^\infty G$ *submanifolds of* $Y$ (*resp*. $Z$) *in general position in* $Y$ (*resp*. $Z$). *Suppose that* $F : \left(\bigcup_{i=1}^k Y_i;\, Y_1, \ldots, Y_k\right) \to \left(\bigcup_{i=1}^k Z_i;\, Z_1, \ldots, Z_k\right)$ *is a definable continuous* $G$ *map*. *If each* $F|Y_i$ *is a definable* $C^\infty G$ *map* $(Y_i;\, Y_i \cap Y_1, \ldots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \ldots, Y_i \cap Y_k) \to (Z_i;\, Z_i \cap Z_1, \ldots, Z_i \cap Z_{i-1}, Z_i \cap Z_{i+1}, \ldots, Z_i \cap Z_k)$, *then there exist a* $G$ *invariant definable open neighborhood* $W$ *of* $\bigcup_{i=1}^n Y_i$ *in* $Y$ *and a definable* $C^\infty G$ *map* $H : (W;\, Y_1, \ldots, Y_k) \to (Z;\, Z_1, \ldots, Z_k)$ *such that* $H|\bigcup_{i=1}^k Y_i = F$.

**Proof of Theorem 1.2.** Using Theorem 3.4 and Proposition 3.6, a similar proof of Theorem 1.1 proves Theorem 1.2. □

**Proof of Theorem 1.3.** By Proposition 3.5, we may assume that $X$ is a bounded definable $C^\infty G$ submanifold of a representation $\Omega$ of $G$ such that $\overline{X}_1 - X_1 = \cdots = \overline{X}_n - X_n = \overline{X} - X = \{0\}$.

Let $f : X \to \mathbb{R}$, $f(x) = \|x\|^{-1}$, where $\|x\|$ denotes the standard norm of $x$ in $\Omega$. Since $f$ is submersive and $G$ invariant and by Theorem 2.1, there exist a sufficiently large positive number $\alpha$ and a definable $C^\infty G$ map $h_1 : f^{-1}((\alpha, \infty)) \to f^{-1}(\alpha)$ such that $h := (h_1, f) : f^{-1}((\alpha, \infty)) \to f^{-1}(\alpha) \times (\alpha, \infty)$ is a definable $C^\infty G$ diffeomorphism.

Let $f_i := f|X_i$. Since $(X;\, X_1, \ldots, X_k)$ satisfies the frontier condition and $X_1, \ldots, X_k$ are in general position in $X$, each $Y_i := f_i^{-1}((\alpha, \infty))$ is a definable $C^\infty G$ submanifold of $Y := f^{-1}((\alpha, \infty))$, $Y_1, \ldots, Y_k$ are in general position in $Y$ and for every $1 \le i_1 < \cdots < i_s \le k$, $f|Y_{i_1} \cap \cdots \cap Y_{i_s} : Y_{i_1} \cap \cdots \cap Y_{i_s} \to (\alpha, \infty)$ is a proper surjective submersion. Since $(\alpha, \infty)$ is definably $C^\infty$ diffeomorphic to $\mathbb{R}$, there exists a $G$ invariant surjective submersive definable $C^\infty$ function $F : (Y; Y_1, \dots, Y_k) \to \mathbb{R}$ satisfying the conditions in Theorem 1.2.

Applying Theorem 1.2 to $F$, there exists a definable $C^\infty G$ diffeomorphism $(f^{-1}((\alpha, \infty)); f_1^{-1}((\alpha, \infty)), \dots, f_k^{-1}((\alpha, \infty))) \to (f^{-1}(\alpha); f_1^{-1}(\alpha), \dots, f_k^{-1}(\alpha)) \times \mathbb{R}$. Thus we have a definable $C^\infty G$ diffeomorphism $H : (f^{-1}((\alpha, \infty)); f_1^{-1}((\alpha, \infty)), \dots, f_k^{-1}((\alpha, \infty))) \to (f^{-1}(\alpha); f_1^{-1}(\alpha), \dots, f_k^{-1}(\alpha)) \times (\alpha, \infty)$. Since $\alpha$ is sufficiently large, $f^{-1}([0, \alpha + 1])$ is a compact definable $C^\infty G$ manifold with boundary $f^{-1}(\alpha + 1)$ and each $f_i^{-1}([0, \alpha + 1])$ is a compact definable $C^\infty G$ submanifold of $f^{-1}([0, \alpha + 1])$ with boundary $f_i^{-1}(\alpha + 1)$. Therefore, using $H$ and Theorem 3.4, $(X; X_1, \dots, X_k)$ is definably $C^\infty G$ diffeomorphic to $(f^{-1}([0, \alpha + 1]); f_1^{-1}([0, \alpha + 1]), \dots, f_k^{-1}([0, \alpha + 1]))$. □

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