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FIXED POINT RESULTS RELATED TO REICH'S PROBLEM

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Abstract

In this note, we present a brief survey on partial answers to Reich's problem and some related results are also given.

1. Introduction

Let (X, d) be a metric space. We use $Cl(X)$ to denote the collection of all nonempty closed subsets of X, $CB(X)$ for the collection of all nonempty closed bounded subsets of X, $P(X)$ for the collection of all nonempty bounded proximinal subsets of X (A subset M of X is called *proximinal* if for each $x \in X$, there exists an element $k \in M$ such that $d(x, k) = d(x, M)$, where $d(x, M) = \inf\{d(x, y) : y \in M\}$ is the distance from the point x to the subset M), $K(X)$ for the collection of all nonempty compact subsets of X, and H the Hausdorff metric on $CB(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \qquad A, B \in CB(X),$$

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where $d(a, B) = \inf\{d(a, b) : b \in B\}$. H on $Cl(X)$ is also a metric except that it takes also the value $+\infty$ if (X, d) is unbounded.

We also use the following notions:

Definition 1.1. We say a multivalued map $T : X \to CB(X)$ is

(a) *contraction* [20] if there exists a constant $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$H(T(x), T(y)) \leq \lambda d(x, y),$$

(b) *generalized contraction* [22] if for all $x, y \in X$,

$$H(Tx, Ty) \leq k(d(x, y))d(x, y),$$

where k is a function of $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r \to t^+} k(r) < 1$, for all $t > 0$.

An element $x \in X$ is said to be a *fixed point* of T if $x \in T(x)$.

Definition 1.2. A real valued function f on X is called *lower semi-continuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

In [12], Kada et al. have introduced a concept of w-distance in the setting of metric spaces as follows:

Definition 1.3. A function $\omega : X \times X \to [0, \infty)$ is called a *w-distance* on X if it satisfies the following:

(w1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$, for all $x, y, z \in X$;

(w2) ω is lower semi-continuous in its second variable;

(w3) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is a w-distance on X. Many other examples of w-distance are given in [12, 27, 28]. The following fundamental lemma was proved in [12], which is crucial for the proofs of results on the existence of fixed points.

Lemma 1.4. *Let X be a metric space with metric d and let* ω *be a w-distance on X. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in X. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, \infty)$ *converging to* 0. *Then the following hold for any* $x, y, z \in X$:

(a) *if* $\omega(x_n, y) \leq \alpha_n$ *and* $\omega(x_n, z) \leq \beta_n$ *for all* $n \geq 1$, *then* $y = z$; *in particular, if* $\omega(x, y) = 0$ *and* $\omega(x, z) = 0$, *then* $y = z$;

(b) *if* $\omega(x_n, y_n) \leq \alpha_n$ *and* $\omega(x_n, z) \leq \beta_n$ *for all* $n \geq 1$, *then* $\{y_n\}$ *converges to* z;

(c) $\omega(x_n, x_m) \leq \alpha_n$ *for any* $n, m \geq 1$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence*;

(d) $\omega(y, x_n) \leq \alpha_n$ *for any* $n \geq 1$, *then* $\{x_n\}$ *is a Cauchy sequence*.

Lin and Du [18] proved the following:

Lemma 1.5. *Let K be a closed subset of X and* ω *be a w-distance on X. Suppose that there exists* $u \in X$ *such that* $\omega(u, u) = 0$. *Then*

$$\omega(u, K) = 0 \Leftrightarrow u \in K.$$

2. Fixed Points Results and Reich's Problem

Using the concept of Hausdorff metric, Nadler [20] has proved the following fixed point result, known as multivalued version of the well-known Banach contraction principle [1].

Theorem 2.1. *Each multivalued contraction map* $T : X \to CB(X)$ *has a fixed point provided X is complete metric space.*

In [21], Reich established the following generalization of the Banach contraction principle, which also generalizes the fixed point result of Boyd and Wong [2].

Theorem 2.2. *Each multivalued generalized contraction* $T : X \to K(X)$ *has a fixed point in X provided X is complete metric space.*

In [22], Reich raised the following problem:

Reich's problem. Does each multivalued generalized contraction has a fixed point?

Note that in Theorem 2.2, the map T assumed to take compact values. For this reason, in [22, 23], Reich posed the question whether or not the range of T in Theorem 2.2 can be relaxed. Specifically, the question is whether or not the range of T, $K(X)$ can be replaced by $CB(X)$. In fact Reich's problem remains unsolved, some partial answers have been obtained.

In [13], Kaneko has obtained the following fixed point result.

Theorem 2.3. *Let X be a complete metric space. Let $T : X \to P(X)$ be a generalized contraction map for which the function k is monotone increasing. Then T has a fixed point.*

The partial affirmative answer to Reich's problem was given by Mizoguchi and Takahashi [19]. They proved the following result, which is also a generalization of Nadler's result (Theorem 2.1).

Theorem 2.4. *Let X be a complete metric space. Let $T : X \to CB(X)$ be a multivalued generalized contraction map for which the function k satisfies $\limsup_{r \to t^+} k(r) < 1$ for all $t \geq 0$. Then T has a fixed point.*

Alternative proofs of this theorem have obtained by Daffer and Kaneko [7], Chang [3], Sastry and Babu [24] and others. In [29], Xu gave a very interesting simple alternative proof of Theorem 2.4 as follows:

Proof. By Lemma 1.2.3 of [29], there exists a sequence x_n in X such that $x_{n+1} \in T(x_n)$ and the sequence of nonnegative real numbers $\{d(x_n, x_{n+1})\}$ is decreasing to zero. By the definition of k, for some $\zeta > 0$ and $b \in (0, 1)$, we get

$$k(\beta) < b^2, \;\; \beta \in (0, \zeta).$$

Note that for some n_0, we have $d(x_{n-1}, x_n) < \zeta$ for all $n \geq n_0$. Thus it follows from the fact $d(x_n, x_{n+1}) \leq \sqrt{kd(x_n, x_{n-1})}\, d(x_{n-1}, x_n)$ that for all $n \geq n_0$,

$$d(x_n, x_{n+1}) \leq \zeta d(x_{n-1}, x_n) \leq \cdots \leq \zeta^{n-n_0+1} d(x_{n_0-1}, x_{n_0}).$$

It follows that $\{x_n\}$ is a Cauchy sequence in X and hence it is convergent because the space X is complete. Now, let the sequence $\{x_n\}$ converge to z. Since for all n, we have $x_n \in T(x_{n-1})$, by taking the limit as $n \to \infty$ and using the continuity of the multivalued map T, we get $z \in T(z)$, that is, z is a fixed point of T.

Most recently, another alternative proof of Theorem 2.4 is given by Suzuki [26, p. 754] as under.

Alternative proof. Define a function $\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = (k(t) + 1)/2$. Then the following hold:

(i) $\limsup_{r \to t+0} \beta(r) < 1$ for all $t \geq 0$.

(ii) For all $x, y \in X$ and $u \in T(x)$, there exists an element $v \in T(x)$ such that $d(u, v) \leq \beta(d(x, y))d(x, y)$.

Thus, we can define a sequence $\{x_n\}$ in X such that for all integers $n \geq 1$, $x_{n+1} \in T(x_n)$ and $d(x_{n+1}, x_{n+2}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1})$. Hence the sequence of nonnegative real numbers $\{d(x_n, x_{n+1})\}$ is non-increasing and thus converges to some nonnegative real number α. Note that there exist some $b \in [0, 1)$ and $\varepsilon > 0$ such that $\beta(r) \leq b$ for all $r \in [\alpha, \alpha + \varepsilon]$. Now we can choose some integer $m \geq 1$ such that $m \leq d(x_n, x_{n+1}) \leq \alpha + \varepsilon$ with $n \geq m$. Note that

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq bd(x_n, x_{n+1}),$$

and thus we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in the complete space X. Let $\{x_n\}$ converge to some $z \in X$. Note that

$$\begin{aligned} d(z, T(z)) &= \lim_{n \to \infty} d(x_{n+1}, T(z)) \\ &\leq \lim_{n \to \infty} H(T(x_n, T(z))) \\ &\leq \lim_{n \to \infty} \beta(d(x_n, z))d(x_n, z) \\ &\leq \lim_{n \to \infty} d(x_n, z) = 0 \end{aligned}$$

and $T(z)$ is closed, we get $z \in T(z)$.

Note that the stronger condition assumed on k in Theorem 2.4, viz., $\limsup_{r \to t^+} k(r) < 1$ for all $t \geq 0$, implies that $k(t) < h$ for some $0 < h < 1$. Thus with this condition, one may get that the map T is a contraction over a region for which $d(x, y)$ is sufficiently small.

In [3], Chang introduced and studied the following notion (also, see [8, 11]).

Definition 2.5. Let $\phi : [0, \infty) \to [0, \infty)$. Then the function ϕ is said to satisfy the condition (Φ) denoted by $\phi \in (\Phi)$ if (i) $\phi(t) < t$ for all $t > 0$; (ii) ϕ is upper semicontinuous from the right on $(0, \infty)$; and (iii) there exists a positive real number r such that ϕ is nondecreasing on $(0, r]$ and $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t \in (0, r]$.

It has been observed by Chang [3] that if k is a function of $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r \to t^+} k(r) < 1$ for all $t \geq 0$, then there exists a function $\phi \in (\Phi)$ such that $k(t)t \leq \phi(t)$ for all $t > 0$.

In [11], Jachymski studied equivalent reformulation of Reich's problem (see [11, Proposition 1]) and proved the following result which generalizes Theorem 2.4 and still gives only a partial answer to Reich' problem.

Theorem 2.6. *Let (X, d) be a complete metric space. Let $T : X \to Cl(X)$ and suppose that there exists a function $\phi \in (\Phi)$ such that*

$$H(Tx, Ty) \leq \phi d(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Chang [3] generalized Theorem 2.4 as follows:

Theorem 2.7. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ and suppose that there exists a function $\phi \in (\Phi)$ such that for all $x, y \in X$,*

$$H(T(x), T(y)) \leq \phi M(x, y),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2}\right\}.$$

Then T has a fixed point.

Recently, Daffer et al. [8] introduced a class of functions that satisfy $\limsup_{r \to t^+} k(r) < 1$ for every $t \in (0, \infty)$ and belong to (Φ). Applying Theorem 2.7, they proved the following fixed point result for multivalued maps which satisfy the conditions required in the Reich's problem.

Theorem 2.8. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued map such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq k(d(x, y))d(x, y),$$

where k is a function of $(0, \infty)$ to $[0, 1]$ such that $k(t) < 1$ for all $t > 0$ and $k(t) \leq 1 - at^{b-1}$, $a > 0$, for some $b \in (1, 2)$ on some interval $[0, s]$, $0 < s < a^{-1/(b-1)}$. Then T has a fixed point.

In [4], Chen obtained the following partial answer to Reich's problem:

Theorem 2.9. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued generalized contraction map. Suppose that T has the following property*:

($*$) *Whenever M is a closed subset of X such that $Tx \cap M \neq \emptyset$ for all $x \in M$, we have $d(x, Tx \cap M) = d(x, Tx)$ for all $x \in M$.*

Then T has a fixed point.

We observe that the condition ($*$) in Theorem 2.9 is very restrictive. Even the constant maps do not satisfy it. The following example is given in [29].

Example 2.10. Let $X = [0, 5]$ be the metric space equipped with the usual metric *d*. Define a map *T* with

$$T(x) = [0, 1] \cup [4, 5] \quad \text{for all } x \in X.$$

Let $M = [1, 3]$. Then *M* is a closed subset of *X* with $T(x) \cap M \neq \emptyset$ for all $x \in X$. But, note that for $x = 3 \in M$, we have

$$d(x, T(x)) = 1 \text{ and } d(x, T(x) \cap M) = 2,$$

thus

$$d(x, T(x)) \neq d(x, T(x) \cap M).$$

In [10], Hu proved a result in which he claims to have affirmative answer to Reich's problem but in fact it is not the case as pointed out by Jachymski [11] that there is a gap in the proof of Theorem 3 of [10].

In [25], Semenov proved a fixed point theorem for a broad class of closed valued generalized contractions with $\limsup_{r \to t^+} k(r) < 1$ for all $t > 0$ and $\limsup_{r \to 0^+} k(r) = 1.$

The Reich's problem is still unsolved and further investigation towards a complete resolution is required.

3. Related Fixed Point Results

Recently, some interesting fixed point results appeared in the literature without using the concept of the Hausdorff metric. Klim and Wardowski [14] proved the following fixed point result which is a generalization of Theorem 2.1 and Theorem 3.1 of Feng and Liu [9].

Theorem 3.1. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued map. If there exists a constant $b \in (0, 1)$ such that for any $x \in X$ there is $y \in T(x)$ satisfying*

$$bd(x, y) \leq d(x, T(x))$$

and

$$d(y, T(y)) \leq k(dx, y)d(x, y),$$

where k is a function from $[0, \infty)$ to $[0, b)$ such that $\limsup_{r \to t^+} k(r) < b$ for all $t \geq 0.$ Then T has a fixed point in X provided a real valued function $f(x) = d(x, T(x))$ on X is lower semicontinuous.

Most recently, Ćirić [5, 6] proved some interesting fixed point results for multivalued nonlinear contractions. In [6], he obtained the following fixed point result which is a generalization of Theorem 2.4 and Theorem 3.1.

Theorem 3.2. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued map. If for any $x \in X$ there is $y \in T(x)$ satisfying*

$$\sqrt{kd(x, y)}\, d(x, y) \leq d(x, T(x))$$

and

$$d(y, T(y)) \leq k(dx, y)d(x, y),$$

where k is a function from $[0, \infty)$ *to* $[a, 1)$, $0 < a < 1$, *satisfying* $\limsup_{r \to t^+} k(r) < 1$ *for all* $t \geq 0$. *Then T has a fixed point in X provided a real valued function* $f(x) = d(x, T(x))$ *on X is lower semicontinuous.*

In the sequel, ω is a *w*-distance on a metric space *X*. Recently, using the concept of *w*-distance, Suzuki and Takahashi [27] improved Nadler's result (Theorem 2.1) as follows:

Theorem 3.3. *Let* (X, d) *be a complete metric space and let* $T : X \to Cl(X)$ *be a multivalued map. If there exists a constant* $\lambda \in [0, 1)$ *such that for each* $x, y \in X$ *and* $u \in T(x)$ *there is* $v \in T(y)$ *satisfying*

$$\omega(u, v) \leq \lambda\omega(x, y),$$

then T has a fixed point.

Without using the concept of the *w*-distance, recently Latif [15] obtained the following fixed point result which is to some extent related to the Reich's problem and generalizes Theorem 3.3.

Theorem 3.4. *Let* (X, d) *be a complete metric space and let* $T : X \to Cl(X)$ *be a multivalued map such that for any* $x, y \in X$ *and* $u \in T(x)$ *there is* $v \in T(y)$ *satisfying*

$$\omega(u, v) \leq k(\omega(x, y))\omega(x, y),$$

where k is a function from $[0, \infty)$ *to* $[0, b)$ *such that* $\limsup_{r \to t^+} k(r) < b$ *for all* $t \geq 0$. *Then T has a fixed point.*

Most recently, Latif and Abdou [16] obtained the following an improved version of Theorem 3.1, which is also a generalization of Theorem 3.3 and Theorem 3.3 of Latif and Albar [17].

Theorem 3.5. *Let* (X, d) *be a complete metric space and let* $T : X \to Cl(X)$ *be a multivalued map such that for a constant* $b \in (0, 1)$ *and for any* $x \in X$ *there is* $y \in T(x)$ *satisfying*

$$b\omega(x, y) \leq \omega(x, T(x))$$

and

$$\omega(y, T(y)) \le k(\omega(x, y))\omega(x, y),$$

where k is a function from $[0, \infty)$ *to* $[0, b)$ *such that* $\limsup_{r \to t^+} k(r) < b$ *for all* $t \ge 0$. *Suppose that a real valued function* $f(x) = \omega(x, T(x))$ *on X is lower semicontinuous*. *Then there exists* $v_0 \in X$ *such that* $f(v_0) = 0$. *Further*, *if* $\omega(v_0, v_0) = 0$, *then* v_0 *is a fixed point of T*.

Proof. Let x_0 be an arbitrary but fixed element of *X*. Using the definition of *T*, we can get a sequence $\{x_n\}$ in *X* such that for each $n \ge 1$,

$$b\omega(x_n, x_{n+1}) \le \omega(x_n, T(x_n)),$$

$$\omega(x_{n+1}, T(x_{n+1})) \le k(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}), \quad k(\omega(x_n, x_{n+1})) < b.$$

Note that

$$\omega(x_n, T(x_n)) - \omega(x_{n+1}, T(x_{n+1})) \ge b\omega(x_n, x_{n+1}) - k(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1})$$

$$= [b - k(\omega(x_n, x_{n+1}))]\omega(x_n, x_{n+1}) > 0,$$

and thus for all *n*,

$$\omega(x_n, T(x_n)) > \omega(x_{n+1}, T(x_{n+1})), \omega(x_n, x_{n+1}) \le \omega(x_{n-1}, x_n).$$

Note that the sequences $\{\omega(x_n, T(x_n))\}$ and $\{\omega(x_n, x_{n+1})\}$ are decreasing, thus convergent. Now, by the definition of the function *k* there exists $\alpha \in [0, b)$ such that

$$\limsup_{n \to \infty} k(\omega(x_n, x_{n+1})) = \alpha.$$

Thus, for any $b_0 \in (\alpha, b)$, there exists $n_0 \ge 1$ such that

$$k(\omega(x_n, x_{n+1})) < b_0, \quad \text{for all } n > n_0$$

and thus for all $n > n_0$, we have

$$k(\omega(x_n, x_{n+1}) \times \cdots \times k(\omega(x_{n_0+1}, x_{n_0+2})) < b_0^{n-n_0}$$

and

$$\omega(x_n, T(x_n)) - \omega(x_{n+1}, T(x_{n+1})) \ge \beta\omega(x_n, x_{n+1}),$$

where $\beta = b - b_0$. Thus for all $n > n_0$, we get

$$\omega(x_{n+1}, T(x_{n+1})) < \left(\frac{b_0}{b}\right)^{n-n_0} \frac{k(\omega(x_{n_0}, x_{n_0+1})) \times \cdots \times k(\omega(x_1, x_2))\omega(x_1, T(x_1))}{b^{n_0}}.$$

Now, since $b_0 < b$, we have $\lim_{n\to\infty}\left(\frac{b_0}{b}\right)^{n-n_0} = 0$, and hence the decreasing sequence $\{\omega(x_n, T(x_n))\}$ converges to 0. Note that for all $n > n_0$,

$$\omega(x_n, x_{n+1}) < \gamma^n \omega(x_0, x_1), \quad n = 0, 1, 2, \ldots,$$

where $\gamma = \frac{b_0}{b} < 1$. Now, for any natural numbers n, m, $m > n > n_0$,

$$\omega(x_n, x_m) \leq \sum_{j=n}^{m-1} \omega(x_j, x_{j+1}) < \frac{\gamma^n}{1-\gamma}\omega(x_0, x_1),$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Hence we obtained that there exists a Cauchy sequence $\{x_n\}$ in X such that the decreasing sequence $\{g(x_n)\} = \{\omega(x_n, T(x_n))\}$ converges to 0. Due to the completeness of X, there exists some $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_0$. Since g is lower semicontinuous, we have

$$0 \leq g(v_0) \leq \liminf_{n\to\infty} g(x_n) = 0,$$

and thus $g(v_0) = \omega(v_0, T(v_0)) = 0$. Since $\omega(v_0, v_0) = 0$, and $T(v_0)$ is closed, we get $v_0 \in T(v_0)$.

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