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A HYBRID EXTRAGRADIENT METHOD FOR FINDING VARIATIONAL INEQUALITY PROBLEMS, FIXED POINT PROBLEMS AND EQUILIBRIUM PROBLEMS

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Abstract

In this paper, we introduce an iterative scheme by the hybrid methods for finding a common element of the set of fixed points of nonexpansive mappings, the set of solutions of equilibrium problems and the set of solutions of variational inequality problems for a monotone, k-Lipschitz continuous mapping in a Hilbert space. Then we obtain a strongly convergence theorem by using hybrid extragradient method to common elements of the set of fixed points of nonexpansive mappings, the set of solutions of equilibrium problems and the set of solutions of variational inequality problems. Our results extend and improve results of Bnouhachem et al. [2] and many others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C

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be the metric projection of H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if $\| Sx - Sy \| \leq \| x - y \|$, for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find

$$x \in C \text{ such that } F(x, y) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of such solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problem. The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au \rangle \geq 0$ for all $v \in C$. The set of solutions of this variational inequality problem is denoted by $VI(C, A)$. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see [1, 4, 12, 14]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. If C is bounded nonempty closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty (see [7]).

We recall that, a mapping $A : C \to H$ is said to be *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \text{ for all } u, v \in C;$$

A is said to be β-*strongly monotone* if there exists a positive real number β such that

$$\langle Au - Av, u - v \rangle \geq \beta \| u - v \|^2, \text{ for all } u, v \in C;$$

A is said to be *k-Lipschitz continuous* if there exists a positive real number k such that

$$\| Au - Av \| \leq k \| u - v \|, \text{ for all } u, v \in C;$$

A is said to be α-*inverse strongly monotone* [1] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \| Au - Av \|^2, \text{ for all } u, v \in C.$$

Remark 1.1. It is obvious that any α-inverse strongly monotone mapping A is monotone and Lipschitz continuous.

It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the most important topics in the study of the variational inequality problem. The variational inequality has extensively been studied in the literature. See, e.g., [17, 18] and the references therein.

In 1976, Korpelevič [8] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \end{cases} \tag{1.2}$$

for all $n \geq 0$, where $\lambda \in \left(0, \frac{1}{k}\right)$, C is a nonempty closed convex subset of $\mathbb{R}^n$, and A is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.2), converge to the same point $z \in VI(C, A)$.

In 2003, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \bigcap VI(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (3.4) converges weakly to some $z \in F(S) \bigcap VI(A, C)$. Recently, Zeng and Yao [19] proved the following strong convergence theorem:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: (i) $\lambda_n k \subset (0, 1 - \delta)$ for some $\delta \in (0, 1)$ and (ii) $\alpha_n \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$. They proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{F(S) \bigcap VI(C, A)} x_0$ provided that $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$.

In 2008, Takahashi et al. [15] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\ C_{n+1} = \{z \in C_n : \| y_n - z \| \leq \| u_n - z \|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

In 2009, Bnouhachem et al. [2] introduced the following new extragradient iterative method for finding an element of $F(S) \cap VI(A, C)$. Let C be a closed convex subset of a real Hilbert space H, A be an α-inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be given by

$$\begin{cases} x_1, u \in C \text{ chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n)), \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (3.8) converges strongly to a common element of $F(S) \cap VI(A, C)$.

In this paper, motivated and inspired by the results of Bnouhachem et al. [2], we introduce a new iterative scheme by using the hybrid extragradient method, as follows: $x_0 = x \in H$ and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S(\beta_n x_n + (1 - \beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \| z_n - z \| \leq \| x_n - z \|\}, \\ x_{n+1} = P_{C_n + 1} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. We prove $\{x_n\}$ and $\{u_n\}$ in (3.1) that strongly convergence common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for *k*-Lipschitz continuous mappings in a Hilbert space. Our results extend and improve that the corresponding ones announced by Bnouhachem et al. [2].

2. Preliminaries

Let *H* be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot, \cdot\rangle$, and let *C* be a nonempty closed convex subset of *H*. For every point $x \in H$, there exists a unique nearest point in *C*, denoted by $P_C x$, such that

$$\| x - P_C x \| \leq \| x - y \| \quad \text{for all} \quad y \in C.$$

P_C is called the *metric projection* of *H* onto *C*. It is well known that P_C is a nonexpansive mapping of *H* onto *C* and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \| P_C x - P_C y \|^2, \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.2}$$

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2 \tag{2.3}$$

for all $x \in H$, $y \in C$. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0. \tag{2.4}$$

Let *A* be a monotone, *k*-Lipschitz continuous mapping of *C* into *H* and let $N_C v$ be the normal cone to *C* at $v \in C$, i.e.,

$$N_C v := \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \varnothing, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. It is also known that H satisfies the *Opial's condition* [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H satisfies the *Kadec-Klee property* [6], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$ together imply $\| x_n - x \| \to 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.1 [11]**.** *There holds the identity in a Hilbert space H*:

(i) $\| x + y \|^2 = \| x \|^2 + \| y \|^2 + 2\langle x, y\rangle, \forall x, y \in H.$

(ii) $\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0, 1]$.

Lemma 2.2 [1]**.** *Let C be a nonempty closed convex subset of H and let F be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $r > 0$ *and* $x \in H$. *Then there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all} \quad y \in C.$$

The following lemma was also given in [3].

Lemma 2.3 [3]**.** *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows*:

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. *Then the following hold*:

1. T_r *is single-valued*;

2. T_r *is firmly nonexpansive*, *i.e.*, *for any* $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

3. $F(T_r) = EP(F)$;

4. $EP(F)$ *is closed and convex*.

3. Strong Convergence Theorems

In this section, we show a strong convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem for a monotone, *k*-Lipschitz continuous mapping in a Hilbert space by using the hybrid extragradient method.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let A be monotone*, *k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that* $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. *Let* $\{x_n\}$, $\{y_n\}$, $\{w_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S(\beta_n x_n + (1 - \beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \| z_n - z \| \leq \| x_n - z \|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \frac{1}{k}\right)$, $\{\alpha_n\}, \{\beta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C, A)\cap EP(F)} x_0$.

Proof. We first show that $F(S) \bigcap EP(F) \bigcap VI(A, C) \subset C_n$ for all $n \in \mathbb{N} \bigcup \{0\}$, we can prove by induction. It is obvious that $F(S) \bigcap EP(F) \bigcap VI(A, C) \subset C_1$. Let $p \in F(S) \bigcap VI(C, A) \bigcap EP(F)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.3. Then $p = P_C(p - \lambda_n Ap) = T_{r_n} p$ and $u_n = T_{r_n} x_n$. Thus, we have

$$\| u_n - p \| = \| T_{r_n} x_n - T_{r_n} p \| \leq \| x_n - p \|. \tag{3.2}$$

Put $v_n = P_C(u_n - \lambda_n Ay_n)$. From (2.3) and the monotonicity of A, we have

$$\begin{aligned}
\| v_n - p \|^2 &\leq \| u_n - \lambda_n Ay_n - p \|^2 - \| u_n - \lambda_n Ay_n - v_n \|^2 \\
&= \| u_n - p \|^2 - \| u_n - v_n \|^2 + 2\lambda_n \langle Ay_n,\ p - v_n \rangle \\
&= \| u_n - p \|^2 - \| u_n - v_n \|^2 \\
&\quad + 2\lambda_n (\langle Ay_n - Ap,\ p - y_n \rangle + \langle Ap,\ p - y_n \rangle + \langle Ay_n,\ y_n - v_n \rangle) \\
&\leq \| u_n - p \|^2 - \| u_n - v_n \|^2 + 2\lambda_n \langle Ay_n,\ y_n - v_n \rangle \\
&= \| u_n - p \|^2 - \| u_n - y_n \|^2 - 2\langle u_n - y_n,\ y_n - v_n \rangle \\
&\quad - \| y_n - v_n \|^2 + 2\lambda_n \langle Ay_n,\ y_n - v_n \rangle \\
&= \| u_n - p \|^2 - \| u_n - y_n \|^2 - \| y_n - v_n \|^2 \\
&\quad + 2\langle u_n - \lambda_n Ay_n - y_n,\ v_n - y_n \rangle.
\end{aligned}$$

Moreover, from $y_n = P_C(u_n - \lambda_n Au_n)$ and (2.2), we have

$$\langle u_n - \lambda_n Au_n - y_n,\ v_n - y_n \rangle \leq 0. \tag{3.3}$$

Since A is k-Lipschitz continuous, it follows that

$$\begin{aligned}
\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n \rangle &= \langle u_n - \lambda_n Au_n - y_n, v_n - y_n \rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n \rangle \\
&\leq \langle \lambda_n Au_n - \lambda_n Ay_n,\ v_n - y_n \rangle \\
&\leq \lambda_n k \| u_n - y_n \| \| v_n - y_n \|.
\end{aligned}$$

So, we have

$$\begin{aligned}\| v_n - p \|^2 &\leq \| u_n - p \|^2 - \| u_n - y_n \|^2 - \| y_n - v_n \|^2 \\ &\quad + 2\lambda_n k \| u_n - y_n \| \| v_n - y_n \| \\ &\leq \| u_n - p \|^2 - \| u_n - y_n \|^2 - \| y_n - v_n \|^2 \\ &\quad + \lambda_n^2 k^2 \| u_n - y_n \|^2 + \| v_n - y_n \|^2 \\ &= \| u_n - p \|^2 + (\lambda_n^2 k^2 - 1) \| u_n - y_n \|^2 \\ &\leq \| u_n - p \|^2, \end{aligned} \tag{3.4}$$

and hence

$$\| v_n - p \| \leq \| u_n - p \| \leq \| x_n - p \|. \tag{3.5}$$

Setting $w_n = \beta_n x_n + (1 - \beta_n) v_n$. Thus, from (3.5), we have

$$\begin{aligned}\| w_n - p \|^2 &= \| \beta_n x_n + (1 - \beta_n) v_n - p \|^2 \\ &= \| \beta_n (x_n - p) + (1 - \beta_n)(v_n - p) \|^2 \\ &= \beta_n \| x_n - p \|^2 + (1 - \beta_n) \| v_n - p \|^2 - \beta_n (1 - \beta_n) \| x_n - v_n \|^2 \\ &\leq \beta_n \| x_n - p \|^2 + (1 - \beta_n) \| v_n - p \|^2 \\ &\leq \beta_n \| x_n - p \|^2 + (1 - \beta_n) \| x_n - p \|^2 \\ &= \| x_n - p \|^2. \end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}\| z_n - p \|^2 &= \| \alpha_n x_n + (1 - \alpha_n) S w_n - p \|^2 \\ &= \| \alpha_n (x_n - p) + (1 - \alpha_n)(S w_n - p) \|^2 \\ &= \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| S w_n - p \|^2 - \alpha_n (1 - \alpha_n) \| x_n - S w_n \|^2 \\ &\leq \alpha_n \| x_n - p \| + (1 - \alpha_n) \| S w_n - p \|^2 \end{aligned}$$

$$\leq \alpha_n \| x_n - p \| + (1 - \alpha_n) \| w_n - p \|^2$$

$$\leq \alpha_n \| x_n - p \| + (1 - \alpha_n) \| x_n - p \|^2$$

$$= \| x_n - p \|^2. \tag{3.7}$$

So, we have $p \in C_{n+1}$ and hence

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_n, \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.8}$$

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $c_j \in C_{m+1} \subset C_m$ with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\| z_m - c_j \| \leq \| c_j - x_m \|$,

$$\| z_m - z \| = \| z_m - c_j + c_j - z \|$$

$$\leq \| z_m - c_j \| + \| c_j - z \|. \tag{3.9}$$

Taking $j \to \infty$,

$$\| z_m - z \| \leq \| z - x_m \|.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1 - \alpha) y$, where $\alpha \in [0, 1]$. Since C_m is convex, $z \in C_m$ and $\| z_m - x \| \leq \| x - x_m \|$, $\| z_m - y \| \leq \| y - x_m \|$, we have

$$\| z_m - z \|^2 = \| z_m - (\alpha x + (1 - \alpha) y) \|^2$$

$$= \| \alpha(z_m - x) + (1 - \alpha)(z_m - y) \|^2$$

$$= \alpha \| z_m - x \|^2 + (1 - \alpha) \| z_m - y \|^2 - \alpha(1 - \alpha) \| (z_m - x) - (z_m - y) \|^2$$

$$\leq \alpha \| z_m - x \|^2 + (1 - \alpha) \| z_m - y \|^2 - \alpha(1 - \alpha) \| y - x \|^2$$

$$\leq \alpha \| x_m - x \|^2 + (1 - \alpha) \| x_m - y \|^2 - \alpha(1 - \alpha) \| (x_m - x) - (x_m - y) \|^2$$

$$= \| x_m - (\alpha x + (1 - \alpha) y) \|^2$$

$$= \| x_m - z \|^2. \tag{3.10}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. From Lemma 2.2, the sequence $\{u_n\}$ is also well-defined.

Since $F(S) \bigcap VI(C, A) \bigcap EP(F)$ is a nonempty closed convex subset of H, there exists a unique $u \in F(S) \bigcap VI(C, A) \bigcap EP(F)$ such that

$$u = P_{F(S) \bigcap VI(C, A) \bigcap EP(F)} x_0.$$

From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y \rangle \geq 0 \quad \text{for all} \quad y \in C_n.$$

Since $F(S) \bigcap VI(C, A) \bigcap EP(F) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u \rangle \geq 0 \text{ for all } u \in F(S) \bigcap VI(C, A) \bigcap EP(F) \text{ and } n \in \mathbb{N}. \quad (3.11)$$

So, for $u \in F(S) \bigcap VI(C, A) \bigcap EP(F)$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - u \rangle \\ &\leq -\| x_n - x_0 \|^2 + \| x_0 - x_n \| \| x_0 - u \|. \end{aligned} \quad (3.12)$$

This implies that

$$\| x_0 - x_n \|^2 \leq \| x_0 - x_n \| \| x_0 - u \|.$$

Hence

$$\| x_0 - x_n \| \leq \| x_0 - u \| \text{ for all } u \in F(S) \bigcap VI(C, A) \bigcap EP(F) \text{ and } n \in \mathbb{N}. \quad (3.13)$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0 \quad \text{for all} \quad n \in \mathbb{N}. \quad (3.14)$$

So, for $x_{n+1} \in C_n$, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\| x_n - x_0 \|^2 + \| x_0 - x_n \| \| x_0 - x_{n+1} \|. \end{aligned} \tag{3.15}$$

This implies that

$$\| x_0 - x_n \|^2 \leq \| x_0 - x_n \| \| x_0 - x_{n+1} \|.$$

Hence

$$\| x_0 - x_n \| \leq \| x_0 - x_{n+1} \| \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.16}$$

From (3.13), we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \| x_n - x_0 \|$ exists. From (3.5) and (3.6), $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also bounded. Next, we show that $\| x_n - x_{n+1} \| \to 0$. In fact, from (3.14), we have

$$\begin{aligned} &\| x_n - x_{n+1} \|^2 \\ &= \| (x_n - x_0) + (x_0 - x_{n+1}) \|^2 \\ &= \| x_n - x_0 \|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \| x_0 - x_{n+1} \|^2 \\ &= \| x_n - x_0 \|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \| x_0 - x_{n+1} \|^2 \\ &= \| x_n - x_0 \|^2 - 2\langle x_0 - x_n, x_0 - x_n \rangle - 2\langle x_0 - x_n, x_n - x_{n+1} \rangle + \| x_0 - x_{n+1} \|^2 \\ &\leq \| x_n - x_0 \|^2 - 2\| x_n - x_0 \|^2 + \| x_0 - x_{n+1} \|^2 \\ &= -\| x_n - x_0 \|^2 + \| x_0 - x_{n+1} \|^2. \end{aligned} \tag{3.17}$$

Since $\lim_{n\to\infty} \| x_n - x_0 \|$ exists, we have

$$\lim_{n\to\infty} \| x_n - x_{n+1} \| = 0. \tag{3.18}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\| z_n - x_{n+1} \| \leq \| x_n - x_{n+1} \|. \tag{3.19}$$

Hence

$$\lim_{n\to\infty} \| z_n - x_{n+1} \| = 0. \tag{3.20}$$

From $x_{n+1} = P_{C_{n+1}} x_0$, we obtain

$$\| x_{n+1} - x_0 \| \leq \| z - x_0 \| \quad \text{for all} \quad z \in C_{n+1} \quad \text{and for all} \quad n \in \mathbb{N}.$$

Since $u \in F(S) \bigcap VI(C, A) \bigcap EP(F) \subset C_{n+1}$, we have

$$\| x_{n+1} - x_0 \| \leq \| u - x_0 \| \quad \text{for all} \quad n \in \mathbb{N} \bigcup \{0\}. \tag{3.21}$$

Since $x_{n+1} \in C_n$, we have

$$\| x_n - z_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| \leq 2 \| x_n - x_{n+1} \|.$$

By (3.18), we obtain

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{3.22}$$

Since

$$\| x_n - z_n \| = \| x_n - \alpha_n x_n - (1 - \alpha_n) S w_n \| = \| (1 - \alpha_n)(x_n - S w_n) \|,$$

it follows by (3.22) that

$$\lim_{n\to\infty} \| x_n - S w_n \| = 0. \tag{3.23}$$

Let $v \in F(S) \bigcap VI(C, A) \bigcap EP(F)$. Then we obtain

$$\begin{aligned} \| u_n - v \|^2 &= \| T_{r_n} x_n - T_{r_n} v \|^2 \\ &\leq \langle T_{r_n} x_n - T_{r_n} v, x_n - v \rangle \\ &= \langle u_n - v, x_n - v \rangle \\ &= \frac{1}{2} (\| u_n - v \|^2 + \| x_n - v \|^2 - \| x_n - u_n \|^2). \end{aligned}$$

Therefore, $\| u_n - v \|^2 \leq \| x_n - v \|^2 - \| x_n - u_n \|^2$. From (3.6) and (3.5), we can calculate

$$\begin{aligned}
\| z_n - v \|^2 &= \| \alpha_n (x_n - v) + (1 - \alpha_n)(Sw_n - v) \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) \| w_n - v \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) [\beta_n \| x_n - v \|^2 + (1 - \beta_n) \| v_n - v \|^2] \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) \beta_n \| x_n - v \|^2 + (1 - \alpha_n)(1 - \beta_n) \| u_n - v \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) \beta_n \| x_n - v \|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n) [\| x_n - v \|^2 - \| x_n - u_n \|^2] \\
&\leq \| x_n - v \|^2 - (1 - \alpha_n)(1 - \beta_n) \| x_n - u_n \|^2
\end{aligned}$$

and hence

$$\| z_n - v \|^2 \leq \| x_n - v \|^2 - (1 - \alpha_n)(1 - \beta_n) \| u_n - x_n \|^2.$$

Since $0 < c \leq \alpha_n, \; \beta_n \leq d < 1$, it follows that

$$\begin{aligned}
(1 - d)(1 - d) \| x_n - u_n \|^2 &\leq (1 - \alpha_n)(1 - \beta_n) \| x_n - u_n \|^2 \\
&= \| x_n - v \|^2 - \| z_n - v \|^2 \\
&= (\| x_n - v \| + \| z_n - v \|)(\| x_n - v \| - \| z_n - v \|) \\
&\leq \| x_n - z_n \| (\| x_n - v \| + \| w_n - v \|). \qquad (3.24)
\end{aligned}$$

From (3.22) and (3.24), we have

$$\lim_{n \to \infty} \| x_n - u_n \| = 0. \qquad (3.25)$$

Since $\liminf_{n \to \infty} r_n > 0$, we obtain

$$\lim_{n \to \infty} \left\| \frac{x_n - u_n}{r_n} \right\| = \lim_{n \to \infty} \frac{1}{r_n} \| x_n - u_n \| = 0. \qquad (3.26)$$

For $v \in F(S) \bigcap VI(C, A) \bigcap EP(F)$, from (3.7), (3.5) and (3.4), we obtain

$$
\begin{aligned}
\| z_n - v \|^2 &= \| \alpha_n (x_n - v) + (1 - \alpha_n)(Sw_n - v) \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) \| Sw_n - v \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n) \| w_n - v \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n)[\beta_n \| x_n - v \|^2 + (1 - \beta_n) \| v_n - v \|^2] \\
&= \alpha_n \| x_n - v \|^2 + (1 - \alpha_n)\beta_n \| x_n - v \|^2 + (1 - \alpha_n)(1 - \beta_n) \| v_n - v \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n)\beta_n \| x_n - v \|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)[\| u_n - v \|^2 + (\lambda_n^2 k^2 - 1) \| u_n - y_n \|^2] \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n)\beta_n \| x_n - v \|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)[\| x_n - v \|^2 - \| x_n - u_n \|^2] \\
&\quad + (1 - \alpha_n)(1 - \beta_n)(\lambda_n^2 k^2 - 1) \| u_n - y_n \|^2 \\
&\leq \alpha_n \| x_n - v \|^2 + (1 - \alpha_n)\beta_n \| x_n - v \|^2 + (1 - \alpha_n)(1 - \beta_n)[\| x_n - v \|^2] \\
&\quad + (1 - \alpha_n)(1 - \beta_n)(\lambda_n^2 k^2 - 1) \| u_n - y_n \|^2 \\
&= \| x_n - v \|^2 + (1 - \alpha_n)(1 - \beta_n)(\lambda_n^2 k^2 - 1) \| u_n - y_n \|^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\| u_n - y_n \| \\
&\leq \frac{1}{(1 - \alpha_n)(1 - \beta_n)(1 - \lambda_n^2 k^2)} (\| x_n - v \|^2 - \| z_n - v \|^2) \\
&= \frac{1}{(1 - \alpha_n)(1 - \beta_n)(1 - \lambda_n^2 k^2)} (\| x_n - v \| + \| z_n - x^* \|)(\| x_n - v \| - \| z_n - v \|) \\
&\leq \frac{1}{(1 - \alpha_n)(1 - \beta_n)(1 - \lambda_n^2 k^2)} \| x_n - z_n \| (\| x_n - v \| + \| z_n - v \|).
\end{aligned}
$$

So, by (3.22), we obtain

$$\lim_{n\to\infty} \| u_n - y_n \| = 0. \tag{3.27}$$

Since $\| x_n - y_n \| \le \| x_n - u_n \| + \| u_n - y_n \|$, from (3.25) and (3.27), we also have

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{3.28}$$

We note that

$$\begin{aligned}
\| y_n - v_n \| &= \| P_C(u_n - \lambda_n A u_n) - P_C(u_n - \lambda_n A y_n) \| \\
&\le \| (u_n - \lambda_n A u_n) - (u_n - \lambda_n A y_n) \| \\
&= \| \lambda_n (A y_n - A u_n) \| \\
&\le \lambda_n k \| y_n - u_n \|.
\end{aligned}$$

Since $\lambda_n \in \left(0, \frac{1}{k}\right)$ and from (3.27), we obtain

$$\lim_{n\to\infty} \| v_n - y_n \| = 0. \tag{3.29}$$

Since

$$\begin{aligned}
\| w_n - x_n \| &= \| \beta_n x_n + (1 - \beta_n) v_n - x_n \| \\
&= (1 - \beta_n) \| v_n - x_n \| \\
&\le (1 - \beta_n) [\| v_n - y_n \| + \| y_n - x_n \|].
\end{aligned}$$

From (3.28) and (3.29), we have

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{3.30}$$

Note that

$$\begin{aligned}
&\| S v_n - v_n \| \\
&\le \| S v_n - S w_n \| + \| S w_n - x_n \| + \| x_n - y_n \| + \| y_n - v_n \| \\
&\le \| v_n - w_n \| + \| S w_n - x_n \| + \| x_n - y_n \| + \| y_n - v_n \| \\
&\le \| v_n - y_n \| + \| y_n - x_n \| + \| x_n - w_n \| + \| S w_n - x_n \| + \| x_n - y_n \| + \| y_n - v_n \| \\
&= 2 \| v_n - y_n \| + 2 \| y_n - x_n \| + \| x_n - w_n \| + \| S w_n - x_n \|.
\end{aligned}$$

From (3.23), (3.28), (3.29) and (3.30), we obtain

$$\lim_{n\to\infty} \| Sv_n - v_n \| = 0. \tag{3.31}$$

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$. Since $v_{n_i} \subset C$ and C is closed and convex, C is weakly closed and hence $z \in C$. From $\| Sv_n - v_n \| \to 0$, we obtain $Sv_{n_i} \rightharpoonup z$. We show that $z \in EP(F)$. Using the same argument as in the proof in [10, Theorem 3.1, p. 273] or [9, Theorem 3, p. 1251], we can prove that $z \in EP(F)$. From Opial's condition, we obtain that $z \in F(S)$. Finally, we can show that $z \in VI(C, A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \varnothing, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, u - Av \rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Ay_n)$, we have $\langle v - v_n, v_n - (u_n - \lambda_n Ay_n) \rangle \geq 0$, and hence $\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + Ay_n \right\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, u \rangle &\geq \langle v - v_{n_i}, Av \rangle \\
&\geq \langle v - v_{n_i}, Av \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\
&= \left\langle v - v_{n_i}, Av - Ay_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i} \rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty}\| v_n - u_n \| = \lim_{n\to\infty}\| v_n - y_n \| = 0,$ $u_{n_i} \rightharpoonup p$ and A is Lipschitz continuous, we obtain that $\lim_{n\to\infty}\| Av_n - Ay_n \| = 0$ and $v_{n_i} \rightharpoonup p.$ From $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty}\| v_n - u_n \| = 0,$ we obtain

$$\langle v - z, u\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Hence, we have $z \in F(S) \bigcap VI(C, A) \bigcap EP(F)$. Finally, we show that $x_n \to z,$ where $z = P_{F(S)\bigcap VI(C, A) EP(F)}x_0$. Since $x_n = P_{C_n}x_0$ and $z \in F(S) \bigcap VI(C, A) \bigcap EP(F) \subset C_n,$ we have $\| x_n - x_0 \| \leq \| z - x_0 \|$. It follows from $z' = P_{F(S)\bigcap VI(C, A)\bigcap EP(F)}x_0$ and the lower semicontinuity of the norm that

$$\begin{aligned}\| z' - x_0 \| &\leq \| z - x_0 \| \\ &\leq \liminf_{i\to\infty}\| x_{n_i} - x_0 \| \\ &\leq \limsup_{i\to\infty}\| x_{n_i} - x_0 \| \\ &\leq \| z' - x_0 \|.\end{aligned}$$

Thus, we obtain that

$$\lim_{k\to\infty}\| x_{n_i} - x_0 \| = \| z - x_0 \| = \| z' - x_0 \|.$$

Using the Kadec-Klee property of H, we obtain that

$$\lim_{i\to\infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\},$ we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{F(T)\bigcap VI(C, A)\bigcap EP(F)}x_0.$ □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone, k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping from C into itself such that $F(S) \bigcap VI(C, A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ be sequences generated by $x_0 \in C$ and let*

$$\begin{cases} y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S(\beta_n x_n + (1-\beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \| z_n - z \| \leq \| x_n - z \|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N} \end{cases} \tag{3.32}$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \frac{1}{k}\right)$, $\{\alpha_n\}, \{\beta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(S) \cap VI(C, A)} x_0$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ in Theorem 3.1. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Corollary 3.2 converges strongly to $P_{F(S) \cap VI(C, A)} x_0$. □

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