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SOME WHEEL RELATED 3-EQUITABLE GRAPHS IN THE CONTEXT OF VERTEX DUPLICATION

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Abstract

In the present investigations, we prove that the graph obtained by duplication of arbitrary rim vertex of wheel W_n and duplication of apex vertex of wheel W_n for even n is 3-equitable and not 3-equitable for odd n, where $n \geq 5$. In addition to this we prove that duplication of vertices of wheel W_n altogether is 3-equitable except $n = 5$.

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1. Introduction

We begin with simple, finite and undirected graph $G = (V, E)$. In the present work, $W_n = C_n + K_1$ $(n \geq 3)$ denotes the wheel. In W_n vertices corresponding to C_n are called *rim vertices* and vertex corresponding to K_1 is called the *apex vertex*. Here $N(v)$ denotes the set of all neighboring vertices of *v*. For all other terminology and notations we follow Harary [5]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. Duplication of a vertex v_k of graph *G* produces a new graph G_1 by adding a vertex v'_k with $N(v'_k) = N(v_k)$.

In other words, a vertex v'_k is said to be *duplication* of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.

Definition 1.2. If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Most interesting graph labeling problems have three important ingredients as follows:

(1) A set of numbers from which the vertex labels are chosen.

(2) A rule that assigns a value to each edge.

(3) A condition that these values must satisfy.

Labeled graph has variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detail study on applications of graph labeling is reported in Bloom and Golomb [2].

For extensive survey on graph labeling one can refer Gallian [4]. Vast amount of literature is available on different types of graph labeling and good number of research papers has been published so far in past three decades. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs.

There are three types of problems that can be considered in this area.

(1) How 3-equitability is affected under various graph operations?

(2) Construct new families of 3-equitable graph by finding suitable labeling.

(3) Given a graph theoretic property P, characterize the class of graphs with property P that are 3-equitable.

This work is aimed to discuss the problems of the first kind.

Definition 1.3. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1, 2\}$ is called *ternary vertex labeling* of *G* and $f(v)$ is called the *label* of the vertex *v* of *G* under *f*.

For an edge $e = uv,$ the induced edge labeling $f^* : E(G) \to \{0, 1, 2\}$ is given by $f^*(e) = | f(u) - f(v)|.$ Let $v_f(0),$ $v_f(1)$ and $v_f(2)$ be the number of vertices of *G* having labels 0, 1 and 2, respectively, under *f* and let $e_f(0),$ $e_f(1)$ and $e_f(2)$ be the number of edges having labels 0, 1 and 2, respectively, under $f^*.$

Definition 1.4. A ternary vertex labeling of a graph *G* is called a 3-*equitable labeling* if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $0 \leq i, j \leq 2.$ A graph *G* is 3-*equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [3]. Many researchers have studied 3-equitability of graphs, e.g., Cahit [3] proved that C_n is 3-equitable except $n \equiv 3 \pmod 6.$ In the same paper he proved that an Eulerian graph with number of edges congruent to $3 \pmod 6$ is not 3-equitable. Youssef [6] proved that W_n is 3-equitable for all $n \geq 4.$

In the present work, we prove that duplication of arbitrary rim vertex of wheel W_n $(n \geq 5)$ and duplication of apex vertex of wheel W_n for even n $(n \geq 5)$ is 3-equitable and not 3-equitable for odd n $(n \geq 5).$ In addition to this we also prove that duplication of vertices of wheel W_n altogether is 3-equitable except for $n = 5.$

2. Main Results

Theorem 2.1. *The graph obtained by duplication of arbitrary rim vertex of wheel* W_n *is* 3-*equitable for* $n \geq 5$ *while duplication of apex vertex is* 3-*equitable for even n and not* 3-*equitable for odd n,* $n \geq 5.$

Proof. Consider the wheel W_n. Let $v_1, v_2, ..., v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating either rim vertex or apex vertex of W_n. Let v'_k be the duplicated vertex of v_k and c'_1 be the duplicated vertex of c_1. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$, we consider the following cases.

Case A. Duplication of arbitrary rim vertex v_k, where $k \in N$, $1 \le k \le n$.

Subcase 1. $n \equiv 0, 1 \pmod 6$.

In this case, we define labeling function f as

$$
\begin{aligned}
f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 2, 3 \pmod 6.\\
&= 2; \text{ if } i \equiv 0, 5 \pmod 6), 1 \le i \le n-k+1.\\
f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 2, 3 \pmod 6.\\
&= 2; \text{ if } i \equiv 0, 5 \pmod 6), n-k+2 \le i \le n.\\
f(v'_k) &= 2; \text{ if } n \equiv 0 \pmod 6.\\
f(v'_k) &= 1; \text{ if } n \equiv 1 \pmod 6.\\
f(c_1) &= 0; \text{ if } n \equiv 0 \pmod 6.\\
f(c_1) &= 2; \text{ if } n \equiv 1 \pmod 6.
\end{aligned}
$$

Subcase 2. $n \equiv 2, 5 \pmod 6$.

In this case, we define labeling function f as

$$
\begin{aligned}
f(v_{k+i-1}) &= 0; \text{ if } i \equiv 0, 3 \pmod 6.\\
&= 1; \text{ if } i \equiv 4, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 1, 2 \pmod 6), 1 \le i \le n-k+1.\\
f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 0, 3 \pmod 6.\\
&= 1; \text{ if } i \equiv 4, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 1, 2 \pmod 6), n-k+2 \le i \le n.
\end{aligned}
$$

$$f(v'_k) = 1; \text{ if } n \equiv 2 \pmod 6.$$

$$f(v'_k) = 2; \text{ if } n \equiv 5 \pmod 6.$$

$$f(c_1) = 0.$$

Subcase 3. $n \equiv 3, 4 \pmod 6$.

In this case, we define labeling function f as:

Subcase 3.1. If $k \leq 2$,

$$\begin{aligned} f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \leq i \leq n-2. \end{aligned}$$

$$f(v_{n-1}) = 1;$$

$$f(v_n) = 2; \text{ if } k = 1.$$

$$f(v_1) = 2;$$

$$f(v_n) = 1; \text{ if } k = 2.$$

$$f(v'_k) = 2;$$

$$f(c_1) = 0.$$

Subcase 3.2. If $k \geq 3$,

$$\begin{aligned} f(v_{k+i-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \leq i \leq n-k+1. \end{aligned}$$

$$\begin{aligned} f(v_{k+i-n-1}) &= 0; \text{ if } i \equiv 1, 4 \pmod 6. \\ &= 1; \text{ if } i \equiv 0, 5 \pmod 6. \\ &= 2; \text{ if } i \equiv 2, 3 \pmod 6, n-k+2 \leq i \leq n-2. \end{aligned}$$

$$f(v_{k-1}) = 1;$$

$$f(v_k) = f(v'_k) = 2;$$

$$f(c_1) = 0.$$

Case B. Duplication of apex vertex c_1.

Subcase 1. $n \equiv 0 \pmod 6$.

In this case, we define labeling f as:

$$
\begin{aligned}
f(v_i) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 0, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n.\\
f(c_1) &= 0;\\
f(c_1') &= 2.
\end{aligned}
$$

Subcase 2. $n \equiv 2 \pmod 6$.

In this case, we define labeling f as:

$$
\begin{aligned}
f(v_i) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 0, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n-2.\\
f(v_{n-1}) &= 1;\\
f(v_n) &= 0;\\
f(c_1) &= f(c_1') = 2.
\end{aligned}
$$

Subcase 3. $n \equiv 4 \pmod 6$.

In this case, we define labeling f as:

$$
\begin{aligned}
f(v_i) &= 0; \text{ if } i \equiv 1, 4 \pmod 6.\\
&= 1; \text{ if } i \equiv 0, 5 \pmod 6.\\
&= 2; \text{ if } i \equiv 2, 3 \pmod 6, 1 \le i \le n-4.\\
f(v_{n-3}) &= f(v_{n-2}) = f(v_{n-1}) = 1.\\
f(v_n) &= f(c_1) = 0.\\
f(c_1') &= 2.
\end{aligned}
$$

Subcase 4. $n \equiv 1 \pmod 6$.

To satisfy the vertex condition it is essential to label $\frac{n+2}{3}$ vertices with 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n+2}{3}$ vertices with label 1 and all the rim vertices are of degree 4 implies that there are at least $3\left(\frac{n+2}{3} - 3\right) + 8 = n + 1$ edges with label 1. As the number of edges in $G = 3n$ and in order to satisfy the edge conditions number of edges with label 1 must be exactly n. Thus edge condition is violated and G is not 3-equitable.

Subcase 5. $n \equiv 3 \pmod 6$.

To satisfy vertex condition it is essential to label $\frac{n}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n}{3}$ vertices with label one and all the rim vertices are of degree 4, it has either $3\left(\frac{n}{3} - 3\right) + 8,$ i.e., $n - 1$ or $3\left(\frac{n}{3} - 1\right) + 4,$ i.e., $n + 1$ edges with label one. As G contains $3n$ edges so number of edges with label one should be exactly n. Thus edge condition is not satisfied. Hence G is not 3-equitable.

Subcase 6. $n \equiv 5 \pmod 6$.

To satisfy vertex condition it is essential to label $\frac{n+1}{3}$ vertices with label 1. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. As G has $\frac{n+1}{3}$ vertices with label one and all the rim vertices are of degree 4, it has either $3\left(\frac{n+1}{3} - 4\right) + 10,$ i.e., $n - 1$ or $3\left(\frac{n+1}{3}\right),$ i.e., $n + 1$ edges with label one. As G contains $3n$ edges so number of edges with label one should be exactly n. Thus edge condition is not satisfied. Hence G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in Table 1 and Table 2, i.e., G admits 3-equitable labeling.

Case A. Let $n = 6a + b$ and $k \in N,\ 1 \le k \le n,\ \ a \in N \cup \{0\}$.

Table 1

b	**Vertex Condition**	**Edge Condition**
0, 3	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
1, 4	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2) + 1$
2, 5	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) + 1 = e_f(1) = e_f(2) + 1$

Case B. Let $n = 6a + b,\ \ a \in N \cup \{0\}$.

Table 2

b	**Vertex Condition**	**Edge Condition**
0	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
2	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
4	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$

Remark 2.2 (For the duplication of rim vertex)**.**

◊ For $n = 3$, $|V(G)| = 5$ and $|E(G)| = 9$. In order to satisfy the vertex conditions it is essential to label two vertices with the same labels, other two vertices with the same labels but with the label different than the label which is used earlier. The label which is spared after the labeling of above referred two pairs of vertices will be the label of the remaining one. For example, if we label two vertices with 0, two vertices with 1, then the remaining vertex will receive the label 2. Such labeling will give rise to exactly two edges with label 0. On the other hand, in order to satisfy the edge conditions at least four edges with label 1 are needed. Thus G fails to satisfy the edge condition to be the 3-equitable graph.

◊ For $n = 4$, as $|V(G)| = 6$ it is essential to label two vertices with label 1 to satisfy the vertex conditions. This constraint will give rise to at least five edges with label 1 because G contains the vertices with degrees 3 and 4. On the other hand, in order to satisfy the edge conditions the number of edges with label 1 should be at most four as $|E(G)| = 11$. Thus G fails to satisfy edge conditions to be the 3-equitable graph.

Remark 2.3 (For the duplication of an apex vertex)**.** For $n = 4$, in order to satisfy the vertex conditions it is essential to label exactly two vertices with label 1 as $|V(G)| = 6$. This constraint will give rise to at least six edges with label 1 as G contains vertices with degree four. On the other hand, in order to satisfy edge conditions it is essential to have exactly four edges with label 1. Thus edge conditions for 3-equitable graph is violated.

For better understanding of the above Theorem 2.1 let us consider few examples:

Illustrations 2.4.

Example 1. Consider a graph obtained by duplicating the vertex v_2 of W_5. This is the example related to Subcase 2 of Case A. The 3-equitable labeling is shown in Figure 1.

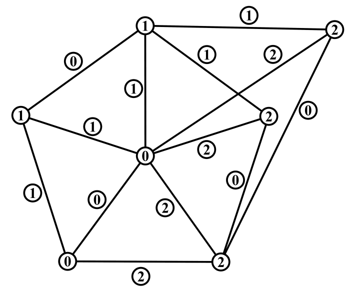

Figure 1

Example 2. Consider a graph obtained by duplicating apex vertex c_1 of W_6. This is the example related to Subcase 1 of Case B. The 3-equitable labeling is shown in Figure 2.

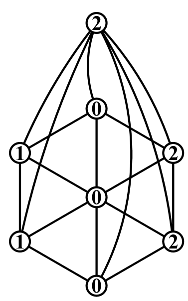

Figure 2

Theorem 2.5. *Duplication of the vertices of wheel* W_n *altogether produces a* 3-*equitable graph except for* $n = 5$, *where* $n \in N$.

Proof. Consider the wheel $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of W_n, c_1 be the apex vertex of W_n and G be the graph obtained by duplicating vertices altogether. Moreover, $v_1', v_2', \ldots, v_n'$ be the duplicated vertices of $v_1, v_2, \ldots, v_n$ respectively and c_1' be the duplicated vertex of c_1. To define vertex labeling $f: V(G) \to \{0, 1, 2\}$, we consider the following cases.

Case 1. $n \equiv 0 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0; \; i \equiv 1, 4 \pmod 6.$$
$$= 1; \; i \equiv 0, 5 \pmod 6.$$
$$= 2; \; i \equiv 2, 3 \pmod 6 \text{ for all } i, \; 1 \le i \le n.$$
$$f(v_i') = 0; \; i \equiv 1, 4 \pmod 6.$$
$$= 1; \; i \equiv 0, 5 \pmod 6.$$
$$= 2; \; i \equiv 2, 3 \pmod 6 \text{ for all } i, \; 1 \le i \le n.$$
$$f(c_1) = 0;$$
$$f(c_1') = 2.$$

Case 2. $n \equiv 1 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0; \; i \equiv 1, 4 \pmod 6.$$
$$= 1; \; i \equiv 0, 5 \pmod 6.$$
$$= 2; \; i \equiv 2, 3 \pmod 6 \text{ for all } i, \; 1 \le i \le n-1.$$
$$f(v_n) = 1;$$
$$f(v_i') = 0; \; i \equiv 1, 4 \pmod 6.$$
$$= 1; \; i \equiv 0, 5 \pmod 6.$$
$$= 2; \; i \equiv 2, 3 \pmod 6 \text{ for all } i, \; 1 \le i \le n-1.$$
$$f(v_n') = f(c_1') = 2;$$
$$f(c_1) = 0.$$

Case 3. $n \equiv 2 \pmod 6$.

In this case, we define labeling f as:

$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n-2.$$
$$f(v_{n-1}) = f(v_n) = 0;$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 0, 5 \pmod 6.$$
$$= 2;\ i \equiv 2, 3 \pmod 6 \text{ for all } i,\ 1 \le i \le n-2.$$
$$f(v_{n-1}') = f(v_n') = 1;$$
$$f(c_1) = f(c_1') = 2.$$

Case 4. $n \equiv 3 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = f(v_2) = 2;$$
$$f(v_3) = 0;$$
$$f(v_i) = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 2, 3 \pmod 6.$$
$$= 2;\ i \equiv 0, 5 \pmod 6,\ 4 \le i \le n.$$
$$f(v_1') = 0;$$
$$f(v_2') = f(v_3') = 1;$$
$$f(v_i') = 0;\ i \equiv 1, 4 \pmod 6.$$
$$= 1;\ i \equiv 2, 3 \pmod 6.$$
$$= 2;\ i \equiv 0, 5 \pmod 6,\ 4 \le i \le n.$$
$$f(c_1) = 2;$$
$$f(c_1') = 0, \text{ if } n \ne 3.$$
$$f(c_1) = 0;$$
$$f(c_1') = 2, \text{ if } n = 3.$$

Case 5. $n \equiv 4 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = 0;$$

$$f(v_2) = f(v_4) = 2;$$

$$f(v_3) = 1;$$

$$\begin{aligned} f(v_i) &= 0;\ i \equiv 2,\ 5 \pmod 6. \\ &= 1;\ i \equiv 3,\ 4 \pmod 6. \\ &= 2;\ i \equiv 0,\ 1 \pmod 6,\ 5 \le i \le n. \end{aligned}$$

$$f(v_1') = 0;$$

$$f(v_2') = f(v_4') = 1;$$

$$f(v_3') = 2;$$

$$\begin{aligned} f(v_i') &= 0;\ i \equiv 2,\ 5 \pmod 6. \\ &= 1;\ i \equiv 3,\ 4 \pmod 6. \\ &= 2;\ i \equiv 0,\ 1 \pmod 6,\ 5 \le i \le n. \end{aligned}$$

$$f(c_1) = 0;$$

$$f(c_1') = 2.$$

Case 6. $n \equiv 5 \pmod 6$.

In this case, we define labeling f as:

$$f(v_1) = f(v_4) = 0;$$

$$f(v_2) = f(v_3) = 1;$$

$$f(v_5) = 2;$$

$$\begin{aligned} f(v_i) &= 0;\ i \equiv 0,\ 3 \pmod 6. \\ &= 1;\ i \equiv 4,\ 5 \pmod 6. \\ &= 2;\ i \equiv 1,\ 2 \pmod 6,\ 6 \le i \le n. \end{aligned}$$

$$f(v_1') = f(v_4') = 1;$$

$$f(v_2') = f(v_3') = 2;$$

$$f(v_5') = 0;$$

$$\begin{aligned} f(v_i') &= 0;\ i \equiv 0,\ 3\ (\bmod 6). \\ &= 1;\ i \equiv 4,\ 5\ (\bmod 6). \\ &= 2;\ i \equiv 1,\ 2\ (\bmod 6),\ 6 \le i \le n. \end{aligned}$$

$$f(c_1) = 0;$$

$$f(c_1') = 2.$$

Case 7. $n = 5$.

W_5 contains 12 vertices. In order to satisfy vertex condition 4 vertices must be labeled one. It is obvious that any edge will have label 1 if it is incident to the vertex with label 1. All the rim vertices are of degree 6 and duplicated vertices are of degree 3. Assign label one to v_1, v_n', v_1' and v_2'. It results minimum 11 edges with label one. As number of edges in W_5 is 30, edge condition is not satisfied. Therefore, for $n = 5$ graph G is not 3-equitable.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ as shown in Table 3, i.e., G admits 3-equitable labeling.

Let $n = 4a + b$ and $a \in N \cup \{0\}$.

Table 3

b	**Vertex Condition**	**Edge Condition**
0, 3	$v_f(0) = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
1	$v_f(0) = v_f(1) + 1 = v_f(2) + 1$	$e_f(0) = e_f(1) = e_f(2)$
2, 5	$v_f(0) = v_f(1) = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$
4	$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$	$e_f(0) = e_f(1) = e_f(2)$

For better understanding of above defined labeling pattern let us consider following illustration:

Illustration 2.6. Consider a graph obtained by duplicating vertices of wheel W_4 altogether. This is example of Case 5. The 3-equitable labeling is shown in Figure 3.

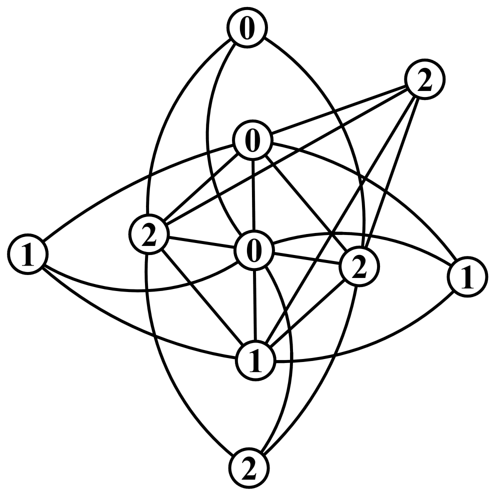

Figure 3

3. Concluding Remarks

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here 3-equitable labeling for duplication of vertices which is one of the graph operations. This approach is novel and contributes two new graphs to the theory of 3-equitable graphs. The derived results are demonstrated by means of sufficient illustrations which provides better understanding. The results reported here are new and will add new dimension to the theory of 3-equitable graphs.

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