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ON THE ETA-PRODUCT OF THE ELLIPTIC ROOT SYSTEM $A_{46}^{(1,1)}$

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Abstract

We calculate the Dirichlet series of the Fourier expansion of the elliptic eta-product $\eta_{A_{46}^{(1,1)}}(\tau) = \eta(\tau)^{48}$ of type $A_{46}^{(1,1)}$.

1. Introduction

In 1985, Saito [3] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belonging to a positive semi-definite quadratic form I whose radical has rank two. Therefore, 2-extended affine root systems are also called *elliptic root systems*. In the cases of 1-codimensional elliptic root systems, Saito [4] described elliptic eta-products and their Fourier coefficients at ∞. In the previous papers [5-18], we examined the elliptic eta-product of type $A_l^{(1,1)}$ $(l \geq 1)$, and more concretely the cases of types $A_{10}^{(1,1)}$, $A_{20}^{(1,1)}$, $A_{24}^{(1,1)}$, $A_{26}^{(1,1)}$, $A_{28}^{(1,1)}$, $A_{30}^{(1,1)}$, $A_{32}^{(1,1)}$, $A_{34}^{(1,1)}$, $A_{36}^{(1,1)}$, $A_{38}^{(1,1)}$, $A_{40}^{(1,1)}$, $A_{42}^{(1,1)}$ and $A_{44}^{(1,1)}$. In this paper,

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we calculate the Dirichlet series of $\eta(\tau)^{48}$ of type $A_{46}^{(1,1)}$ according to the theory of Hecke operators due to van Lint [19], Rankin [2] and Rangachari [1].

2. Elliptic Eta-product of Type $A_{46}^{(1,1)}$

Dedekind's η-function, defined by the infinite product $\eta(\tau) := q^{1/24} \times \prod_{n=1}^{\infty}(1-q^n)$, $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ (= the upper half of the complex plane) is a modular form of weight $\frac{1}{2}$. The elliptic eta-product of type $A_{46}^{(1,1)}$ is given by [5]; $\eta_{A_{46}^{(1,1)}}(\tau) = \eta(\tau)^{48}$, which is a cusp form of weight $k = 24$ and level $N = 1$. Therefore, $\eta(\tau)^{48} \in S_{24}(\Gamma_0(1))$, and the space $S_{24}(\Gamma_0(1))$ is 2-dimensional (see [1]). From the result of [1], $F(\tau) = E_6^2\eta^{24} + \alpha\eta^{48} = \sum_{n=1}^{\infty} r(n)q^n$, is a normalized eigenfunction of the Hecke operators T_n for some value of α. Here E_6 is Eisenstein series given by

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

We have

$$\eta^{48} = \sum_{n=1}^{\infty} a(n)q^{\frac{n}{24}} = \sum_{n=1}^{\infty} a(24+24n)q^{\frac{24+24n}{24}}$$

$$= q^2 - 48q^3 + 1080q^4 - 15040q^5 + 143820q^6 + \cdots,$$

$$E_6^2\eta^{24} = \sum_{n=1}^{\infty} b(n)q^{\frac{n}{24}} = \sum_{n=1}^{\infty} b(24n)q^{\frac{24n}{24}}$$

$$= q - 1032q^2 + 254196q^3 + 10965568q^4 + 60177390q^5 + \cdots.$$

We recall the result [19]. For Hecke operator T_p, we set $T_p = (-1)^{\frac{k(p-1)}{2}} \times p^{-1}T(p)$, then for $f(\tau) = \sum_{n=0}^{\infty} a(n)q^{\frac{n}{24}}$, we have

$$f(\tau)|T(p) = \sum_{n=0}^{\infty}\left\{p^k a\left(\frac{n}{p}\right) + (-1)^{\frac{k(p-1)}{2}} pa(np)\right\} q^{\frac{n}{24}},$$

$(a(x) = 0$ if x is not an integer). From this we obtain the following.

Lemma 2.1.

$$\eta^{48}|T(p) = \frac{p^{24} + pa(48p^2) - pa(24p)b(48p)}{a(48p)}\eta^{48}$$

$$+ pa(24p)E_6^2\eta^{24} \quad (p \equiv 1, 5 \pmod 6),$$

$$E_6^2\eta^{24}|T(p) = \frac{p^{24} + pb(24p^2) - p(b(24p))^2}{a(24p)}\eta^{48}$$

$$+ pb(24p)E_6^2\eta^{24} \quad (p \equiv 1, 5 \pmod 6).$$

Proof. It is easily proved from the formula for $T(p)$ and the expressions of η^{48} and $E_6^2\eta^{24}$. ∎

From the fact that $E_6^2\eta^{24} + \alpha\eta^{48}$ is an eigenfunction for Hecke operators and Lemma 2.1, for $p \equiv 1, 5 \pmod 6$, we see that $\alpha^2 - 3144\alpha - 18289152 = 0$, i.e., $\alpha = -1572 \pm 12\sqrt{144169}$, and its eigenvalue is $pb(24p) + \alpha pa(24p)$. We set $E_6^2\eta^{24} + (-1572 \pm 12\sqrt{144169})\eta^{48} = \sum A_{\pm}(n)q^n$. Then we obtain the following.

Proposition 2.2. *We have* $24\sqrt{144169}\,\eta^{48}(\tau) = \sum(A_+(n) - A_-(n))q^n :=$ $\sum e(n)q^n$ *and its Dirichlet series is given as follows*:

$$\sum e(n)\cdot n^{-s} = \sum(A_+(n) - A_-(n))n^{-s},$$

where

$$\sum A_{\pm}(n)\cdot n^{-s}$$

$$= \prod_{p\equiv 1, 5 \pmod 6}(1 - (b(24p) + (-1572 \pm 12\sqrt{144169})a(24p))\,p^{-s} + p^{23-2s})^{-1}$$

(*where p is prime number*).

Proof. It is easily proved from the following result [19]. If $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{24}}$ is eigenfunction for $T(p)$ with eigenvalue c, then Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ is given by

$$\varphi(s) = \left(\sum_{n \not\equiv 0 \pmod p} a(n) n^{-s} \right) \left(1 - (-1)^{\frac{k(p-1)}{2}} c p^{-s-1} + (-1)^{\frac{k(p-1)}{2}} p^{k-1-2s}\right)^{-1}. \qquad \square$$

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