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# ON THE ETA-PRODUCT OF THE ELLIPTIC ROOT SYSTEM $A_{40}^{(1,1)}$

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**Abstract**

We obtain the Dirichlet series of the Fourier expansion of the elliptic eta-product $\eta_{A_{40}^{(1,1)}}(4\tau) = \eta(4\tau)^{42}$ of type $A_{40}^{(1,1)}$.

## 1. Introduction

In 1985, Saito [3] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belonging to a positive semi-definite quadratic form $I$ whose radical has rank two. Therefore, 2-extended affine root systems are also called *elliptic root systems*. In the cases of 1-codimensional elliptic root systems, Saito [4] described elliptic eta-products and their Fourier coefficients at $\infty$. In the previous papers [5-15], we examined the elliptic eta-product of type $A_l^{(1,1)}$ $(l \geq 1)$, and more concretely the cases of types $A_{10}^{(1,1)}$, $A_{20}^{(1,1)}$, $A_{24}^{(1,1)}$, $A_{26}^{(1,1)}$, $A_{28}^{(1,1)}$, $A_{30}^{(1,1)}$, $A_{32}^{(1,1)}$, $A_{34}^{(1,1)}$, $A_{38}^{(1,1)}$ and $A_{42}^{(1,1)}$. In this paper, we obtain the

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Dirichlet series of $\eta(4\tau)^{42}$ of type $A_{40}^{(1,1)}$ according to the theory of Hecke operators due to van Lint [16], Rankin [2] and Rangachari [1].

## 2. Elliptic Eta-product of Type $A_{40}^{(1,1)}$

Dedekind's $\eta$-function, defined by the infinite product $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$, $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$ (= the upper half of the complex plane) is a modular form of weight $\frac{1}{2}$. The elliptic eta-product of type $A_{40}^{(1,1)}$ is given by [5]; $\eta_{A_{40}^{(1,1)}}(4\tau)$ $= \eta(4\tau)^{42}$, which is a cusp form of weight $k = 21$ and level $N = 16$. Therefore, $\eta(4\tau)^{42} \in S_{21}(\Gamma_0(16), \epsilon)$, and the space $S_{21}(\Gamma_0(16), \epsilon)$ is 4-dimensional (see [1]). From the result of [1],

$$F(\tau) = E_6^3\eta^6 + \alpha E_6^2\eta^{18} + \beta E_6\eta^{30} + \gamma\eta^{42} = \sum_{n=1}^{\infty} r(n) q^{\frac{n}{4}}$$

is a normalized eigenfunction of the Hecke operators $T_n$ for some values $\alpha$, $\beta$ and $\gamma$. Here $E_6$ is Eisenstein series given by $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}$. We have

$$\begin{aligned} \eta^{42} &= \sum_{n=1}^{\infty} a(n) q^{\frac{n}{24}} \\ &= \sum_{n=1}^{\infty} a(18 + 24n) q^{\frac{18+24n}{24}} \\ &= q^{\frac{7}{4}} - 42q^{\frac{11}{4}} + 819q^{\frac{15}{4}} - 9758q^{\frac{19}{4}} + 78351q^{\frac{23}{4}} \\ &\quad - 437346q^{\frac{27}{4}} + 1650824q^{\frac{31}{4}} + \cdots, \end{aligned}$$

$$E_6\eta^{30} = \sum_{n=1}^{\infty} b(n) q^{\frac{n}{24}}$$

$$= \sum_{n=1}^{\infty} b(6+24n) q^{\frac{6+24n}{24}}$$

$$= q^{\frac{5}{4}} - 534q^{\frac{9}{4}} - 1107q^{\frac{13}{4}} + 168674q^{\frac{17}{4}} - 1955988q^{\frac{21}{4}}$$

$$+ 9719460q^{\frac{25}{4}} - 17972719q^{\frac{29}{4}} + \cdots,$$

$$E_6^2\eta^{18} = \sum_{n=1}^{\infty} c(n) q^{\frac{n}{24}}$$

$$= \sum_{n=1}^{\infty} c(-6+24n) q^{\frac{-6+24n}{24}}$$

$$= q^{\frac{3}{4}} - 1026q^{\frac{7}{4}} + 239031q^{\frac{11}{4}} + 12408978q^{\frac{15}{4}}$$

$$+ 132490269q^{\frac{19}{4}} - 450081846q^{\frac{23}{4}} + \cdots,$$

$$E_6^3\eta^{6} = \sum_{n=1}^{\infty} d(n) q^{\frac{n}{24}}$$

$$= \sum_{n=1}^{\infty} d(-18+24n) q^{\frac{-18+24n}{24}}$$

$$= q^{\frac{1}{4}} - 1518q^{\frac{5}{4}} + 721233q^{\frac{9}{4}} - 82384334q^{\frac{13}{4}}$$

$$- 10999249662q^{\frac{17}{4}} - 429940296576q^{\frac{21}{4}} + \cdots.$$

We recall the result [16]. For Hecke operator $T_p$, we set $T_p = (-1)^{\frac{k(p-1)}{2}} p^{-1} T(p)$, then for $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{24}}$, we have

$$f(\tau)|T(p) = \sum_{n=0}^{\infty} \left\{ p^k a\left(\frac{n}{p}\right) + (-1)^{\frac{k(p-1)}{2}} pa(np) \right\} q^{\frac{n}{24}},$$

($a(x) = 0$ if $x$ is not an integer). From this we obtain the following.

**Lemma 2.1.**

$$\eta^{42} \mid T(p) = \frac{p^{21} + pa(42p^2) - pa(18p)c(42p)}{a(42p)} \eta^{42}$$

$$+ pa(18p) E_6^2 \eta^{18} \quad (p \equiv 1, 5 \pmod{12}),$$

$$\eta^{42} \mid T(p) = -pa(30p) E_6 \eta^{30} - pa(6p) E_6^3 \eta^6 \quad (p \equiv 7, 11 \pmod{12}),$$

$$E_6\eta^{30} \mid T(p) = \frac{p^{21} + pb(30p^2) - pd(30p)b(6p)}{b(30p)} E_6\eta^{30}$$

$$+ pb(6p) E_6^3 \eta^4 \quad (p \equiv 1, 5),$$

$$E_6\eta^{30} \mid T(p) = \frac{p^{21} - pb(30p^2) + pb(18p)c(30p)}{a(30p)} \eta^{42}$$

$$- pb(18p) E_6^2 \eta^{18} \quad (p \equiv 7, 11),$$

$$E_6^2\eta^{18} \mid T(p) = \frac{p^{21} + pc(18p^2) - p(c(18p))^2}{a(18p)} \eta^{42}$$

$$+ pc(18p) E_6^2 \eta^{18} \quad (p \equiv 1, 5),$$

$$E_6^2\eta^{18} \mid T(p) = \frac{p^{21} - pc(18p^2) + pc(6p)d(18p)}{b(18p)} E_6\eta^{30}$$

$$- pc(6p) E_6^3 \eta^6 \quad (p \equiv 7, 11),$$

$$E_6^3\eta^6 \mid T(p) = \frac{p^{21} + pd(6p^2) - p(d(6p))^2}{b(6p)} E_6\eta^{30}$$

$$+ pd(6p) E_6^3 \eta^6 \quad (p \equiv 1, 5),$$

$$E_6^3\eta^6 \mid T(p) = \frac{p^{21} - pd(6p^2) + pc(6p)d(18p)}{a(6p)} \eta^{42}$$

$$- pd(18p) E_6^2 \eta^{18} \quad (p \equiv 7, 11).$$

**Proof.** It is easily proved from the formula for $T(p)$ and the expressions of $\eta^{42}$, $E_6\eta^{30}$, $E_6^2\eta^{18}$ and $E_6^3\eta^6$. □

From the fact that $E_6^3\eta^6 + \alpha E_6^2\eta^{18} + \beta E_6\eta^{30} + \gamma\eta^{42}$ is an eigenfunction for Hecke operators and Lemma 2.1, for $p \equiv 1, 5 \pmod{12}$, we see that

$$\beta^2 - 9722496\beta - 87163318480896 = 0,$$

i.e., $\beta = 4861248 \pm 13440\sqrt{613369}$, $\alpha\beta = 12410496\alpha + 819\gamma$ and its eigenvalue is $pc(18p) + \frac{\gamma}{\alpha} pa(18p)$. Further, for $p \equiv 7, 11 \pmod{12}$, we have $\alpha^2 = -534\beta - 3486063168$, i.e., $\alpha = \pm 24\sqrt{-35(301685 \pm 356\sqrt{613369})}$, and its eigenvalue is $-\alpha pc(6p) - \gamma pa(6p)$. We choose

$$\beta = 4861248 + 13440\sqrt{613369},$$

$$\alpha = 24\sqrt{-35(301685 + 356\sqrt{613369})},$$

$$\gamma = \frac{128}{273}(226297 + 840\sqrt{613369})\sqrt{-35(301685 + 356\sqrt{613369})},$$

$$\tilde{\beta} = 4861248 - 13440\sqrt{613369},$$

$$\tilde{\alpha} = 24\sqrt{-35(301685 - 356\sqrt{613369})},$$

and

$$\tilde{\gamma} = \frac{128}{273}(226297 - 840\sqrt{613369})\sqrt{-35(301685 - 356\sqrt{613369})}.$$

We set

$$A = E_6^3\eta^6, \quad B = E_6^2\eta^{18}, \quad C = E_6\eta^{30}, \quad D = \eta^{42},$$

$$A + \alpha B + \beta C + \gamma D = \sum a_1(n) q^{\frac{n}{4}},$$

$$A - \alpha B + \beta C - \gamma D = \sum a_2(n) q^{\frac{n}{4}},$$

$$A + \tilde{\alpha} B + \tilde{\beta} C + \tilde{\gamma} D = \sum a_3(n) q^{\frac{n}{4}},$$

$$A - \tilde{\alpha} B + \tilde{\beta} C - \tilde{\gamma} D = \sum a_4(n) q^{\frac{n}{4}}.$$

Then we obtain the following.

**Proposition 2.2.** *We have*

$$\begin{aligned}
&-\frac{2184806400}{13}\sqrt{594948915561}\,\eta^{42}(\tau) \\
&= 2(\tilde{\alpha}\gamma - \alpha\tilde{\gamma})\eta^{42}(\tau) \\
&= \sum (\tilde{\alpha}(a_1(n) - a_2(n)) - \alpha(a_3(n) - a_4(n))) q^{\frac{n}{4}} \\
&:= \sum e(n) q^{\frac{n}{4}},
\end{aligned}$$

*and its Dirichlet series is given as follows*:

$$\sum e(n) \cdot n^{-s} = \sum (\tilde{\alpha}(a_1(n) - a_2(n)) - \alpha(a_3(n) - a_4(n))) n^{-s},$$

*where*

$$\begin{aligned}
\sum a_1(n) \cdot n^{-s} = &\prod_{p \equiv 1,\, 5 \pmod{12}} \left(1 - \left(c(18p) + \frac{\gamma}{\alpha} a(18p)\right) p^{-s} + p^{20-2s}\right)^{-1} \\
&\times \prod_{p \equiv 7,\, 11 \pmod{12}} (1 - (\alpha c(6p) + \gamma a(6p)) p^{-s} - p^{20-2s})^{-1},
\end{aligned}$$

(*where p is prime number*).

$\sum a_2(n) n^{-s}$, $\sum a_3(n) n^{-s}$ *and* $\sum a_4(n) n^{-s}$ *are similarly given.*

**Proof.** It is easily proved from the following result [16]. If $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{24}}$ is eigenfunction for $T(p)$ with eigenvalue $c$, then Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ is given by

$$\varphi(s) = \left( \sum_{n \not\equiv 0 \pmod p} a(n) n^{-s} \right) (1 - (-1)^{\frac{k(p-1)}{2}} c p^{-s-1} + (-1)^{\frac{k(p-1)}{2}} p^{k-1-2s})^{-1}. \quad \square$$

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