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DIAMOND OPERATOR RELATED TO BIHARMONIC EQUATION

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Abstract

In this paper, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2} u(x, t) - c^2 (\Diamond)^k u(x, t) = 0$$

with the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{dt} u(x, 0) = g(x),$$

where $u(x, t) \in \mathbb{R}^n \times [0, \infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Diamond^k$ is the Diamond operator iterated k-times defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k,$$

$\Diamond$ can be written as the product of the operators in the form $\Diamond = \Delta \Box = \Box \Delta$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$

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is the ultra-hyperbolic. $p + q = n$, c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x, t)$ as a solution for such equation. Moreover, by ε-approximation we also obtain the asymptotic solution $u(x, t) = O(\varepsilon^{-n/2k})$. In particularly, if we put $n = 1$, $k = 2$ and $p = 0$, then the $u(x, t)$ reduces to the solution of the biharmonic wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\Delta)^4 u(x, t) = 0.$$

1. Introduction

It is well known that for 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t), \tag{1.1}$$

we obtain $u(x, t) = f(x + ct) + g(x - ct)$ as a solution of the equation, where f and g are continuous.

Also for n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 \Delta u(x, t) = 0, \tag{1.2}$$

with the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x),$$

where f and g are continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos(2\pi |\xi|) t + \hat{g}(\xi) \frac{\sin(2\pi |\xi|) t}{2\pi |\xi|},$$

where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$, $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$ (see [1, p. 177]). By using the inverse Fourier transform, we obtain $u(x, t)$ in the convolution form, that is,

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x), \tag{1.3}$$

where Φ_t is an inverse Fourier transform of $\hat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|)t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\hat{\Psi}_t(\xi) = \cos(2\pi|\xi|)t = \frac{\partial}{\partial t}\hat{\Phi}(\xi)$.

In 1997, Kananthai [2] introduced the *Diamond operator* $\Diamond$ defined by

$$\Diamond = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \quad p + q = n$$

or $\Diamond$ can be written as the product of the operators in the form $\Diamond = \Delta\Box = \Box\Delta$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourier transform of the Diamond operator has also been studied and the elementary solution of such operator, see [3]. Next, Sritantatana and Kananthai studied the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2(-\Delta)^k u(x, t) = 0$$

see [7, pp. 23-29], where

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^k.$$

Next, Satsanit and Kananthai studied the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box)^k u(x, t) = 0$$

see [6], where

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

we obtain the solution related to the beam equation.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\lozenge)^k u(x, t) = 0 \tag{1.4}$$

with $u(x, 0) = f(x)$ and $\frac{\partial}{\partial t} u(x, 0) = g(x)$, where c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable. Equation (1.4) is motivated by the heat equation of the form

$$\frac{\partial}{\partial t} u(x, t) = -c^2 (\lozenge)^k u(x, t)$$

(see [4, 1-4]). We obtain

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{1.5}$$

as a solution of (1.4), where Φ_t is an inverse Fourier transform of $\hat{\Phi}_t(\xi) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k}$ and Ψ_t is an inverse Fourier transform of $\hat{\Psi}_t(\xi) = \cos c(\sqrt{s^4 - r^4})^k t = \frac{\partial}{\partial t} \hat{\Phi}_t(\xi)$, where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$. Moreover, if we put $k = 2$ and $p = 0$ in (1.4), then (1.5) reduces to the solution of the n-dimensional biharmonic wave equation and also if $k = 1$, $n = 1$ and $p = 0$ in (1.4), then (1.5) reduces to the solution of beam equation.

We also study the asymptotic form of $u(x, t)$ in (1.5) by using ε-approximation and obtain $u(x, t) = O(\varepsilon^{-n/2k})$.

2. Preliminaries

We shall need the following definitions:

Definition 2.1. Let $f \in L_1(\mathbb{R}^n)$ be the space of integrable function in $\mathbb{R}^n$. The *Fourier transform* of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, ..., \xi_n)$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(x) dx. \tag{2.2}$$

Lemma 2.1. *Given the function*

$$f(x) = \exp\left[- \sqrt{ - \left(\sum_{i=1}^{p} x_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^2 } \right],$$

where $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. *Then*

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\Omega_p \Omega_q}{8} \cdot \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)},$$

where Γ *denotes the Gamma function. That is,* $\int_{\mathbb{R}^n} f(x) dx$ *is bounded.*

Proof. First note that

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} \exp\left[- \sqrt{ - \left(\sum_{i=1}^{p} x_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^2 } \right] dx.$$

Now, we transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, ..., \quad x_p = r\omega_p,$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, ..., \quad dx_p = rd\omega_p$$

and

$$x_{p+1} = s\omega_{p+1}, \quad x_{p+2} = s\omega_{p+2}, ..., \quad x_{p+q} = s\omega_{p+q},$$

$$dx_{p+1} = sd\omega_{p+1}, \quad dx_{p+2} = sd\omega_{p+2}, ..., \quad dx_{p+q} = sd\omega_{p+q},$$

where $\omega_1^2 + \omega_2^2 + \cdots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \cdots + \omega_{p+q}^2 = 1.$ Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \int_{\mathbb{R}^n} \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q.$$

By a direct computation, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^s \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drds,$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drds.$$

Put $r^2 = s^2 \sin\theta$, $2rdr = s^2 \cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$, to have

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s e^{-\sqrt{s^4 - s^4\sin^2\theta}} s^{p-2}(\sin\theta)^{\frac{p-2}{2}} s^{q+1}\cos\theta d\theta ds$$

$$= \frac{\Omega_p\Omega_q}{2} \int_0^\infty \int_0^s e^{-s^2\cos\theta} s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}} \cos\theta d\theta ds.$$

Put $y = s^2\cos\theta$, $ds = \dfrac{dy}{2s\cos\theta}$, to have

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y}\left(\frac{y}{\cos\theta}\right)^{\frac{n-2}{2}} (\sin\theta)^{\frac{p-2}{2}} \cos\theta d\theta \frac{dy}{\cos\theta}$$

$$= \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y} y^{\frac{n-2}{2}} (\cos\theta)^{\frac{2-n}{2}} (\sin\theta)^{\frac{p-2}{2}} dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{4} \Gamma\left(\frac{n}{2}\right) \int_0^{\pi/2} (\cos\theta)^{\frac{2-n}{2}} (\sin\theta)^{\frac{p-2}{2}} d\theta$$

$$= \frac{\Omega_p\Omega_q}{8} \Gamma\left(\frac{n}{2}\right) \beta\left(\frac{p}{4}, \frac{4-n}{4}\right),$$

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p\Omega_q}{8} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Thus it follows that $\int_{\mathbb{R}^n} f(x)dx$ is bounded.

3. Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\lozenge)^k u(x, t) = 0 \tag{3.1}$$

with initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x), \tag{3.2}$$

where $u(x, t) \in \mathbb{R}^n \times [0, \infty)$, $\lozenge^k$ *is the Diamond operator iterated k-times, c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable for* $x \in \mathbb{R}^n$. *Then* (3.1) *has a unique solution*

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{3.3}$$

and satisfies the condition (3.2), *where* Φ_t *is the inverse Fourier transform of*

$$\hat{\Phi}_t(\xi) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k}$$

and Ψ_t *is the inverse Fourier transform of*

$$\hat{\Psi}_t(\xi) = \cos c(\sqrt{s^4 - r^4})^k t = \frac{\partial}{\partial t} \hat{\Phi}(\xi),$$

with $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ *and* $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Proof. By applying the Fourier transform defined by (2.1) to (3.1), we obtain

$$\frac{\partial^2}{\partial t^2} \hat{u}(\xi, t) + c^2 \left(-\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \hat{u}(\xi, t) = 0.$$

Let $s > r$. Thus

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi, t) + c^2(s^4 - r^4)^k \hat{u}(\xi, t) = 0,$$

$$\hat{u}(\xi, t) = A(\xi)\cos c(\sqrt{s^4 - r^4})^k t + B(\xi)\sin c(\sqrt{s^4 - r^4})^k t.$$

By (3.2), $\hat{u}(\xi, 0) = A(\xi) = \hat{f}(\xi)$,

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} = -c(\sqrt{s^4 - r^4})^k A(\xi)\sin c(\sqrt{s^4 - r^4})^k t$$

$$+ c(\sqrt{s^4 - r^4})^k B(\xi)\cos c(\sqrt{s^4 - r^4})^k t,$$

$$\frac{\partial \hat{u}(\xi, 0)}{\partial t} = 0 + c(\sqrt{s^4 - r^4})^k B(\xi) = \hat{g}(\xi),$$

$$B(\xi) = \frac{\hat{g}(\xi)}{c(\sqrt{s^4 - r^4})^k},$$

$$\hat{u}(\xi, t) = \hat{f}(\xi)\cos c(\sqrt{s^4 - r^4})^k t + \frac{\hat{g}(\xi)}{c(\sqrt{s^4 - r^4})^k}\sin c(\sqrt{s^4 - r^4})^k t. \tag{3.4}$$

By applying the inverse Fourier transform (3.4), we obtain the solution $u(x, t)$ in the convolution form of (3.1). Now, we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Consider the Fourier transforms

$$\widehat{\Phi_t}(x) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k} \quad \text{and} \quad \Psi_t(x) = \cos c(\sqrt{s^4 - r^4})^k t.$$

These are all tempered distributions not lying in the space $L_1(\mathbb{R}^n)$ of integrable functions. So we cannot compute the inverse Fourier transforms $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of ε-approximation.

Define

$$\widehat{\phi_t^{\varepsilon}}(\xi) = e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}\widehat{\phi_t}(\xi) = e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}\frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k} \quad \text{for} \quad \varepsilon > 0. \tag{3.5}$$

We see that $\phi_t^{\varepsilon}(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^{\varepsilon}}(x) \to \widehat{\phi_t}(x)$ uniformly as $\varepsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^{\varepsilon}(x)$. Now

$$\Phi_t^{\varepsilon}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \widehat{\Phi_t^{\varepsilon}}(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} e^{-\varepsilon c(\sqrt{s^4 - r^4})^k} \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k} d\xi,$$

$$|\Phi_t^{\varepsilon}(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}}{c(\sqrt{s^4 - r^4})^k} d\xi. \tag{3.6}$$

By changing to bipolar coordinates and putting

$$\xi_1 = rw_1, \quad \xi_2 = rw_2, \ldots, \quad \xi_p = rw_p$$

and

$$\xi_{p+1} = sw_{p+1}, \quad \xi_{p+2} = sw_{p+2}, \ldots, \quad \xi_p = sw_{p+q}, \quad p + q = n,$$

where

$$w_1^2 + w_2^2 + \cdots + w_p^2 = 1 \text{ and } w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1,$$

we obtain

$$|\Phi_t^{\varepsilon}(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}}{c(\sqrt{s^4 - r^4})^k} r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit spheres in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, with $\Omega_p = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{(2\pi)^{q/2}}{\Gamma(q/2)}$. Now,

$$|\Phi_t^{\varepsilon}(x)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^{n/2}} \int_0^{\infty} \int_0^{s} \frac{e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}}{c(\sqrt{s^4 - r^4})^k} r^{p-1} s^{q-1} dr ds.$$

Putting $r^2 = s^2 \sin\theta$, $2rdr = s^2 \cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$, we get

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}} \int_0^{\infty}\int_0^{\pi/2} \frac{e^{-\varepsilon c(\sqrt{s^4 - s^4\sin^2\theta})^k}}{c(\sqrt{s^4 - s^4\sin^2\theta})^k} (\sin\theta)^{\frac{p-2}{2}} s^{p+q-1}\cos\theta d\theta ds$$

$$= \frac{\Omega_p\Omega_q}{2c(2\pi)^{n/2}} \int_0^{\infty}\int_0^{\pi/2} \frac{e^{-\varepsilon c(s^2\cos\theta)^k}}{(s^2\cos\theta)^k} s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta ds.$$

Putting $y = \varepsilon c(s^2\cos\theta)^k = \varepsilon c s^{2k}\cos^k\theta$, $s^{2k} = \frac{y}{c\varepsilon\cos^k\theta}$, $ds = \frac{sdy}{2ky}$, it follows that

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{4c(2\pi)^{n/2}} \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-y}s^{n-1}}{y/(\varepsilon c)} (\sin\theta)^{\frac{p-2}{2}}\cos\theta \frac{s}{ky} dy d\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-y}\varepsilon}{ky^2}\left(\frac{y}{c\varepsilon\cos^k\theta}\right)^{n/2k} (\sin\theta)^{\frac{p-2}{2}}\cos\theta dy d\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-y}y^{n/2k-2}}{c^{n/2k}k\varepsilon^{n/2k-1}} (\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}} dy d\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}} \frac{\Gamma\left(\frac{n}{2k}-1\right)}{k\varepsilon^{\frac{n}{2k}-1}c^{n/2k}} \int_0^{\pi/2} (\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}} d\theta$$

$$= \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\varepsilon^{n/2k-1}} \Gamma\left(\frac{n}{2k}-1\right)\beta\left(\frac{p}{4}, \frac{4-n}{4}\right),$$

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\varepsilon^{n/2k-1}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Similarly, we define $\widehat{\Psi_t^{\varepsilon}}(\xi) = e^{-\varepsilon c(\sqrt{s^4 - r^4})^k}\cos c(\sqrt{s^4 - r^4})^k t$ and

$$\Psi_t^{\varepsilon}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)}\widehat{\Psi_t^{\varepsilon}}(\xi) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} e^{-\varepsilon c(\sqrt{s^4 - r^4})^k} \cos c(\sqrt{s^4 - r^4})^k t d\xi,$$

$$|\Psi_t^{\varepsilon}(x)| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\varepsilon c(\sqrt{s^4 - r^4})^k} d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \int_0^{s} e^{-\varepsilon c(\sqrt{s^4 - r^4})^k} r^{p-1} s^{q-1} drds.$$

Putting $r^2 = s^2 \sin\theta$, $2rdr = s^2 \cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$, we obtain

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{2(2\pi)^{n/2}} \int_0^{\infty} \int_0^{\pi/2} e^{-\varepsilon c(s^2 \cos\theta)^k} (\sin\theta)^{\frac{p-2}{2}} s^{p+q-1} \cos\theta d\theta ds$$

$$= \frac{\Omega_p \Omega_q}{2(2\pi)^{n/2}} \int_0^{\infty} \int_0^{\pi/2} e^{-\varepsilon c(s^2 \cos\theta)^k} s^{p+q-1} (\sin\theta)^{\frac{p-2}{2}} \cos\theta d\theta ds.$$

Next, putting $y = \varepsilon c(s^2 \cos\theta)^k$, $ds = s\frac{dy}{2ky}$, we have

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-y}}{y} \left(\frac{y}{c\varepsilon \cos^k \theta}\right)^{n/2k} (\sin\theta)^{\frac{p-2}{2}} \cos\theta dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-y} y^{n/2k-1}}{c^{n/2k} \varepsilon^{n/2k}} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k}} \Gamma\left(\frac{n}{2k}\right) \int_0^{\pi/2} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} d\theta,$$

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right) \Gamma\left(\frac{p}{4}\right) \Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Set

$$u^{\varepsilon}(x, t) = f(x) * \Psi_t^{\varepsilon}(x) + g(x) * \Phi_t^{\varepsilon}(x) \tag{3.7}$$

which is an ε-approximation of $u(x, t)$ in (3.7) for $\varepsilon \to 0$, $u^{\varepsilon}(x, t) \to u(x, t)$ uniformly. Now

$$u^{\varepsilon}(x, t) = \int_{\mathbb{R}^n} f(r) \Psi_t^{\varepsilon}(x - r) dr + \int_{\mathbb{R}^n} g(r) \Phi_t^{\varepsilon}(x - r) dr.$$

Thus

$$|u^{\varepsilon}(x, t)| \leq |\Psi_t^{\varepsilon}(x-r)| \int_{\mathbb{R}^n} |f(r)| dr + |\Phi_t^{\varepsilon}(x-r)| \int_{\mathbb{R}^n} |g(r)| dr$$

$$\leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M$$

$$+ \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k-1}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{2-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N,$$

$$\varepsilon^{n/2k} |u^{\varepsilon}(x, t)| \leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M$$

$$+ \frac{\Omega_p \Omega_q \varepsilon}{8(2\pi)^{n/2} k c^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N,$$

where $M = \int_{\mathbb{R}^n} |f(r)| dr$ and $N = \int_{\mathbb{R}^n} |g(r)| dr$. Since f and g are absolutely integrable,

$$\lim_{\varepsilon \to 0} \varepsilon^{n/2k} |u^{\varepsilon}(x, t)| \leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} = K.$$

It follows that $u(x, t) = O(\varepsilon^{-n/2k})$ for $n \neq k$ as $\varepsilon \to 0$.

In particular, if we put $k = 2$, $n = 1$ and $p = 0$, then (3.1) reduces to the solution of the beam equation, see [5, p. 47],

$$u(x, 0) = f(x) \text{ and } \frac{\partial}{\partial t} u(x, 0) = g(x),$$

where f and g are continuous and absolutely integrable for $x \in \mathbb{R}^n$.

Thus we obtain $u(x, t) = O(\varepsilon^{-1/4})$ which is a solution of such a biharmonic wave equation.

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