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GAUSS FORMULA AND CURVATURE TENSOR OF DUAL SPACE D^n

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Abstract

In this paper, we investigate the Gauss equation and curvature tensor in the dual space $D^n = \{\overline{X} \mid \overline{X} = x_i + \eta x_i^*,\ x_i,\ x_i^* \in R^n\}$. We obtain Gauss formula in D^n and show that the curvature tensor of D^n would be zero.

1. Introduction

In the real n-dimensional space R^n, lines combined with one of their two directions can be represented by unit dual vectors over the ring of dual numbers. The properties of real vector analysis are valid for the dual vectors also if d and d^* are real numbers, then the combination

$$D = d + \eta d^* \tag{1.1}$$

is called a *dual number* and the symbol η denotes the dual unit with the property that $\eta \cdot \eta = (0, 1)(0, 1) = \eta^2 = 0$. In analogy of the complex numbers, Clifford has

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defined the dual numbers and shown them to form an algebra which is not a field [1, 5]. The pure dual numbers are ηa^*. Corresponding to this definition, pure dual numbers ηd^* are zero divisors. The number ηd^* has no inverse in algebra, as a consequence $i^2 = -1$ [1, 4]. In this article, we investigate further Gauss formula and curvature tensor on the dual space D^n. In Section 2, basic definitions are given and Section 3 deals with the investigation of Gauss equation and curvature tensor of D^n.

2. Preliminaries

The set D is defined by

$$D = \overline{x} = x + \eta x^* \tag{2.1}$$

and these dual numbers form a commutative ring with respect to the following operations. For $\overline{x}, \overline{y} \in D^n$, we have

$$\overline{x} = (x_1 + \eta x_1^*, \ldots, x_n + \eta x_n^*), \tag{2.2}$$

$$\overline{y} = (y_1 + \eta y_1^*, \ldots, y_n + \eta y_n^*); \tag{2.3}$$

(1)

$$\overline{x} + \overline{y} = ((x_1 + \eta x_1^*, \ldots, x_n + \eta x_n^*) + (y_1 + \eta y_1^*, \ldots, y_n + \eta y_n^*))$$

$$= ((x_1 + y_1, \ldots, x_n + y_n) + \eta(x_1^* + y_1^*, \ldots, x_n^* + y_n^*)).$$

(2)

$$\overline{x} \cdot \overline{y} = (x_1 + \eta x_1^*, \ldots, x_n + \eta x_n^*) \cdot (y_1 + \eta y_1^*, \ldots, y_n + \eta y_n^*)$$

$$= \{x_1 y_1 + \eta(x_1 y_1^* + x_1^* y_1), \ldots, x_n y_n + \eta(x_n y_n^* + y_n x_n^*)\}.$$

(3) The division $\overline{x}/\overline{y}$ exists if $\overline{y} \neq 0$ $(y \neq 0,\ y^* \neq 0)$. We define the division $\overline{x}/\overline{y}$ such that

$$\frac{\overline{x}}{\overline{y}} = \frac{x_1 + \eta x_1^*, \ldots, x_n + \eta x_1^*}{y_1 + \eta y_1^*, \ldots, y_n + \eta y_n^*} \tag{2.4}$$

and see that

$$\frac{\bar{x}}{\bar{y}} = \frac{x_1}{y_1} + \eta\frac{x_1^* y_1 - y_1^* x_1}{y_1^2}, \ldots, \frac{x_n}{y_n} + \eta\frac{x_n^* y_n - y_n^* x_n}{y_n^2}. \tag{2.5}$$

$$D^n = \{\bar{x} \mid \bar{x} = (x_1 + \eta x_1^*, \ldots, x_n + \eta x_n^*)\} \tag{2.6}$$

is a module over the ring D. It is clear that each of dual vector $\bar{x}$ in D^n consists of any two real vectors x and x^*, which are expressed in the orthonormal frame in the n-dimensional Euclidean space R^n. We call the elements of D^n to be the dual vectors. If $\bar{x} \neq 0$ $(x \neq 0, x^* \neq 0)$, then the norm of $\bar{x}$ is defined by $\| \bar{x} \| = \sqrt{\langle \bar{x}, \bar{x} \rangle}$. It may be recalled that ∇^* is a derivative operator and acts like ∇. So $\nabla(x^* y^*) = y^{*2} + 2x^* y^*$ and also $\nabla^*(x^* y^{*2}) = y^{*2} + 2x^* y^*$. Finally, we have $\nabla^{*(xy^2)} = y^2 + 2xy$. We know that an $(n-1)$-submanifold is called a *hypersurface* [4], and an $(n-1)$-dual submanifold is called a *dual hypersurface*.

3. Main Results

Definition 3.1. Let M be a hypersurface of R^n and $\overline{\nabla}$ be the natural connection on R^n. Suppose N is a unit normal vector field that is C^∞ on M, and let $S(X) = \overline{\nabla}_X N$ be X tangent to M. If Y is a C^∞ vector field about p in M, and X in M_p, then define $\nabla_X Y$ by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle S(X), Y \rangle N, \tag{3.1}$$

is a Gauss equation [4].

Definition 3.2. Let X, Y and Z be C^∞ vector fields on an open set A in R^n and R be curvature tensor of R^n. Define $R(X, Y)Z$ by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z = 0. \tag{3.2}$$

This is called *curvature tensor of* R^n. Now, we introduce the dual vector fields $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ such that $\overline{X}, \overline{Y}, \overline{Z} \in \chi(D^n)$, where $\chi(D^n)$ is the space of all vector fields on D^n. Utilising the Gauss equation and curvature tensor for $\overline{X}, \overline{Y}, \overline{Z} \in \chi(D^n)$, we give the following theorems.

Theorem 3.3. *Let $\overline{M}$ be dual hypersurface in D^n and S be Weingarten map on $\overline{M}$. If $\nabla' = \nabla + \eta\nabla^*$ is covariant derivative in D^n and $\overline{\nabla}$ is covariant derivative on $\overline{M}$, then the Gauss equation given by*

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \sum_{i=1}^{n}(x_i(\nabla y_1) + \eta(x_i(\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*))), \ldots,$$

$$\sum_{i=1}^{n}(x_i(\nabla y_n) + \eta(x_i(\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*)))$$

$$+ \sum_{i=1}^{n}(k_i x_i y_i a_i + \eta(k_i^* x_i y_i a_i + x_i^* k_i y_i a_i)).$$

Proof. Let $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n)$ such that $\overline{x}_1 = x_1 + \eta x_1^*, \ldots, \overline{x}_n = x_n + \eta x_n^*$ and $\overline{Y} = (\overline{y}_1, \ldots, \overline{y}_n)$ so that $\overline{y}_1 = y_1 + \eta y_1^*, \ldots, \overline{y}_n = y_n + \eta y_n^*$ and we denote the inner product by $\langle\ ,\ \rangle$. Then

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \nabla'_{\overline{X}}\overline{Y} + \langle S(\overline{X}), \overline{Y}\rangle \overline{N}, \tag{3.3}$$

$$\nabla'_{\overline{X}}\overline{Y} = (\nabla + \eta\nabla^*)_{\overline{X}}\overline{Y}$$

$$= (\langle \overline{X}, (\nabla + \eta\nabla^*)\overline{y}_1\rangle, \ldots, \langle \overline{X}, (\nabla + \eta\nabla^*)\overline{y}_n\rangle)$$

$$= \langle \overline{X}, (\nabla + \eta\nabla^*)(y_1 + \eta y_1^*)\rangle, \ldots, \langle \overline{X}, (\nabla + \eta\nabla^*)(y_n + \eta y_n^*)\rangle$$

$$= \{\langle (x_i + \eta x_i), (\nabla + \eta\nabla^*)(y_1 + \eta y_1^*)\rangle, \ldots, \langle (x_i + \eta x_i), (\nabla + \eta\nabla^*)(y_n + \eta y_n^*)\rangle\}$$

$$= \sum_{i=1}^{n}(x_i(\nabla y_1) + \eta(x_i(\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*))), \ldots,$$

$$\sum_{i=1}^{n}(x_i(\nabla y_n) + \eta(x_i(\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*))).$$

We know that $\overline{X}$ is a principal vector and S is a Weingarten map. So that

$$S(\overline{X}) = \overline{k}\overline{x}, \quad S(\overline{X}) = \sum_{i=1}^{n}\overline{k}_i\overline{x}_i X_i, \quad x_i \in C^\infty(\overline{M}, R),$$

$$\bar{k}_i = k_i + \eta k_i^*, \quad \bar{x}_i = x_i + \eta x_i^*, \quad \bar{k}_i \bar{x}_i = k_i x_i + \eta(k_i^* x_i + x_i^* k_i),$$

$$\bar{Y} = (\bar{y}_1, \ldots, \bar{y}_n) = (y_1 + \eta y_1^*, \ldots, y_n + \eta y_n^*).$$

To see this $\langle S(\bar{X}), \bar{Y}\rangle \bar{N}$, first we find $\langle S(\bar{X}), \bar{Y}\rangle$. So

$$\langle S(\bar{X}), \bar{Y}\rangle = (k_i x_i + \eta(k_i^* x_i + x_i^* k_i))(y_i + \eta y_i^*) = k_i x_i y_i + \eta(k_i^* x_i y_i + x_i^* k_i y_i)$$

and $\bar{N} = a_i + \eta a_i^*$,

$$\begin{aligned}\langle S(\bar{X}), \bar{Y}\rangle \bar{N} &= (k_i x_i y_i + \eta(k_i^* x_i y_i + x_i^* k_i y_i))(a_i + \eta a_i^*)\\ &= k_i x_i y_i a_i + \eta(k_i^* x_i y_i a_i + x_i^* k_i y_i a_i)\\ &= \sum_{i=1}^{n} (k_i x_i y_i a_i + \eta(k_i^* x_i y_i a_i + x_i^* k_i y_i a_i)).\end{aligned}$$

Finally

$$\begin{aligned}\bar{\nabla}_{\bar{X}} \bar{Y} &= \nabla'_{\bar{X}} \bar{Y} + \langle S(\bar{X}), \bar{Y}\rangle \bar{N}\\ &= \sum_{i=1}^{n} (x_i(\nabla y_1) + \eta(x_i(\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*))), \ldots,\\ &\quad \sum_{i=1}^{n} (x_i(\nabla y_n) + \eta(x_i(\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*)))\\ &\quad + \sum_{i=1}^{n} (k_i x_i y_i a_i + \eta(k_i^* x_i y_i a_i + x_i^* k_i y_i a_i)).\end{aligned}$$

Theorem 3.4. *Let D^n be a dual space and $\bar{X}, \bar{Y}, \bar{Z}$ be dual vector fields on the D^n. Let R be a curvature tensor of D^n and $\bar{\nabla} = \nabla + \eta\nabla^*$ be a dual covariant derivative (connection) on D^n. Then*

$$R(\bar{X}, \bar{Y})\bar{Z} = \bar{\nabla}_{\bar{X}}(\bar{\nabla}_{\bar{Y}}\bar{Z}) - \bar{\nabla}_{\bar{Y}}(\bar{\nabla}_{\bar{X}}\bar{Z}) - \bar{\nabla}_{[\bar{X}, \bar{Y}]}\bar{Z} = 0. \tag{3.4}$$

Proof.

$$\begin{aligned}\bar{\nabla}_{\bar{X}} \bar{Y} &= \sum_{i=1}^{n} (y_i(\nabla z_1) + \eta(y_i(\nabla z_1^* + \nabla^* z_1) + \nabla z_1(y_i^*))), \ldots,\\ &\quad \sum_{i=1}^{n} (y_i(\nabla Z_n) + \eta(y_i(\nabla z_n^* + \nabla^* z_n) + \nabla z_n(y_i^*))),\end{aligned}$$

$$\overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{Y}}\overline{Z}) = (x_i + \eta x_i^*)(\nabla + \eta\nabla^*)(\overline{\nabla}_{\overline{Y}}\overline{Z})$$

$$= \sum_{i=1}^{n}(x_i + \eta x_i^*)\{\nabla y_i(\nabla z_1) + \eta(\nabla y_i(\nabla z_1^* + \nabla^* z_1) + \nabla^2 z_1(y_i^*))$$

$$+ \eta\nabla^*(\nabla y_i)(\nabla z_1)\}, \ldots,$$

$$\sum_{i=1}^{n}(x_i + \eta x_i^*)\{\nabla y_i(\nabla z_n) + \eta(\nabla y_i(\nabla z_n^* + \nabla^* z_n) + \nabla^2 z_n(y_i^*))$$

$$+ \eta\nabla^*(\nabla y_i)(\nabla z_n)\}$$

$$= \sum_{i=1}^{n}\{x_i(\nabla y_i)(\nabla z_1) + \eta(x_i(\nabla y_i)(\nabla z_1^* + \nabla^* z_1) + x_i(\nabla^2 z_1)(y_i^*))$$

$$+ \eta x_i^*\nabla^*(\nabla y_i)(\nabla z_1)\}, \ldots,$$

$$\sum_{i=1}^{n}\{x_i(\nabla y_i)(\nabla z_n) + \eta(x_i(\nabla y_i)(\nabla z_n^* + \nabla^* z_n) + x_i(\nabla^2 z_n) + (y_i^*))$$

$$+ \eta x_i^*\nabla^*(\nabla y_i)(\nabla z_n)\}. \tag{3.5}$$

On the other hand, we have

$$\overline{\nabla}_{\overline{Y}}(\overline{\nabla}_{\overline{X}}\overline{Z})$$

$$= \sum_{i=1}^{n}\{y_i(\nabla x_i)(\nabla z_1) + \nabla[y_i(\nabla x_i)(\nabla z_1^* + \nabla^* z_1) + y_i(\nabla^2 z_1)(x_i^*) + y_i^*\nabla^*(\nabla x_i)(\nabla z_1)]\}, \ldots,$$

$$\sum_{i=1}^{n}\{y_i(\nabla x_i)(\nabla z_n) + \nabla[y_i(\nabla x_i)(\nabla z_n^* + \nabla^* z_n) + y_i(\nabla^2 z_n)(x_i^*) + y_i^*\nabla^*(\nabla x_i)(\nabla z_n)]\}. \tag{3.6}$$

We know that

$$\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z} = \nabla_{[\overline{X},\overline{Y}]}\overline{Z} + \eta\nabla^*{}_{[\overline{X},\overline{Y}]}\overline{Z} \tag{3.7}$$

or

$$\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z} = (\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X})(\nabla + \eta\nabla^*)(z_i + \eta z_i^*),$$

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \sum_{i=1}^{n}(x_i(\nabla y_1) + \eta(\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*)), \ldots,$$

$$\sum_{i=1}^{n}(x_i(\nabla y_n) + \eta(\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*)),$$

$$\overline{\nabla}_{\overline{Y}}\overline{X} = \sum_{i=1}^{n}(y_i(\nabla x_1) + \eta(\nabla x_1^* + \nabla^* x_1) + \nabla x_1(y_i^*)), \ldots,$$

$$\sum_{i=1}^{n}(y_i(\nabla x_n) + \eta(\nabla x_n^* + \nabla^* x_n) + \nabla x_n(y_i^*)),$$

$$(\nabla + \eta\nabla^*)(z_i + \eta z_i^*) = \nabla z_i + \eta(\nabla z_i^* + \nabla^* z_i)$$

$$= (\nabla + \eta\nabla^*)(z_1 + \eta z_1^*, \ldots, z_n + \eta z_n^*).$$

Using above relations, we get

$$(\overline{\nabla}_{\overline{X}}\overline{Y})(\nabla z_i + \eta(\nabla z_i^* + \nabla^* z_i))$$

$$= \sum_{i=1}^{n}(x_i(\nabla y_1) + \eta(x_i(\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*)))(\nabla z_i + \eta(\nabla Z_i^* + \nabla^* z_i)), \ldots,$$

$$\sum_{i=1}^{n}(x_i(\nabla y_n) + \eta(x_i(\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*)))(\nabla z_i + \eta(\nabla Z_i^* + \nabla^* z_i))$$

$$\times \sum_{i=1}^{n}(x_i(\nabla y_1)(\nabla z_1) + \eta[x_i(\nabla y_1(\nabla z_1^* + \nabla^* z_1) + (\nabla y_1^* + \nabla^* y_1) + \nabla y_1(x_i^*))]), \ldots,$$

$$\sum_{i=1}^{n}(x_i(\nabla y_n)(\nabla z_n) + \eta[x_i(\nabla y_n(\nabla z_n^* + \nabla^* z_n) + (\nabla y_n^* + \nabla^* y_n) + \nabla y_n(x_i^*))]).$$

Also using same manner, we have

$$(\overline{\nabla}_{\overline{y}}\overline{z})(\nabla z_1 + \eta(\nabla z_i^* + \nabla^* z_i))$$

$$= \sum_{i=1}^{n}(y_i(\nabla x_1)(\nabla z_1) + \eta[y_i(\nabla x_1(\nabla z_1^* + \nabla^* z_1) + (\nabla x_1^* + \nabla^* x_1) + \nabla x_1(y_i))]), \ldots,$$

$$\sum_{i=1}^{n}(y_i(\nabla x_n)(\nabla z_n) + \eta[y_i(\nabla x_n(\nabla z_n^* + \nabla^* z_n) + (\nabla x_n^* + \nabla^* x_n) + \nabla x_n(y_i))]). \quad (3.8)$$

Substituting equations (12), (13) and (14) in equation (11), we obtain the following relation

$$R(\overline{X}, \overline{Y})\overline{Z} = 0.$$

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