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RIESZ-MARTIN REPRESENTATION FOR POSITIVE POLYSUPERHARMONIC FUNCTIONS IN A HARMONIC SPACE

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Abstract

In the context of the axiomatic potential theory, we introduce the notions of polyharmonic Green domains and polyharmonic functions of order m on a Brelot space Ω. For these functions, we prove that if u is a positive polyharmonic function in a polyharmonic Green domain ω, then u has a representation analogous to the Riesz-Martin representation for positive harmonic functions on Ω.

1. Introduction

In [6], Anandam and Othman proved that in a biharmonic Green domain ω in a Riemannian manifold R, a locally dx-integrable function v on ω which satisfies the conditions $v \geq 0$, $\Delta v \leq 0$ and $\Delta^2 v \geq 0$ has Riesz-Martin representation. We initiate in this note a similar study in the framework of the axiomatic potential theory. After defining polyharmonic functions of order m on a Brelot harmonic space, we obtain Riesz-Martin representation for these functions analogous to those functions studied in Riemannian manifolds and Riemann surfaces.

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2. Preliminaries

Let Ω be a locally compact space provided with a sheaf of harmonic functions satisfying the axioms 1, 2 and 3 of Brelot [7]. Fix a Radon measure λ on Ω such that each superharmonic function on a domain ω in Ω is λ-integrable. Such measures can be constructed by using the harmonic measures on Ω (see [4]). Let us assume also that the axiom of local proportionality (see [7]) and the axiom A^* of quasi-analyticity (see de La Pradelle [9]) are verified on Ω, and the constants are harmonic on Ω. With these restrictions, we call $\Omega = (\Omega, H, \lambda)$ a *harmonic space*.

As examples of Brelot harmonic spaces, we can cite $\Re^n$, $n \geq 1$: parabolic, hyperbolic Riemann surfaces and Riemannian manifolds, also domains in $\Re^n$, $n \geq 2$, with harmonic functions as the solutions of second-order elliptic differential operators with smooth coefficients (see Hervé [10]).

Among these harmonic spaces Ω, some have potentials > 0 in Ω (like, $\Re^n$, $n \geq 3$, hyperbolic Riemann surfaces and hyperbolic Riemannian manifolds) and some others do not have potentials > 0 in Ω (like $\Re^2$ and parabolic Riemann surfaces). We say that a harmonic space Ω is a *B.P.* or *B.S. space* depending on whether there exist or not potentials > 0 in Ω.

Lemma 2.1 [3]. *Let μ be a positive Radon measure on an open set ω in a harmonic space $\Omega = (\Omega, H, \lambda)$. Then there exists a superharmonic function s on ω such that μ is the measure associated with s in a local Riesz representation.* (*We represent this correspondence by the equation $(-L)s = \mu$ on ω.*)

A domain ω in Ω is called a *Green domain* if the Green function $G(x, y)$ is well defined on Ω. On a Green domain ω in Ω, we can construct the Martin compactification $\overline{\Omega}$ of Ω as in [8]. Some of the important points to remember here are the following: fix a point y_0 in a Green domain ω.

If $G(x, y)$ is the Green function on Ω, write

$$k_y(x) = k(x,\ y) = \frac{G(x,\ y)}{G(x,\ y_0)}$$

with the convention $k(y_0,\ y_0) = 1$. Then there exists only one (metrizable) compactification $\overline{\Omega}$ up to homeomorphism such that

(i) Ω is dense open in the compact space $\overline{\Omega}$;

(ii) $k_y(x),\ y \in \Omega,$ extends as a continuous function of x on $\overline{\Omega}$;

(iii) the family of these extended continuous functions on $\overline{\Omega}$ separates the points $x \in \Delta = \overline{\Omega} \setminus \Omega$.

$\overline{\Omega}$ is called the *Martin compactification* of Ω and $\Delta = \overline{\Omega} \setminus \Omega$ is called the *Martin boundary*. A positive harmonic function $u > 0$ is called *minimal* if and only if for any harmonic function $v,\ 0 \le v \le u,$ we should have $v = \alpha u$ for a constant $\alpha,\ 0 \le \alpha \le 1.$ It can be proved that every minimal harmonic function $u(y)$ on Ω is of the form $u(y_0)k(x,\ y)$ for some $x \in \Delta$, and the points $x \in \Delta$ corresponding to these minimal harmonic functions are called the *minimal points* of Δ, and the set of minimal points of Δ is denoted by Δ_1, called the *minimal boundary*.

Martin Representation Theorem 2.2. *For any harmonic function* $u \ge 0$ *on* Ω, *there exists a unique Radon measure* $\mu \ge 0$ *on* Δ *with support in the minimal boundary* $\Delta_1 \subset \Delta$ *such that* $u(y) = \int_{\Delta_1} k(x,\ y)d\mu(x).$

Definition 2.3. Let $(u_i)_{m \ge i \ge 1}$ be m functions defined on an open set ω in a harmonic space $\Omega = (\Omega,\ H,\ \lambda)$ such that $(-L)u_{j+1} = u_j,\ 1 \le j \le m - 1$. We say that $u = (u_i)_{m \ge i \ge 1}$ is a *polysuperharmonic function of order* m or shortly m*-superharmonic* (resp., m*-subharmonic*, resp., m*-harmonic*) if u_1 is superharmonic (resp., subharmonic, resp., harmonic). We say that $u \ge 0$ if each $u_i \ge 0$.

Let ω be a Green domain in a harmonic space Ω, with $G(x,\ y)$ as the Green function on ω. For an integer $m \ge 2,$ we will denote

$$G^m(x,\ y) = \int G(x,\ z_{m-1})G(z_{m-1},\ z_{m-2})\cdots G(z_1,\ y)dz_1 \cdots dz_{m-1}$$

and say that a positive Radon measure μ on ω is in π_m if $u(x) = \int G^m(x, y) d\mu(y) \not\equiv \infty$ on ω, in which case u is a potential on ω and $(-L)^m u = \mu$; also $(-L)^j u \geq 0$ for $0 \leq j \leq m$.

Let $\overline{\Omega}$ be the Martin compactification of Ω and let $k(x, y)$ be the Martin kernel. For any i, $1 \leq i \leq m-1$, let Λ_i denote the set of positive Radon measures v_i on $\Delta = \overline{\Omega} \setminus \Omega$ with support in the minimal boundary Δ_1, such that

$$v_i(x) = \int G(x, z_i) G(z_i, z_{i-1}) \cdots G(z_2, z_1) \left[\int_{\Delta_1} k(X, z_1) dv(X) \right] dz_1 \cdots dz_i$$

$$\not\equiv \infty.$$

In that case, $v_i(x)$ is a potential on ω, $(-L)^i v_i \equiv 0$; also $(-L)^j v_i \geq 0$ for $0 \leq j \leq i$. Let us write for $X \in \Delta_1$ and $x \in \omega$,

$$k_i(X, x) = \int G(x, z_i) \cdots G(z_2, z_1) k(X, z_1) dz_1 \cdots dz_i.$$

Then, if $v \in \Lambda_i$, $v_i(x) = \int_{\Delta_1} k_i(X, x) dv(X)$ is well defined on ω with the above properties.

Definition 2.4. A domain ω in Ω is called *m-harmonic Green domain* if there exists a polysuperharmonic function of order m in ω.

Example. $\Re^n$, $n \geq 2m+1$, [2] is an m-harmonic Green domain since the function $u = (u_m, u_{m-1}, \ldots, u_1)$, $u_m(x) = |x|^{2m-n}$ is a polysuperharmonic function of order m.

3. Integral Representation in a Harmonic Space

Theorem 3.1. *Let ω be an m-harmonic domain in a harmonic space Ω. Let $m \geq 1$ be an integer. Then the following are equivalent*:

(i) $s = (s_m, s_{m-1}, \ldots, s_1) \geq 0$ *is a polysuperharmonic function of order m in* ω.

(ii) *For any* j, $1 \leq j \leq m$, *there exist unique measures* $\mu \in \pi_j$ *and* $v_i \in \Lambda_i$ *for* $0 \leq i \leq j-1$ *such that*

$$s_j(x) = \int_{\Omega} G^j(x, y) d\mu(y) + \sum_{i=0}^{j-1} \int_{\Delta_1} k_i(X, x) dv_i(X) \text{ } a.e. \text{ on } \omega.$$

(iii) *The above property* (ii) *is satisfied for* $j = m$.

Proof. (i) $\Rightarrow$ (ii) Fix j, $1 \leq j \leq m$. Then $(s_j, s_{j-1}, ..., s_1)$ is a polysuperharmonic function of order j on Ω, since $(-L)s_{i+1} = s_i$ for $1 \leq i \leq j-1$ and s_1 is superharmonic. Moreover, since $(-L)s_{i+1} \geq 0$, each s_i is a positive superharmonic function. Write $s_1 = p_1 + h_1$ as the unique sum of a potential p_1 and a positive harmonic function h_1. Let $(-L)p_1^* = p_1$ and $(-L)h_1^* = h_1$. Then p_1^* and h_1^* are superharmonic on ω and $(-L)s_2 = p_1 + h_1 = (-L)p_1^* + (-L)h_1^*$.

That is, $s_2 = p_1^* + h_1^* +$ (a harmonic function) on ω. Since $s_2 \geq 0$, p_1^* has a subharmonic minorant on ω and hence $p_1^* =$ (a potential p_2) + (the greatest harmonic minorant of p_1^*, which may not necessarily be positive).

Then $s_2 = p_2 + u_2$, where u_2 is superharmonic on ω. Since $s_2 \geq 0$, $p_2 \geq -u_2$. Since p_2 is a potential and $-u_2$ is subharmonic, $-u_2 \leq 0$. Hence $s_2 = p_2 + u_2$, where p_2 is a potential on ω such that $(-L)p_2 = p_1$ and $u_2 \geq 0$ is superharmonic such that $(-L)u_2 = h_1$.

Thus proceeding, we can write

$$(s_j, s_{j-1}, ..., s_1) = (p_j, p_{j-1}, ..., p_1) + (u_j, u_{j-1}, ..., u_2, h_1),$$

where $(-L)p_{i+1} = p_i$ for $1 \leq i \leq j-1$, and $p_1, ..., p_j$ are all potentials; $(-L)u_{i+1} = u_i$ for $2 \leq i \leq j-1$ and $(-L)u_2 = h_1$.

Now take $(u_j, u_{j-1}, ..., u_2, h_1)$ and proceed as before. Note now h_1 is positive harmonic, so that we can write

$$(u_j, u_{j-1}, \ldots, u_2, h_1) = (q_j, q_{j-1}, \ldots, h_1) + (f_j, f_{j-1}, \ldots, f_3, h_2, 0),$$

where $(-L)q_{i+1} = q_i$ for $2 \le i \le j-1$, $(-L)q_2 = h_1$, and each q_i is a potential; $(-L)f_{i+1} = f_i \ge 0$ for $3 \le i \le j-1$, $(-L)f_3 = h_2$, and $(-L)h_2 = 0$, so that h_2 is positive harmonic.

Then take $(f_j, f_{j-1}, \ldots, f_3, h_2, 0)$ and follow the same procedure, so that

$$(f_j, f_{j-1}, \ldots, f_3, h_2, 0) = (r_j, r_{j-1}, \ldots, r_3, h_2, 0) + (g_j, g_{j-1}, \ldots, g_4, h_3, 0, 0),$$

where $(-L)r_{i+1} = r_i$ for $3 \le i \le j-1$, $(-L)r_3 = h_2$ and each r_i is a potential; $(-L)g_{i+1} = g_i \ge 0$ for $4 \le i \le j-1$, $(-L)g_4 = h_3$ and $(-L)h_3 = 0$, so that h_3 is harmonic ≥ 0.

Thus proceeding, we finally arrive at the decomposition

$$(s_j, s_{j-1}, \ldots, s_1) = (p_j, p_{j-1}, \ldots, p_1) + (q_j, q_{j-1}, \ldots, q_2, h_1)$$
$$+ (r_j, r_{j-1}, \ldots, r_3, h_2, 0) + \cdots + (h_j, 0, 0, \ldots, 0).$$

Let $(-L)p_1 = \mu$ and let v_j $(1 \le i \le j)$ be the positive Radon measure on Δ_1, associated with the positive harmonic function h_i in the Martin representation. Then $s_j = p_j + q_j + r_j + \cdots + h_j$ has the integral representation

$$s_j(x) = \int_\Omega G^j(x, y)d\mu(y) + \sum_{i=0}^{j-1} \int_{\Delta_1} k_i(X, x)dv_i(X) \text{ a.e. on } \omega.$$

(ii) $\Rightarrow$ (iii) $j = m$ is a particular case of (ii).

(iii) $\Rightarrow$ (i) By the assumption

$$s_m(x) = \int_\omega G^m(x, y)d\mu(y) + \sum_{i=0}^{m-1} \int_{\Delta_1} k_i(X, x)dv_i(X) \text{ a.e.}$$

Hence we can express s_m in the form $s_m(x) = p_m(x) + \sum_{j=0}^{m-1} q_j(x)$. We can calculate to find that $(-L)^i p_m$ is a potential for $1 \le i \le m-1$ and

$(-L)^m p_m = \mu$, a positive Radon measure; and $(-L)^i q_j$ is a potential for $1 \leq i \leq j-1$ and $(-L)^j q_j = 0$.

Write now $(-L)s_m = s_{m-1}$, $(-L)s_{m-1} = s_{m-2}$, ..., $(-L)s_2 = s_1$. We can see that each s_i $(1 \leq i \leq m)$ is a positive superharmonic function and $(-L)s_{i+1} = s_i$ for $1 \leq i \leq m-1$.

Hence $s = (s_m, s_{m-1}, ..., s_1) \geq 0$ is a polyharmonic function of order m.

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