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ON THE TWO EQUIVALENT DEFINITIONS OF MODULAR LATTICES WITH UNIT ELEMENT

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Abstract

For modular lattices with unit element, the paper is on the basis of deleting the conditions of $1 \vee a = 1$, $a \wedge 1 = a$ and simplifying equation M_{21} in [4]. We give two much simpler equivalent conditions.

Definitions of two and three conditions of modular lattices with unit element were given in paper [4]. According to flexibility of operations $\vee$ and $\wedge$, we again obtain two more simplified definitions of modular lattices with unit element.

1. Original Definition with Unit Modular Lattice

Original definition (1) with unit modular lattice is denoted by the following six conditions:

L_1 : $a \vee a = a$, $a \wedge a = a$ (idempotent law).

L_2 : $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ (commutative law).

L_3 : $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associative law).

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L_4 : $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ (absorptive law).

L_5 : $a \wedge (b \vee (a \wedge c)) = (a \wedge b) \vee (a \wedge c)$ (modular law).

L_6 : being unit element 1 with $a \vee 1 = 1 \vee a = 1$, $a \wedge 1 = 1 \wedge a = a$. (1)

2. Definition with Unit Modular Lattice Denoted by Three Conditions

The following is definition (2) with unit modular lattice denoted by three conditions:

M_{11} : $a \wedge (b \vee (a \wedge c)) = (c \wedge a) \vee (b \wedge a)$.

M_{12} : $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

M_{13} : being unit element 1 with $a \vee 1 = 1$, $1 \wedge a = a$. (2)

Lemma 2.1. *Original definition* (1) *with unit modular lattice denoted by six conditions is equivalent with definition* (2) *with unit modular lattice denoted by three conditions* (*additional conditions* $1 \vee a = 1$ *and* $a \wedge 1 = a$), *i.e.*, (3) *in proof of Theorem* 2.1.

Proof. See paper [4].

Theorem 2.1. *The definition of a nonempty set with two binary operations* $\vee$, $\wedge$ *is a modular lattice with unit if and only if condition* (2) *holds on* L.

Proof. (2) $\Rightarrow$ (1) Set $a = 1$, $b = c = a$ in M_{11}, we have $1 \wedge (a \vee (1 \wedge a)) = (a \wedge 1) \vee (a \wedge 1)$, then using $1 \wedge a = a$, we have

$$a \vee a = (a \wedge 1) \vee (a \wedge 1). \tag{I}$$

Set $c = 1$, $b = 1$ in M_{11}, by $1 \wedge a = a$, we have

$$a \wedge (1 \vee (a \wedge 1)) = (1 \wedge a) \vee (1 \wedge a) = a \vee a. \tag{II}$$

Set $b = a$, $a = c = 1$ in M_{11}, by $a \vee 1 = 1$, $1 \wedge a = a$, we have

$$1 \wedge (a \vee (1 \wedge 1)) = (1 \wedge 1) \vee (a \wedge 1) \Leftrightarrow 1 = 1 \vee (a \wedge 1). \tag{III}$$

In this case, by (III), we know that $1 \vee (a \wedge 1)$ can be substituted with 1 in (II), then we have $a \wedge 1 = a \vee a$, by (I), we have $a \vee a = (a \vee a) \vee (a \vee a)$, then we regard $a \vee a$ as a', we have $a' = a' \vee a'$, invoking it, we use (I) again, we get $a = a \wedge 1$, invoking it and (III), we have $1 = 1 \vee a$. From the above, we can substitute M_{13} with M'_{13}, its form is as follows:

$$M'_{13} \;:\; \text{being unit element 1 with } a \vee 1 = 1 \vee a = 1,\ a \wedge 1 = 1 \wedge a = a,$$

we assume

$$M_{11} \;:\; a \wedge (b \vee (a \wedge c)) = (c \wedge a) \vee (b \wedge a).$$

$$M_{12} \;:\; (a \wedge b) \wedge c = a \wedge (b \wedge c).$$

$$M'_{13} \;:\; \text{being unit element 1 with } a \vee 1 = 1 \vee a = 1,\ a \wedge 1 = 1 \wedge a = a. \tag{3}$$

From the above proof, we know (2) $\Leftrightarrow$ (3).

So we only need to prove (3) $\Leftrightarrow$ (1), however it can easily be obtained from Lemma 2.1.

(1) $\Rightarrow$ (2) Because (1) $\Rightarrow$ (2) is equivalent with (1) $\Rightarrow$ (3), however the proof of (1) $\Rightarrow$ (3) can be obtained from Lemma 2.1. This completes the proof.

3. Definition with Unit Modular Lattice Denoted by Two Conditions

Definition (4) with unit modular lattice is denoted by two conditions:

$$M_{21} \;:\; a \wedge ((b \wedge d) \vee (a \wedge (c \wedge d))) = (d \wedge (c \wedge a)) \vee (d \wedge (b \wedge a)).$$

$$M_{22} \;:\; \text{being unit element 1 with } a \vee 1 = 1,\ 1 \wedge a = a. \tag{4}$$

Theorem 3.1. *The definition of a nonempty set L with two binary operations* $\vee$, $\wedge$ *is a modular lattice with unit if and only if condition* (4) *holds on L.*

Proof. In fact, we only need to prove that definition of two conditions is equivalent with the definition of three conditions.

(2) $\Rightarrow$ (4) In Theorem 2.1, if (2) holds, then we have proven that commutative law holds. Invoking it, we have

$$\begin{aligned} a \wedge ((b \wedge d) \vee (a \wedge (c \wedge d))) &= ((c \wedge d) \wedge a) \vee ((b \wedge d) \wedge a) \text{ (by } M_{11}) \\ &= ((d \wedge c) \wedge a) \vee ((d \wedge b) \wedge a) \text{ (by commutative law)} \\ &= (d \wedge (c \wedge a)) \vee (d \wedge (b \wedge a)) \text{ (by } M_{12}). \end{aligned}$$

It means that equation M_{21} holds, i.e., (4) holds.

(4) $\Rightarrow$ (2) Set $a = b = d = 1$, $c = a$ in M_{21}, by M_{22}, we have

$$1 \wedge ((1 \wedge 1) \vee (1 \wedge (a \wedge 1))) = (1 \wedge (a \wedge 1)) \vee (1 \wedge (1 \wedge 1)), \text{ i.e., } 1 \vee (a \wedge 1) = 1. \quad (5)$$

Set $b = c = d = 1$ in M_{21}, by M_{22}, we have

$$a \wedge ((1 \wedge 1) \vee (a \wedge (1 \wedge 1))) = (1 \wedge (1 \wedge a)) \vee (1 \wedge (1 \wedge a)),$$

i.e.,

$$a \wedge (1 \vee (a \wedge 1)) = a \vee a,$$

by (5), we have

$$a \wedge 1 = a \vee a. \quad (6)$$

Set $a = b = c = 1$, $d = a$ in M_{21}, we have

$$1 \wedge ((1 \wedge a) \vee (1 \wedge (1 \wedge a))) = (a \wedge (1 \wedge 1)) \vee (a \wedge (1 \wedge 1)),$$

by M_{21}, we have

$$a \vee a = (a \wedge 1) \vee (a \wedge 1),$$

by (6), we have

$$a \vee a = (a \vee a) \vee (a \vee a),$$

if we regard $a \vee a$ as a', we have

$$a' = a' \vee a',$$

again by (6), we have

$$a \wedge 1 = a. \quad (7)$$

Set $d = 1$ in M_{21}, by M_{22}, we have

$$a \wedge ((b \wedge d) \vee (a \wedge (c \wedge d))) = a \wedge ((b \wedge 1) \vee (a \wedge (c \wedge 1))) = a \wedge (b \vee (a \wedge c)),$$

$$(d \wedge (c \wedge a)) \vee (d \wedge (b \wedge a)) = (1 \wedge (c \wedge a)) \vee (1 \wedge (b \wedge a)) = (c \wedge a) \vee (b \wedge a),$$

so from the above, we know $a \wedge (b \vee (a \wedge c)) = (c \wedge a) \vee (b \wedge a)$, i.e., equation M_{11} holds.

From the proof of Theorem 2.1, we know that M_{11} and M_{13} yield that idempotent law, absorptive law and commutative law hold. Set $c = b$ in M_{21}, we have

$$a \wedge ((b \wedge d) \vee (a \wedge (c \wedge d))) = a \wedge ((b \wedge d) \vee (a \wedge (b \wedge d))) = a \wedge (b \wedge d),$$

$$(d \wedge (c \wedge a)) \vee (d \wedge (b \wedge a)) = (d \wedge (b \wedge a)) \vee (d \wedge (b \wedge a)) = (a \wedge b) \wedge d.$$

Then we know $a \wedge (b \wedge d) = (a \wedge b) \wedge d$, i.e., equation M_{12} holds.

From the above, equation (2) holds. So we can conclude that definition of two conditions is equivalent with the definition of three conditions. This completes the proof.

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