Advances and Applications in Fluid Mechanics
Volume 6, Issue 1, 2009, Pages 81-101
This paper is available online at http://www.pphmj.com

FLOW OVER SHARP CRESTED WAVES ON DEEP WATER. PART 1: THEORY

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Abstract

Croft and Sajjadi [3] successfully constructed a mathematical model for separated laminar flow over a wavy boundary with sharp crests. In order to adapt this model to sharp-crested water waves the dynamic condition at the surface has to be satisfied. Thus, the aim of this paper is to show how the dynamic condition can be applied to problems involving sharp-crested water waves. To illustrate the technique, the simpler problem of unseparated flow will be considered. The method consists of using a new non-orthogonal coordinates system (ξ, η) where $\eta = 0$ is the sharp-crested boundary. A separable solution is obtained in the new coordinate system for the water which satisfies the kinematic condition at the interface exactly. The velocity potentials φ_a, φ_w and $(\boldsymbol{\nabla}\varphi_a)^2, (\boldsymbol{\nabla}\varphi_w)^2$, where a and w refer to air and water respectively, together with the equation of the interface, are expressed as a Fourier series. The dynamic condition at the interface enables the phase speed for each mode to be calculated.

1. Introduction

The mechanism involved in the generation of wave by wind, for the air-sea interface, has received much attention in recent years. One of

2000 Mathematics Subject Classification: 76B15.
Keywords and phrases: sharp-crested water waves, unseparated flow, Fourier series.
Communicated by K. K. Azad
Received April 3, 2009

the earliest mechanisms is the sheltering hypothesis by Jeffreys [4], which states the air-flow over a wave separates somewhere on the downwind side of the crest and reattaches on the upwind face of the next crest. This causes a pressure asymmetry with respect to the wave crest which results in wave growth.

However, as was surmised by Barnett and Kenyon [1], 'Jeffreys theory may yet emerge as being important since most recent theories (though not completely evaluated yet) based on perturbation techniques has not yet yielded the major growth mechanism for wind waves'.

More recently, Croft and Sajjadi [3] constructed a model for the flow of a high speed wind over a rigid periodic boundary with sharp crests, where the undisturbed flow was perpendicular to the crests. In that model, the flow separated at each crest creating a region of vortical flow on the leeward side of each wave. This vortical flow was modelled by means of a line vortex parallel to the crest and situated below it. The theory predicted the strength and position of each vortex and the resulting flow field.

In the present contribution, Croft and Sajjadi's model is extended for the case of the laminar flow of air over sharp crested water waves when the undisturbed wind speed is sufficiently high for separation to take place at each crest. This additionally requires the dynamic condition at the free surface to be satisfied.

In this paper, a strategy for linking the free surface dynamic condition to the earlier work of Croft and Sajjadi will be established. Here we shall therefore consider the simpler problem of laminar flow over the sharp crested wave without separation. The calculation details together with the more complex case of separated flow will be given in the part two of this paper.

The novel feature of the present method is the use of a non-orthogonal coordinate system (ξ, η) which is described in detail in Section 2. In this coordinate system the equation of the free surface is given by $\eta = 0$ which means that the boundary conditions on the interface are satisfied exactly. This is, of course, in contrast to the standard approach of working in Cartesian coordinates (x, y) and using a perturbation scheme, together with a Taylor expansion about $y = 0$, which effectively makes $y = 0$ the boundary. Also, in such an approach, the ordering parameter ε, which is usually a measure of the wave slope will not be small in the neighbourhood of the sharp crests. The new coordinate system allows the kinematic condition for the water to be satisfied exactly. However, by expressing the terms in the dynamic condition as Fourier series, we shall be able to satisfy this boundary condition to an order of accuracy which will depend upon the number of harmonics taken.

Another objective of our analysis is related to the stability of sharp crested water waves. In a pioneering work Benjamin and Feir [2] discussed the side-band instability of Stokes waves on deep water. In that work they represented the free surface by the first two harmonics of Stokes wave which consequently meant that their free surface was not represented by the sharp crest which is usually associated with a Stokes wave. In addition the effect of air flow over the water wave was not considered. Our ultimate goal is, therefore, to undertake a similar study using the present model. Since we are considering air flow over sharp crested water waves, there is the potential for offering better explanation for the stability of such waves in more physically realistic situations.

2. Equations of Motion

Following Miles [6], the Euler and continuity equations for a viscous incompressible fluid may be written as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}, \qquad (2.1a)$$

$$\frac{\partial u_i}{\partial x_j} = 0, \qquad (2.1b)$$

where x_i denotes a Cartesian coordinate, u_i is a velocity component, p_{ij} is a stress tensor component, ρ is the fluid density. We decompose the velocity and stress tensor according to

$$u_i = U_i + u_i' + u_i'', \qquad p_{ij} = P_{ij} + p_{ij}' + p_{ij}'', \qquad (2.2a,b)$$

where $U_i + u_i'$ and $P_{ij} + p_{ij}'$ represent a solution to (2.1a,b) having two dimensional (x_1 and x_2) mean values (with respect to either x_3 or t) U_i and P_{ij} plus turbulent fluctuations u_i' and p_{ij}', and u_i'' and p_{ij}'' representing a small perturbation with respect to this solution. Substituting (2.2a,b) into (2.1a), neglecting terms of second order in perturbation flow, and invoking the requirement that the unperturbed flow satisfy (2.1a), we obtain

$$\frac{\partial u_i''}{\partial t} + (U_j + u_j')\frac{\partial u_i''}{\partial x_j} + u_j''\frac{\partial}{\partial x_j}(U_i + u_i') = \frac{1}{\rho}\frac{\partial p_{ij}''}{\partial x_j}, \qquad (2.3a)$$

$$\frac{\partial u_i''}{\partial x_j} = 0. \qquad (2.3b)$$

Taking mean values with respect to x_3, we may place the result in the form

$$\frac{\partial \overline{u_i''}}{\partial t} + U_j \frac{\partial \overline{u_i''}}{\partial x_j} + \overline{u_j''} \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} (\overline{p_{ij}''} - \overline{r_{ij}''}), \tag{2.4a}$$

$$\frac{\partial \overline{u_i''}}{\partial x_j} = 0, \tag{2.4b}$$

where we have introduced the perturbation Reynolds stress, by invoking the equations of continuity for both u_i' and u_i'' in its derivation

$$\overline{r_{ij}''} = \rho(\overline{u_i' u_j''} - \overline{u_j' u_i''}). \tag{2.5}$$

The system (2.4a,b) represent equations of motion governing the two-dimensional perturbed shear flow. Note that they differ from the equations of motion for a laminar perturbation flow only in the presence of the $\overline{r_{ij}''}$ which represent the interaction between the fluctuations in the original and perturbation flows. Note further $\overline{r_{ij}''}$ are the first order perturbations of the usual Reynolds stresses $\rho\overline{u_i' u_j'}$.

Now taking $x_1 = x, x_2 = y, U_1 = U(y), U_2 = U_3 = 0, \overline{u_1''} = u, \overline{u_2''} = v, \overline{p_{ij}''} = -\delta_{ij} p$, and $\overline{r_{ij}''} = 0$, and assuming an inviscid fluid, we obtain

$$\rho(u_t + U u_x + v U_y) = -p_x, \tag{2.6a}$$

$$\rho(v_t + U v_x) = -p_y, \tag{2.6b}$$

$$u_x + v_y = 0. \tag{2.6c}$$

Introducing the velocity potential φ, under the assumption that the flow is irrotational, $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$, such that $\boldsymbol{u} = \boldsymbol{\nabla}\varphi$, equations (2.6) reduces to

$$\varphi_{xx} + \varphi_{yy} = 0. \tag{2.7}$$

3. Coordinate Systems

We follow the earlier work of Croft and Sajjadi [3] and map four consecutive sharp-crested waves in the z-plane into the real axis of the t-plane by a series of conformal transformations. Thus, the airflow is mapped into the upper half of the t-plane.

Croft and Sajjadi [3] showed the line vortex in the lee of each wave, which they used to model the vortical flow generated by the separated flow, is equivalent to a simple image system in the t-plane. Thus, they obtained the velocity potential representing the separated flow over the four waves.

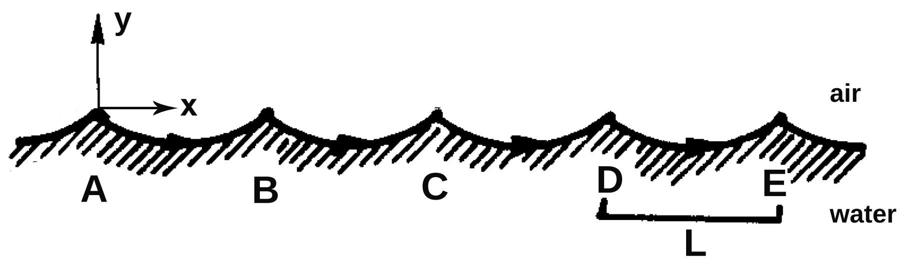

Figure 1. The schematic diagram showing the sharp-crested wave.

The first conformal transformation used by Croft and Sajjadi was that given by Longuet-Higgins [5] for which he derived the Cartesian equation of a single wave depending upon how many waves is present in the wave-train. In the present case we consider four waves, and following Longuet-Higgins [5], we take the following equation

$$\lambda(y - y_0) = \ln\sec[\lambda(x - x_0) - n\pi/2], \tag{3.1}$$

where

$$\lambda y_0 = -\ln\sqrt{2}, \quad x_0 = \frac{\pi}{4\lambda}, \quad \lambda = \frac{\pi}{2L}, \quad n = 0, 1, 2, 3$$

to represent the sharp crested waves. Note that, the four values of n correspond to the arcs AB, BC, CD and DE, respectively, see Figure 1.

In the present analysis, we shall consider waves on infinitely deep water and introduce a new coordinate system (ξ, η) such that $\eta = 0$ represents the surface of the wave. Even though only four consecutive waves are considered, periodicity implies that $\eta = 0$ represents an infinite wavetrain. Therefore, in the (ξ, η) plane, the ξ-axis represents the entire wave surface. We shall define η equal to a positive constant as a line parallel to the ξ-axis in the upper half plane, while negative values will pertain to the lower half of the plane. The two coordinate systems are shown in Figure 2.

From (3.1) η is, defined by

$$\eta = y - y_0 - \frac{1}{\lambda}\ln\sec[\lambda(x - x_0) - n\pi/2], \quad n = 0, 1, 2, 3 \tag{3.2}$$

with $\eta = 0$ represents the wave surface.

If $\xi =$ constant is the family of curves orthogonal to $\eta =$ constant, then

$$\xi = y\tan[\lambda(x - x_0) - n\pi/2] + f(x), \tag{3.3}$$

where $f(x)$ is an arbitrary function of x.

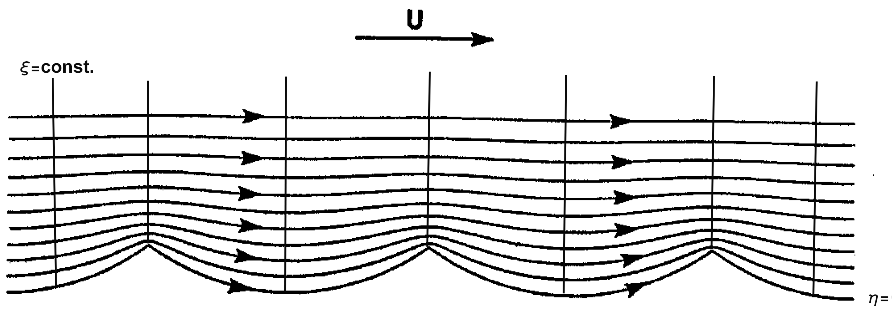

Figure 2. The two system of coordinates.

The two variables ξ and η must then satisfy the following relationship

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = 1. \tag{3.4}$$

Integrating (3.4) with respect to x, we get

$$\xi = x + F(y),$$

where $F(y)$ is an arbitrary function of y. Hence, for (3.3) to satisfy the condition (3.4) we must set $F(y) = 0$, which yields $\eta = \xi = 0$ at the origin. Thus, we see that

$$\xi = x \tag{3.5}$$

which identifies the two families of curves given by (3.2) and (3.3) as non-orthogonal.

4. Induced Velocity Field in the Water

The model we consider here is that of a uniform inviscid wind flowing over an infinite flat sheet of water. As the wind speed increases a wave train is established and propagates along with a speed c in the direction of the wind. One effect of a high speed wind is characterized by increasing the curvature of the wave at the crest. This is idealized in the present model by waves having established a sharp crest. Here, we shall assume the ratio of the wavelength to the height of the wave is approximately 0.2, thereby creating perturbations both in the airflow and in the water. The purpose of this paper is therefore to ascertain the velocity potential for the water on the basis that the velocity potential for the airflow is known. In fact, the velocity potential for the airflow is the same as that found by Croft and Sajjadi [3] for the rigid sharp crested wavy surface.

Consider now the set of axes shown in Figure 1 moving with speed c in the direction of the positive x-axis. Then relative to such axes, the surface of water waves may be assumed fixed. Now, if the velocity potential for the airflow is denoted by φ_a, then the kinematic condition at the interface may be expressed as

$$\left(c + \frac{\partial \varphi_a}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{\partial \varphi_a}{\partial y} \tag{4.1}$$

with a similar expression for φ_w, where φ_w is the velocity potential for the water.

In order to satisfy the boundary condition (4.1) on the free surface, the normal practice requires a perturbation of velocity potential in powers of ε, where ε is a measure of the wave slope, followed by a Taylor expansion about $y = 0$. Thus, the velocity potential for the water is obtained as a series in ε whose coefficients are separable solutions of Laplace's equation. This methodology enables the velocity potential for the water to be obtained up to the required order in ε.

However, in the present contribution, the undisturbed free surface is transformed into the (ξ, η) coordinates (discussed in the previous section) to the line $\eta = 0$ as shown in Figure 3. Thus any difficulties in satisfying the boundary condition (4.1), which may be associated with the sharp crests of the undisturbed free surface, are removed. Furthermore, the subsequent stability analysis will be more tractable as a result of perturbing a flat surface rather than a wavy surface having sharp crests.

We shall further assume at some significant height above the waves the airflow is a uniform stream parallel to the x-axis, and that for the water the liquid is at rest at an infinite depth. Thus, for the perturbed airflow, in the present case, this uniform flow may simply be subtracted from the solution obtained by Croft and Sajjadi [3].

In order to obtain the velocity potential for the water, we require a solution to Laplace's equation in the new non-orthogonal (ξ, η) coordinate system, which may be expressed in tensor form as

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j}\right) = 0, \tag{4.2}$$

where in the new coordinate system $x^1 = \xi$ and $x^2 = \eta$.

The covariant components of the metric tensor are given by

$$g_{rs} = \frac{\partial x^m}{\partial y^r}\frac{\partial x^m}{\partial y^s},$$

where the x's and y's are the Cartesian and curvilinear coordinates respectively. Also

$$g = J^{\mathsf{T}} J,$$

with J representing the Jacobian of the transformation.

Substituting from (3.2) and (3.5) we obtain

$$g = \begin{vmatrix} \sec^2[\lambda(\xi - x_0) - n\pi/2] & \tan[\lambda(\xi - x_0) - n\pi/2] \\ \tan[\lambda(\xi - x_0) - n\pi/2] & 1 \end{vmatrix} = 1.$$

Using $g^{rs} = G^{rs}/g$, where G^{rs} is the cofactor of g_{rs} in g, and replacing f by φ_w in (4.2), we obtain

$$\nabla^2 \varphi_w = \frac{\partial^2 \varphi_w}{\partial \xi^2} - \lambda \sec^2[\lambda(\xi - x_0) - n\pi/2] \frac{\partial \varphi_w}{\partial \eta},$$

$$-2\tan[\lambda(\xi - x_0) - n\pi/2] \frac{\partial^2 \varphi_w}{\partial \xi \partial \eta} + \sec^2[\lambda(\xi - x_0) - n\pi/2] \frac{\partial^2 \varphi_w}{\partial \eta^2} = 0 \quad (4.3)$$

as the governing equation for the flow field in the water.

Note that, in a perturbation expansion (of φ_w) in Cartesian coordinates the disturbance decays exponentially from the interface and vanishes as $y \downarrow -\infty$. Thus, the lowest order (in ε) for y-variation in the velocity potential for the water is proportional to e^{my}, where m is the wavenumber and y is measured vertically upwards. Since $\lambda = \pi/2L$, and we are considering a group of four waves, each of wavelength L, then it is reasonable to assume a trial solution to $\nabla^2 \varphi_w = 0$ in the form

$$\varphi_w = e^{\lambda \eta} F(\xi), \quad (4.4)$$

for $-\infty < \eta < 0$ for the water, since it is situated in the lower half of the (ξ, η) plane.

Substituting (4.4) into (4.1) we obtain

$$\frac{d^2 F}{d\xi^2} - 2\lambda \tan[\lambda(\xi - x_0) - n\pi/2] \frac{dF}{d\xi} = 0, \quad (4.5)$$

whose solution may be expressed as

$$F(\xi) = E \tan[\lambda(\xi - x_0) - n\pi/2] + D,$$

where D and E are arbitrary constants. Now, due to the periodicity of the interface we expect φ_w to be periodic also, therefore we put D equal to zero. Thus, (4.4) becomes

$$\varphi_w = E e^{\lambda \eta} \tan[\lambda(\xi - x_0) - n\pi/2]. \quad (4.6)$$

We expect the velocity field in the water is composed of an infinite sum of such terms. Thus we generalize the above analysis by replacing η with η/m, where m is a positive integer, in (4.6) and sum over m. Hence, we have [1]

$$\varphi_w = \sum_{m=1}^{\infty} E_m e^{\lambda\eta/m} \tan[\lambda(m\xi - x_0) - n\pi/2].$$

Note that, by virtue of the above generalization $x = m\xi$. Furthermore, we remark that a line $\eta =$ const. will still represent a streamline.

Next, from (4.6) we see that

$$\left.\frac{\partial\varphi_w}{\partial\eta}\right|_{\eta=0} = \lambda E \tan[\lambda(\xi - x_0) - n\pi/2],$$

and referring to (3.1), we see that the velocity component normal to the interface is zero at each trough and is maximum at the crest.

For the remainder of the analysis we shall assume a fixed set of axes (rather than those which move with the wave). Accordingly (4.6) becomes

$$\varphi_w = E e^{\lambda\eta} \tan[\lambda(\xi - ct - x_0) - n\pi/2]. \tag{4.7}$$

It is to be noted that this modification does not affect any of the above analysis. Also, because the wavetrain is moving, then even though the interface is a streamline, in the sense that there is no fluid flowing across it, the component of velocity parallel to the normal at any point does not vanish.

Since the coordinate η is the stream function ψ, the normal component of velocity into the air at the interface is $\partial\psi/\partial\xi$. To match this to the normal component of velocity for the water we must have

$$\frac{\partial\psi}{\partial\xi} = -\left.\frac{\partial\varphi_w}{\partial\eta}\right|_{\eta=0}.$$

Using (3.2), with x replaced by $\xi - ct$, and (4.7) we find $E = 1/\lambda$. Hence, the velocity potential for the water becomes

$$\varphi_w = \frac{1}{\lambda} e^{\lambda\eta} \tan[\lambda(\xi - ct - x_0) - n\pi/2]. \tag{4.8}$$

[1] Although conceptually more satisfactory, this generalization poses a serious problem. In order to be able to utilize this form of solution the eigenvectors $E_m(\xi)$ must be orthogonal. Of course, this is not the case here since (4.5) cannot be cast into the standard Sturm-Liouville form. Thus, we are forced to use (4.6) as the solution governing the motion of the water.

It can be seen from (4.8) that both components of velocity for the water vanish as $\eta \downarrow -\infty$.

Thus, the benefit of using the non-orthogonal system of coordinates is that we can satisfy the interface kinematic condition exactly. The other interfacial boundary condition, namely the dynamic condition at the interface, will yield the dispersion relation (see Section 8).

5. Velocity Field in the Water

Following Croft and Sajjadi [3], we shall use a series of conformal transformation to map the interface in the physical z-plane into the real axis in the t-plane. However, in the present paper the interface in the physical z-plane is assumed to be a sharp crested water wave. Thus, in addition to the kinematic condition that was automatically satisfied for a sharp crested rigid wavy surface in Croft and Sajjadi's model, the interface dynamic condition must also be satisfied.

Since in this paper we shall confine ourselves to the case of the unseparated flow over a sharp crested water waves, we would expect that the velocity potential for the airflow may be obtained by a method similar to that used for the water.

Thus, using (4.5) the equation satisfied for the velocity potential for airflow above the wave may be expressed as

$$\frac{\partial^2 \varphi_a}{\partial \xi^2} - \lambda \sec^2[\lambda(\xi - x_0) - n\pi/2] \frac{\partial \varphi_a}{\partial \eta}$$

$$-2 \tan[\lambda(\xi - x_0) - n\pi/2] \frac{\partial^2 \varphi_a}{\partial \xi \partial \eta} + \sec^2[\lambda(\xi - x_0) - n\pi/2] \frac{\partial^2 \varphi_a}{\partial \eta^2} = 0, \quad (5.1)$$

where φ_a is the velocity potential for the airflow due to the perturbation effect of the interface.

We seek a solution of (5.1) in the form

$$\varphi_a = e^{-\lambda \eta} F(\xi). \quad (5.2)$$

Note that, $\varphi_a = \varphi_w$ at $\eta = 0$ which indicates the condition for the continuity of velocities at the interface is satisfied.

Substituting (5.2) into (5.1), we get

$$\frac{d^2 F}{d\xi^2} + 2\lambda \tan[\lambda(\xi - x_0) - n\pi/2] \frac{dF}{d\xi} + 2\lambda^2 \sec^2[\lambda(\xi - x_0) - n\pi/2] F = 0$$

or

$$\frac{d^2 F}{d\xi^2} + 2\lambda \frac{d}{d\xi} \left\{ \tan[\lambda(\xi - x_0) - n\pi/2] F \right\} = 0.$$

Solving this equation and using (5.2), we obtain

$$\varphi_a = e^{-\lambda\eta}\left\{A\sin[2\lambda(\xi - x_0) - n\pi/2] + B\cos^2[\lambda(\xi - x_0) - n\pi/2]\right\},$$

where A and B are arbitrary constants. The solution for φ_w is periodic with period $\lambda(\xi - x_0) - n\pi/2$ and since φ_a must have the same periodicity we take A to be zero. Thus

$$\varphi_a = Be^{-\lambda\eta}\cos^2[\lambda(\xi - x_0) - n\pi/2]. \tag{5.3}$$

Using (5.3), with ξ replaced with $\xi - ct$, we see that

$$\frac{\partial\psi}{\partial\xi} \neq -\left.\frac{\partial\varphi_a}{\partial\eta}\right|_{\eta=0}.$$

Thus, φ_a, given by (5.3), neither satisfies the interfacial kinematic boundary condition nor the condition of a uniform stream as $\eta \uparrow \infty$. To circumvent these difficulties, we must revert to the method of conformal mapping outlined by Croft and Sajjadi [3]). To do this it would be appropriate here to give a brief outline of their methodology.

6. Method of Conformal Mapping

The physical plane, in which we consider the four crests A, B, C and D, is related to the w-plane by the transformation

$$w = e^{-i(\lambda z - \pi/4)}, \qquad \lambda L = \frac{\pi}{2}. \tag{6.1}$$

This transformation maps the four crests into A_1, B_1, C_1 and D_1, respectively and preserves the angle at the crests. Note that, the arcs in the z-plane do not map into the corresponding sides of the square in the w-plane, but into arcs, joining the corners, the largest error being 6%. The region below the interface (water) maps into the interior and the region above the interface (air) maps into the exterior of the square in the w-plane, as shown in Figure 3.

The second transformation, given by

$$w = 2\int_0^t \frac{\sqrt{(\tau^2 - c^2)(\tau^2 - 1/c^2)}}{(\tau^2 + 1)^2}\, d\tau \tag{6.2}$$

maps the exterior of the square in the w-plane into the upper half of the t-plane, the vertices corresponding to $t = \pm c, t = \pm 1/c$ on the real axis. Thus the flow field above the wave profile, in the absence of flow

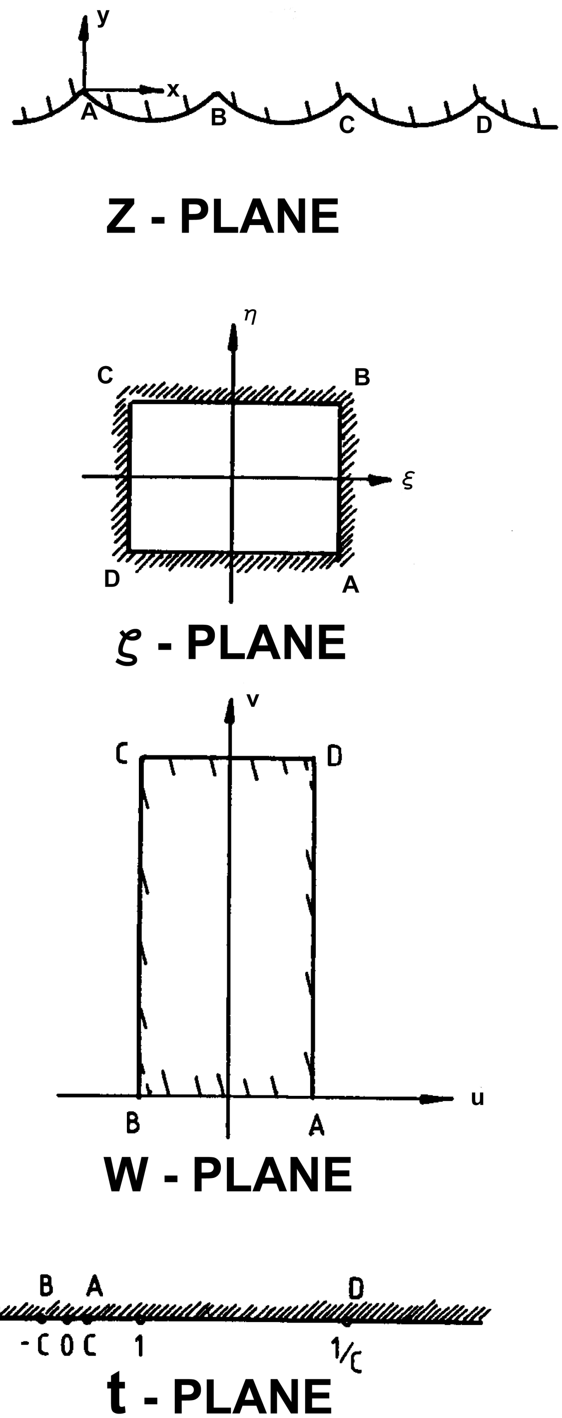

Figure 3. The conformal mappings which transforms the wave in the z-plane to a flat surface in the upper half of the t-plane.

separation, is transformed into a flow in the upper half of the t-plane whose streamlines are parallel to the real axis.

It can be shown that (see Croft and Sajjadi [3])

$$w = \frac{1}{k}\left[\mathrm{Zn}(\zeta) + \left(\frac{\mathsf{E}}{\mathsf{K}} - k'^2\right)\zeta\right], \tag{6.3}$$

where k' and K are the complementary modulus and the quarter period of the Jacobian elliptic function respectively, and Zn is the Jacobian zeta function of complex argument. In (6.3), K and E are given by

$$\mathsf{K} = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right) \quad \text{and} \quad \mathsf{E} = \frac{\pi}{4\mathsf{K}} + \frac{\mathsf{K}}{2}.$$

Finally, Croft and Sajjadi [3] found the following relation (in terms of Jacobian elliptic functions)

$$t = \frac{1 - \mathrm{dn}\,\zeta}{k\,\mathrm{sn}\,\zeta}, \tag{6.4}$$

which maps the ζ-plane to t-plane. This transformation maps the points $(\pm c, 0)$ and $(\pm 1/c, 0)$ in the t-plane into $(\pm\mathsf{K}, 0)$ and $(\pm\mathsf{K}, 2\mathsf{K}')$ in the ζ-plane, respectively. Here K' is the associate quarter period of the Jacobian elliptic function. Furthermore, this transformation maps the interior of the rectangle in the ζ-plane, having these points as vertices, into the upper half of the t-plane.

Now, if W denotes the complex potential, then

$$\frac{dW}{dz} = \frac{dW}{dt}\frac{dt}{dw}\frac{dw}{dz},$$

and if U_0 is the the speed of the flow in the t-plane, we have

$$W = -U_0 t.$$

Using the above relationships in conjunction with (6.1) and (6.2), we get

$$\frac{dW}{dz} = \frac{i\lambda U_0(t^2+1)^2 e^{-i(\lambda z - \pi/4)}}{2\sqrt{(t^2-c^2)(t^2-1/c^2)}}. \tag{6.5}$$

Note that the expression (6.5) is singular when $t = \pm c$ and $t = \pm 1/c$. These points in the t-plane correspond to the points into which the crest of the four waves map. Thus, from (6.5) we see that the speed of the wind at the crest is infinite which of course is not physically acceptable. However, in the present model, this does not pose any serious problem.

In the case of separated flow, discussed by Croft and Sajjadi [3], the sharp crests were used simply to locate the points of separation in the t-plane. They showed that the points $(\pm c, 0)$ and $(\pm 1/c, 0)$ on the real axis in the t-plane are indeed the stagnation points. This led to four equations which were used, together with the equilibrium of the vortices, to obtain the location of the vortices, together with their strengths, in the t-plane.

7. Relationship between U and U_0

It now remains to obtain the relationship between the undisturbed free stream speed U in the physical plane and the corresponding speed U_0 in the t-plane. In the physical plane, as $y \to \infty$, the flow is a uniform stream flowing in the positive x-direction with speed U, therefore we may write

$$W \sim -Uz,$$

where W is the complex potential. Using (6.1) we then have

$$\frac{dW}{dw} \sim -\frac{iU}{\lambda w}. \tag{7.1}$$

As was explained in Section 6, the region above the wave maps into the exterior of the square in the w-plane, the four waves transforming into the square itself. Therefore, as $y \to \infty$ the corresponding region in the w-plane moves farther away from the square. The asymptotic form of dW/dw in (7.1) shows that the flow is that of a vortex located at the origin in the w-plane. In other words, the streamlines associated with the undisturbed flow in the physical plane map into concentric circles having their centres at $w = 0$.

Consider now the relationship between the flow for large $|w|$ and that in the upper half of the t-plane. For large t (6.2) behaves like $w \sim 1/t$, which implies that w is small. On the other hand, small t yields $w \sim t$, which again implies w is small. However, since the region in the immediate neighbourhood of $t = i$ corresponds to large w, we therefore need to consider the limit as $t \to i$ such that the ratio of U/U_0 remains a finite non-zero constant.

Let $\tau = i + w$, where $|w| \ll 1$, and substituting it into (6.2) yields

$$w = -\frac{1}{2}\sqrt{(c^2+1)\left(\frac{1}{c^2}+1\right)}\lim_{t\to i}\int_0^t \frac{d\tau}{w^2}.$$

As $t \to i$ the domain around $t = i$ shrinks such that the variation of the integrand becomes negligible, and therefore we may approximate the above integral as

$$w = -\frac{1}{2}\sqrt{(c^2+1)\left(\frac{1}{c^2}+1\right)}\lim_{w\to 0}\frac{1}{w^2}\lim_{t\to i}\int_0^t d\tau,$$

and thence, we obtain

$$w = -\frac{i}{2}\sqrt{(c^2+1)\left(\frac{1}{c^2}+1\right)}\lim_{w\to 0}\frac{1}{w^2}. \tag{7.2}$$

Also, as

$$\frac{dW}{dt} = \frac{dW}{dw}\frac{dw}{dt},$$

then using (7.1), we have asymptotically

$$\frac{dW}{dt} = -\lim_{w\to 0}\frac{iU}{\lambda w}\lim_{t\to i}\frac{dw}{dt}.$$

Combining (7.2) and (6.2), we have

$$\frac{dW}{dt} = \frac{2U}{\lambda}\lim_{w\to 0}\left\{\frac{w^2}{\sqrt{(c^2+1)\left(\frac{1}{c^2}+1\right)}}\,\frac{2\sqrt{(c^2+1)\left(\frac{1}{c^2}+1\right)}}{-4w^2}\right\}, \tag{7.3}$$

whence

$$\frac{dW}{dt} = -\frac{U}{\lambda},$$

and since $W = -U_0 t$, we obtain the desired relationship between U in the z-plane and its corresponding value, U_0, in the t-plane, namely

$$U_0 = \frac{U}{\lambda}. \tag{7.4}$$

8. Calculation of the Fourier Coefficients

In Section 4 we outlined a methodology for satisfying the interfacial dynamic boundary condition which required a number of Fourier coefficients to be calculated.

We begin with the Fourier coefficients associated with the airflow, and in particular, the Fourier series representation of q_a^2, where q is the speed and the suffix a denoting air.

Referring to (6.5) we see that

$$\frac{dW}{dz} = \frac{i\lambda U_0(t^2+1)^2 e^{\lambda y} e^{-i(\lambda x - \pi/4)}}{2\sqrt{(t^2-c^2)(t^2-1/c^2)}}.$$

Now since

$$q_a^2 = \frac{dW}{dz}\frac{dW}{d\overline{z}},$$

where the overbar symbol indicates the complex conjugate, then using (7.4) we have

$$q_a^2 = \frac{U^2(t_r+1)^4 e^{2\lambda y}}{4(t_r^2-c^2)(t_r^2-1/c^2)}, \tag{8.1}$$

where in general $t = t_r + it_i$. The above result has been derived from Croft and Sajjadi [3] whom considered a rigid boundary.

Since the speed of undisturbed free stream, U, is the same for the interface $\eta = 0$, from (3.1), and noting that $x = \xi$ in the new coordinate system, (8.1) may be cast as

$$q_a^2 = \frac{U^2(t_r+1)^4 \sec^2[\lambda(\xi - x_0) - n\pi/2]}{8(t_r^2-c^2)(t_r^2-1/c^2)}. \tag{8.2}$$

The main difficulty that now arises is the inverse transformation from the t-plane to the physical z-plane. This, in general, is a numerical task which we will report in part 2 of this paper. For now, if we assume that this inverse transformation for the fraction that consists of t_r in (8.2) is denoted by $\mathscr{G}(\xi)$, i.e.,

$$\mathscr{G}(\xi) \leftarrow \frac{(t_r+1)^4}{(t_r^2-c^2)(t_r^2-1/c^2)},$$

then (8.2) may be written as

$$q_a^2 = U^2\mathscr{G}(\xi)\sec^2[\lambda(\xi - x_0) - n\pi/2].$$

We emphasize that this expression is the value of q_a^2 at any point on the interface. As was mentioned in Section 3, the interface is moving with speed c, thus the above expression must be modified accordingly, i.e.,

$$q_a^2 = U^2\mathscr{G}(\xi - ct)\sec^2[\lambda(\xi - ct - x_0) - n\pi/2]. \tag{8.3}$$

Expressing (8.3) as a complex Fourier series we have

$$U^2\mathscr{G}(\xi - ct)\sec^2[\lambda(\xi - ct - x_0) - n\pi/2] = \sum_{p=-\infty}^{\infty} e_p \exp\left[\frac{ip\pi}{2L}(\xi - c_p t)\right], \tag{8.4}$$

where coefficients e_p are given by

$$e_p = \frac{U^2}{2L}\int_0^{4L} \mathscr{G}(\xi)\sec^2[\lambda(\xi - x_0) - n\pi/2]\exp\left(-\frac{ip\pi}{2L}\xi\right)\,d\xi. \tag{8.5}$$

We now focus our attention to the Fourier series representation of the velocity potential for the airflow, namely φ_a. Since the real axis in the t-plane corresponds to the interface in the physical plane, and that

$$W = -Ut,$$

then φ_a, evaluated on the interface, gives

$$\varphi_a = -\lambda U t_r.$$

Similarly, by the same argument leading to (8.3), the velocity potential for the air becomes

$$\varphi_a = -\lambda U \mathscr{H}(\xi - ct), \tag{8.6}$$

where $\mathscr{H}(\xi - ct)$ is an appropriate inverse transformation from the t-plane to the z-plane. Expressing (8.6) as a complex Fourier series, we have

$$-\lambda U \mathscr{H}(\xi - ct) = \sum_{p=-\infty}^{\infty} d_p \exp\left[\frac{ip\pi}{2L}(\xi - c_p t)\right], \tag{8.7}$$

with the coefficients given by

$$d_p = -\frac{\lambda U}{2L}\int_0^{4L} \mathscr{H}(\xi)\exp\left(-\frac{ip\pi}{2L}\xi\right)\,d\xi.$$

We next consider the corresponding Fourier series for the fluid below the interface. In this case we need to obtain the gradient and scalar product of φ_w. In tensor notation, the velocity field in the water is expressed in terms of the potential φ_w by

$$\boldsymbol{q}_w = -\left(\frac{\partial\varphi_w}{\partial\xi}, \frac{\partial\varphi_w}{\partial\eta}\right) = -\left(\frac{\partial\varphi_w}{\partial y^j}\right).$$

Note that, this is a covariant tensor, therefore with the usual notation, the contravariant tensor is $g^{ij}\partial\varphi_w/\partial y^j$. Using the outer product, given by

$$g^{ij}\frac{\partial\varphi_w}{\partial y^j}\frac{\partial\varphi_w}{\partial y^j},$$

and contracting the indicies i and j by putting $i = j$ we obtain the inner product

$$g^{ii}\left(\frac{\partial\varphi_w}{\partial y^i}\right)^2 = g^{11}\left(\frac{\partial\varphi_w}{\partial\xi}\right)^2 + g^{22}\left(\frac{\partial\varphi_w}{\partial\eta}\right)^2 = q_w^2.$$

Since $g^{11} = 1$ and $g^{22} = \sec^2[\lambda(\xi - ct - x_0) - n\pi/2]$ (see Section 4), replacing ξ by $\xi - ct$, using (4.8), and letting $X = \lambda(\xi - ct - x_0) - n\pi/2$, we obtain

$$q_w^2 = e^{2\lambda\eta}\left\{\sec^4(X) + \sec^2(X)\tan^2(X)\right\}$$

or

$$q_w^2 = \left\{1 + \tan^2(X)\right\}\left\{1 + 2\tan^2(X)\right\}. \tag{8.8}$$

Expressing (8.8) as a complex Fourier series, we have

$$\left\{1 + \tan^2(X)\right\}\left\{1 + 2\tan^2(X)\right\} = \sum_{p=-\infty}^{\infty} b_p \exp\left[\frac{ip\pi}{2L}(\xi - c_p t)\right], \tag{8.9}$$

where

$$b_p = \frac{1}{2L}\int_0^{4L}\{1 + 3\tan^2 Y\tan^4 Y\}\exp\left(-\frac{ip\pi}{2L}\xi\right)\,d\xi, \tag{8.10}$$

and $Y = \lambda(\xi - x_0) - n\pi/2$.

Similarly from (4.8) and the argument presented in Section 4 we have, at the interface,

$$\varphi_w(\xi, 0) = \lambda^{-1}\tan[\lambda(\xi - ct - x_0) - n\pi/2] = \sum_{p=-\infty}^{\infty} a_p \exp\left[\frac{ip\pi}{2L}(\xi - c_p t)\right], \tag{8.11}$$

where

$$a_p = \frac{1}{2\lambda L}\int_0^{4L}\tan[\lambda(\xi - x_0) - n\pi/2]\exp\left(-\frac{ip\pi}{2L}\xi\right)\,d\xi. \tag{8.12}$$

Although we have transformed the flow field into the (ξ, η) plane, by stretching the interface $\eta = 0$ along the ξ-axis, we emphasize that this is a purely kinematic process thereby allowing the flow field for the water to be obtained in a closed form and simultaneously satisfying the interface kinematic condition exactly. However, the strategy used above cannot be applied to the dynamic condition due the presence of the gravitational term. We circumvent this by expressing y, that appears in the gravity term [see equation (9.1) below], in terms of ξ from (3.2). Thus, putting

$\eta = 0$ in (3.2) and referring to Section 4, we have, upon replacing ξ with $\xi - ct$,

$$y = y_0 + \lambda^{-1} \ln \sec[\lambda(\xi - ct - x_0) - n\pi/2] = \sum_{p=-\infty}^{\infty} a_p \exp\left[\frac{ip\pi}{2L}(\xi - c_p t)\right], \tag{8.13}$$

where

$$a_p = \frac{1}{2L}\int_0^{4L} \{y_0 + \lambda^{-1} \ln \sec[\lambda(\xi - x_0) - n\pi/2]\} \exp\left(-\frac{ip\pi}{2L}\xi\right) d\xi. \tag{8.14}$$

Since the dynamic condition has to be applied in the physical plane, we need to determine whether the surface tension can be neglected. Considering for clarity of argument only one arc of the four consecutive waves as a stationary curve then we see from (3.2) that the curvature at the trough is zero, while as $\xi \to L-$, the curvature $\to \pi/2L$. If the surface tension were to be included in the interface dynamic condition then the curvature would have to be multiplied by T/ρ, where T is the surface tension and ρ the density. Typically this factor is of $O\,(10^{-5})$. On the other hand, (8.8) shows that as $\xi \to L-$, then $\frac{1}{2}q_w^2 \to 4$, therefore invoking this argument, we can justify neglecting the surface tension from the interfacial dynamic boundary condition.

9. Dynamic Condition and Dispersion Relation

In Cartesian coordinates the dynamic condition at the interface is,

$$\rho_w \left(\frac{\partial \varphi_w}{\partial t} - \frac{1}{2}q_w^2 - gy\right) = \rho_a \left(\frac{\partial \varphi_a}{\partial t} - \frac{1}{2}q_a^2 - gy\right), \tag{9.1}$$

where suffices w and a refer to water and air, respectively and g is the acceleration due to gravity.

The strategy for dealing with (9.1) is to express each term as a Fourier series. This can be justified on the grounds that a more general solution for the velocity potential can be written in the form

$$\varphi = \int_{-\infty}^{\infty} F(K) \exp[iKx - iW(K)]\, dK,$$

where $W(K)/K = c_q$ is the phase speed, being a superposition of elementary solutions. This can be justified *a priori* since the Fourier transform can be obtained from Fourier series by a limiting process. We shall use the complex form of the Fourier series, and assign a phase speed to each harmonic due to the dispersive nature of the system. In (9.1) y will be obtained from (3.2) with η equated to zero and x replaced by $\xi - ct$. Therefore, the dynamic condition, for each harmonic, is satisfied and then the associate phase speed can be determined. Hence, we write

$$\left.\begin{aligned}
\varphi_w &= \sum_{p=-\infty}^{\infty} a_p \exp\left[i\frac{p\pi}{2L}(\xi - c_p t)\right]\\
q_w^2 &= \sum_{p=-\infty}^{\infty} b_p \exp\left[i\frac{p\pi}{2L}(\xi - c_p t)\right]\\
\varphi_a &= \sum_{p=-\infty}^{\infty} d_p \exp\left[i\frac{p\pi}{2L}(\xi - c_p t)\right]\\
q_a^2 &= \sum_{p=-\infty}^{\infty} e_p \exp\left[i\frac{p\pi}{2L}(\xi - c_p t)\right]\\
y &= \sum_{p=-\infty}^{\infty} f_p \exp\left[i\frac{p\pi}{2L}(\xi - c_p t)\right]
\end{aligned}\right\} \text{ on } \eta = 0,$$

where a_p, b_p, d_p, e_p and f_p are known coefficients.

Thus, as a result of satisfying the dynamic condition at the interface, the dispersion relation can be determined. Note incidentally, the c on the left hand side of (8.13) is the speed of the interface, while the c_p on the right hand side are the speeds of the constituent harmonics. Therefore, we may refer to c as the group velocity of the envelope. Note further, for a general progressive wavetrain $\varphi = ae^{i(Kx-nt)}$, the group velocity c_g is given by $c_g = dn/dK$.

10. Conclusions

The problem of unseparated flow over sharp-crested wave is considered. The method consists of using a non-orthogonal coordinates system (ξ, η) where $\eta = 0$ represents the sharp-crested boundary. A separable solution is obtained in the new coordinate system for the water which satisfies the kinematic condition at the interface exactly. The velocity potentials φ_a and φ_w together with the equation of the interface, are expressed as a Fourier series. Then by substituting these expressions in the dynamic condition at the interface the phase speed for individual harmonics are calculated. The detail of these calculations will be reported in the part 2 of this paper.

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