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A NOTE ON SOLUTION OF BLASIUS EQUATION BY FOURIER SERIES

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Abstract

In this paper, we solve the equation of Blasius through an expansion in Fourier series and the results are compared with the other methods of solution found in the literature. The results show that the classical expansion in Fourier series delivers a solution with very good accuracy and rapid convergence, considering only six terms in the Fourier series leads to error minor than 1% accuracy of initial slope.

Introduction

The Blasius equation is perhaps one of the most famous equations of fluid dynamics. This equation is obtained from the analysis of the fluid dynamic problem of an incompressible fluid that passes on a flat plate.

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Blasius [1] was first to solve this equation, making patching between two asymptotic solutions. Subsequently, several authors [3-8] have studied this equation, mainly for use as a test of precision for new algorithms and methods of solution for nonlinear differential equations with the boundary conditions. Despite the abundant literature on the Blasius equation, there are no reports on solutions through classical Fourier series. In this paper, we solve the Blasius equation through an expansion in classical Fourier series.

Analysis

The Blasius equation, which is a laminar viscous flow over a semi-infinite flat plate, is

$$f''' + \frac{1}{2} ff'' = 0 \tag{1}$$

with boundary conditions

$$f(\eta = 0) = 0, \quad f'(\eta = 0) = 0, \quad f'(\eta \to \infty) = 1. \tag{2}$$

We postulate the following solution for $f(\eta)$:

$$f(\eta) = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{k_i} \sin(k_i L) \cos(k_i \eta), \tag{3}$$

where the number of wavelengths k_i is given by $k_i = \frac{\pi}{2L}(2i - 1)$.

The boundary conditions become

$$\sum_{i=1}^{\infty} a_i = -1, \quad a_0 + \sum_{i=1}^{\infty} \frac{a_i}{k_i} \sin(k_i L) = 0. \tag{4}$$

Replacing (3) in (1), one has

$$\sum_{i=1}^{\infty} a_i k_i^2 \sin(k_i L) \sin(k_i \eta) - \frac{1}{2}\left(a_0 + \sum_{i=1}^{\infty} \frac{a_i}{k_i} \sin(k_i L) \cos(k_i \eta)\right) \sum_{i=1}^{\infty} a_i k_i \sin(k_i L) \cos(k_i \eta) = 0. \tag{5}$$

Multiplying equation (5) by $\sin(k_p \eta)$ and integrating from $\eta = 0$ to L, one obtains the following system of algebraic nonlinear equations:

$$k_p^2 a_p \sin(k_p L)\frac{L}{2} - \frac{1}{2}\left(a_0 \sum_{i=1}^{\infty} a_i A_{ip} + \sum_{i,\, j=1}^{\infty} a_i a_j B_{ij}^p\right) = 0, \tag{6}$$

where

$$A_{ip} = k_i \sin(k_i L)\int_0^L \cos(k_i \eta)\sin(k_p \eta)\, d\eta,$$

$$B_{ij}^p = \frac{k_j}{k_i}\sin(k_i L)\sin(k_j L)\int_0^L \cos(k_i \eta)\cos(k_j \eta)\sin(k_p \eta)\, d\eta.$$

Finally, the boundary conditions (4) and the set of equations (6) with $p = 1, \ldots, N - 2,$ form a set of N nonlinear algebraic equations for the coefficients of the Fourier series.

Numerical Solution

The numerical solution of the systems (4) and (6) was resolved with the usual method of Newton-Raphson, the solution is found through an iterative method for coefficients of the Fourier series, defining the vectors $\vec{a}^T = [a_0, a_1, \ldots, a_N]$ and $\vec{G}^T = [G_0, G_1, \ldots, G_N],$ where

$$G_0 = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{k_i}, \tag{7}$$

$$G_1 = 1 + \sum_{i=1}^{\infty} a_i \sin(k_i L), \tag{8}$$

$$G_{p-1} = k_{p-2}^2 a_{p-2}\frac{L}{2} - \frac{1}{2}\left(a_0 \sum_{i=1}^{\infty} a_i A_{ip-2} + \sum_{i,\, j=1}^{\infty} a_i a_j B_{ij}^{p-2}\right), \quad p = 3, \ldots, N. \tag{9}$$

So, the coefficients a_i of the Fourier series may recursively be determined by the following equation:

$$\vec{a}^{(n+1)} = \vec{a}^{(n)} - J^{-1}(\vec{a}^{(n)})\vec{G}(\vec{a}^{(n)}), \tag{10}$$

where J is the Jacobian matrix $J_{ij} = \dfrac{\partial G_i}{\partial a_j}$ and $\vec{a}^{(n+1)}$ represents the value of $\vec{a}$ in the $n + 1$ iteration. The iterations for $\vec{a}^{(n+1)}$ from equation (10) may be repeated until $\| \vec{a}^{(n+1)} - \vec{a}^{(n)} \| \le \varepsilon,$ for some prescribed error tolerance ε.

The elements of the Jacobian matrix are given by

$$J = \begin{bmatrix} 1 & \frac{\sin(k_1 L)}{k_1} & \cdot & \cdots & \frac{\sin(k_N L)}{k_N} \\ 0 & 1 & \cdot & \cdots & 1 \\ \sum_{i=1}^{\infty} a_i A_{i1} & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N-1} \\ \sum_{i=1}^{\infty} a_i A_{i2} & \gamma_{21} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \sum_{i=1}^{\infty} a_i A_{iN} & \gamma_{N-2,1} & \cdot & \cdots & \gamma_{N-2,N-1} \end{bmatrix}, \tag{11}$$

where

$$\gamma_{pr} = k_p^2 a_p \frac{L}{2} \delta_{pr} - \frac{1}{2}\left(a_0 A_{rp} + \sum_{j=1}^{\infty} a_j B_{rj}^p + \sum_{i=1}^{\infty} a_i B_{ir}^p \right). \tag{12}$$

Results

Equation (10) was resolved using an error tolerance $\varepsilon = 10^{-10}$, in this case equation (10) converges in no more than eight iterations. We impose the far field condition $(\eta \to \infty)$ at $L = 7$. The results for $f'(\eta)$, which are shown in Figure 1, compare very well, with a numerical solution of Blasius equation provided by Howarth [6]. In Figure 1, the points represented the numerical solution [6] and the solid line represented the Fourier series solution with only six terms $N = 6$.

Table 1 shows the initial slope $f''(0) = -\sum_{i=1}^{N} a_i k_i \sin(k_i L)$ obtained by the expansion in Fourier series for the different values of *N* and compared with a highly accurate numerical value obtained by Boyd [2] $(f''(0) = 0.33205733)$. The classical numerical value obtained for Howarth [6] $(f''(0) = 0.33206)$ is obtained with $N = 40$ terms. To get seven decimal places correct, the number of terms $N = 200$ is required.

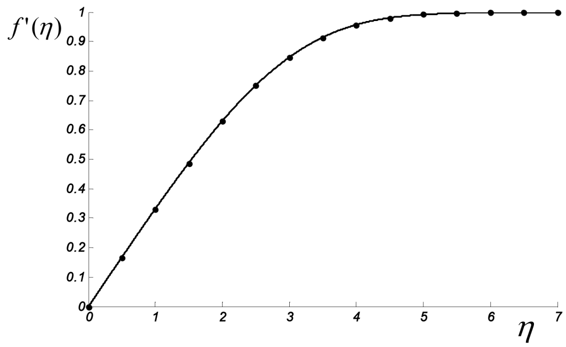

Figure 1. $f'(\eta)$ obtained from Fourier series with $N = 6$ (solid line), point represented numerical solution [6].

Table 1. Initial slope $f''(0)$ obtained by the expansion in Fourier series and compared with the value obtained numerically by Boyd [2]

N	$f''(0)$	Error %
3	0.352083648	6.03
4	0.359092941	8.14
5	0.344145196	3.64
6	0.335350023	0.99
7	0.333053303	0.30
8	0.332733377	0.20
9	0.332618519	0.17
10	0.332486682	0.13
15	0.332206110	0.04
20	0.332115130	0.02
25	0.332085506	8.5E-03
30	0.332073161	4.8E-03
40	0.332063777	1.9E-03
50	0.332060582	9.8E-04
100	0.332057802	1.4E-04
150	0.332057578	7.3E-05
200	0.332057382	1.4E-05
250	0.332057359	6.8E-06
350	0.332057343	2.1E-06

Conclusions

The results show that the classical expansion in Fourier series delivers a solution with very good accuracy and rapid convergence, considering only six terms in the Fourier series leads to error minor than 1% accuracy of initial slope. This method can easily be generalized to boundary layer flows governed by the equation of Falkner-Skan including heat and mass transfer.

References

[1] H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung, Z. Math. Physik. 56 (1908), 1-37.

[2] John P. Boyd, The Blasius function in the complex plane, Experiment. Math. 8(4) (1999), 381-394.

[3] R. Cortell, Numerical solutions of the classical Blasius flat-plate problem, Appl. Math. Comput. 170(1) (2005), 706-710.

[4] J. H. He, Approximate analytical solution of Blasius equation, Commun. Nonlinear Sci. Numer. Simulation 3(4) (1998), 260-263.

[5] J. H. He, A simple perturbation approach to Blasius equation, Appl. Math. Comput. 140 (2003), 217-222.

[6] L. Howarth, On the solution of the laminar boundary layer equation, Proc. Roy. Soc. London A 164 (1938), 547-579.

[7] I. K. Khabibrakhmanov and D. Summers, The use of generalized Laguerre polynomials in spectral methods for nonlinear differential equations, Comput. Math. Appl. 36(2) (1998), 65-70.

[8] Lei Wang, A new algorithm for solving classical Blasius equation, Appl. Math. Comput. 157(1) (2004), 1-9.