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ON A KdV TYPE EQUATION OF VISCOUS LONG RANGE WATER WAVES

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Abstract

A generalization of the famous KdV equation is investigated where applications to unidirectional propagation of long water waves on a viscous liquid are considered. The emphasis of this paper is on rigorous mathematical analysis to prove the existence, uniqueness and like properties of the solution of this partial differential equation. In addition, a second generalization equation, KdV with constant coefficients, is considered and like results are provided.

1. Introduction

In 1975, a ground breaking work (1) was published by J. L. Bona and R. Smith where the equation

$$u_t + u_x(u+1) + u_{xxx} = 0 \tag{1}$$

was investigated with applications to long range water wave on an inviscous liquid. They analyzed a basic theory of equation (1) and its extension

$$u_t + u_x(u+1) + u_{xxx} = f(x, t) \tag{2}$$

with the term $f(x, t)$ treated as an external force in the physical sense.

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They showed that equations (1) and (2) have wholly satisfactory mathematical properties. In particular, they discussed existence, uniqueness, regularity and established the stability of solutions with respect to variations in prescribed influences is a favorable property with regards to applications of the model.

As it is well known when doing mathematical modeling for any physical problem there are often a great deal of assumptions made, conditions overlooked and/or simplified. This is indeed the case when modeling long range water waves. Namely, if all of the assumptions and simplifications are made, then one will obtain equation (1) as the mathematical model. However, it is often necessary for some of these assumptions to be modified. In these cases one will often obtain a similar partial differential equation as the original model but with a new term that is not including the dependent variable, hence, the prior interest of equation (2). In some cases this equation will not suffice and the new f term will also need to include some appearance of the dependent variable; for example the equation

$$u_t + u_x(u+1) + u_{xxx} = F(u) \tag{3}$$

may be of need.

In the following work, we will examine how the equation and results are changed when the problem of long range water waves is considered in viscous liquids. The model considered relates to the initial-value problem for the equation

$$u_t + u_x(u+1) + u_{xxx} = \upsilon u_{xx}, \tag{4}$$

where υ is nondimensional kinematic viscosity of the liquid; moreover, the solution $u(x, t)$ is considered in a class of real nonperiodic functions defined for $-\infty < x < \infty$, $t \geq 0$. The notation $u^{(m,n)} = \frac{\partial u^{m+n}}{\partial x^m \partial t^n}$ will be frequently utilized, and the appropriate function spaces will be discussed in the statement of the results. In addition, several other results will be given for similar partial differential equations which appear during the analysis. An interesting note for further research is to either consider the general equation (3) in order to gain more information regarding what kind of forcing terms $F(u)$ could be introduced and/or consider equation (3) with coefficients that are functions of the independent variables x and t.

2. Statement of Results

In the following section, our main partial differential equation of interest will be

$$u^{(1,0)} + u^{(1,0)}(u+1) + u^{(3,0)} = vu^{(2,0)} \tag{5}$$

with the initial value conditions $u(x, 0) = g(x)$. With $g(x)$ begin considered as a continuous and bounded nonperiodic function and the solutions $u(x, t)$ are considered in a class of real nonperiodic functions defined for $-\infty < x < \infty,\ t \geq 0$.

Theorem 1. *Let* $g(x)$ *be a continuous function such that*

$$\sup_{x \in \Re} |g(x)| \leq b < \infty.$$

Then there exists a $t_0(b)$ *such that the integral equation*

$$u(\bar{x}, \bar{t}) = g(\bar{x}) + \int_0^{\bar{x}} \int_{-\infty}^{+\infty} K(\bar{x} - \xi)\left(u(\xi, \tau) + \frac{1}{2}u^2(\xi, \tau) - vu^{(1,0)}(\xi, \tau)\right) d\xi d\tau \tag{6}$$

which can alternatively be written as

$$u(\bar{x}, \bar{t}) = g(\bar{x}) - \int_0^{\bar{x}} vu(\bar{x}, \tau)$$

$$+ \int_{-\infty}^{+\infty} K(\bar{x} - \xi)\left[\left(u(\xi, \tau) + \frac{1}{2}u^2(\xi, \tau)\right)\right] - ve^{-|x-\xi|}u(\xi, \tau) d\xi d\tau$$

has a solution that is bounded and continuous for $\bar{x} \in Re$ *and* $0 \leq b < \infty$ *such that* $u(\bar{x}, 0) = g(\bar{x})$, *where* $K(\bar{x}) = \frac{1}{2} sgn(x) e^{-\bar{x}}$. *The coordinates* $(\bar{x}, \bar{t})$ *in which the integral equation is given in can be obtained from the original coordinate system* (x, t), *where equation* (5) *was given in by the transformation* $x = \varepsilon^{1/2}(\bar{x} - \bar{t})$ *and* $t = \varepsilon^{3/2}\bar{t}$. *Moreover, if* $u(\bar{x}, 0) = g(\bar{x})$ *and* $g \in C^2(R^2)$, *then any solution of the integral equation* (6) *is a classical solution of the partial differential equation* (5).

Theorem 2. *Let* $g(x)$ *be a continuous function such that*

$$\sup_{x \in \Re} | g(x) | \leq b < \infty.$$

Then there exists a $t_0(b)$ *such that the integral equation*

$$u(x, t) = g(x) + \int_0^x \int_{-\infty}^{+\infty} K(x - \xi)\left(u(\xi, \tau) + \frac{1}{2}u^2(\xi, \tau) - \upsilon u^{(1,0)}(\xi, \tau)\right) d\xi d\tau$$

has a solution that is bounded and continuous for $x \in Re$ *and* $0 \leq b < \infty$ *such that* $u(x, 0) = g(x)$, *where* $K(x) = \frac{1}{2} sgn(x) e^{-\bar{x}}$. *Moreover, if* $u(x, 0)$ $= g(x)$ *with* $g \in C^2(R^2)$, *then there is a unique classical solution of the partial differential equation*

$$a(t)u^{(0,1)} + b(t)u^{(1,0)} + c(t)u^{(1,0)}u - d(t)u^{(2,1)} = 0, \tag{7}$$

where provided that $a(t) = d(t) \neq 0$.

3. Proofs and Auxiliary Statements

In [3] it is shown that the partial differential equation

$$u^{(0,1)} + u^{(1,0)}(u + 1) + u^{(3,0)} - \varepsilon u^{(2,1)} = 0$$

can be rewritten as

$$u_{\bar{t}} + u_{\bar{x}}(u + 1) - u_{\bar{x}\bar{x}\bar{t}} = 0$$

by utilizing the standard change of variables process with $x = \varepsilon^{1/2}(\bar{x} - \bar{t})$ and $t = \varepsilon^{3/2}\bar{t}$, hence, $\bar{x} = \varepsilon^{1/2}x + \varepsilon^{-3/2}t$ and $\bar{t} = \varepsilon^{-3/2}t$. Now, by following the same scheme one sees the equation

$$u^{(0,1)} + u^{(1,0)}(u + 1) + u^{(3,0)} - \varepsilon u^{(2,1)} = \upsilon u^{(2,0)} \tag{8}$$

can also be rewritten in a similar manner. However, one must note by the chain rule that

$$u^{(2,0)} = \left(u_{\bar{x}\bar{x}} \frac{\partial \bar{x}}{\partial x} + u_{\bar{x}\bar{t}} \frac{\partial \bar{t}}{\partial x}\right) \frac{\partial \bar{x}}{\partial x} + u_{\bar{x}} \bar{x}_{xx} + (u_{\bar{t}\bar{x}} \bar{x}_x + u_{\bar{t}\bar{t}} \bar{t}_x) \bar{t}_x + u_{\bar{t}} \bar{x}_{xx} = \varepsilon(u_{\bar{x}\bar{x}}).$$

Doing so one obtains that equation (8) can be rewritten as

$$u_{\bar{t}} + u_{\bar{x}}(u+1) - u_{\bar{x}\bar{x}\bar{t}} = v\varepsilon u_{\bar{x}\bar{x}}. \tag{9}$$

Proof of Theorem 1. To begin we note that (9) can be rewritten as

$$u^{(0,1)} - u^{(2,1)} = \varepsilon v u^{(2,0)} - u^{(1,0)} - u^{(1,0)}u$$

which can be rewritten as

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x\left[u + \frac{1}{2}u^2 - \varepsilon v u^{(1,0)}\right].$$

Now viewing the above equation as differential equation for $u^{(0,1)}$ one obtains that

$$u^{(0,1)} = \int_{-\infty}^{+\infty} K(x-\xi)\left(u(\xi, t) + \frac{1}{2}u^2(\xi, t) - \varepsilon v u^{(1,0)}(\xi, t)\right)d\xi,$$

where the Kernel is defined as $K(x) = \frac{1}{2}(sgnx)e^{-|x|}$.

The above differential equation can easily be rewritten as an integral equation as

$$u(x, y) = g(x) + \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)\left(u(\xi, \tau) + \frac{1}{2}u^2(\xi, \tau) - \varepsilon v u^{(1,0)}(\xi, \tau)\right)d\xi d\tau. \tag{10}$$

Now, we will write (10) in operator notation as

$$u = \overline{A}[u] = g + \overline{B}[u],$$

and for our considerations let us denote ζ_{t_0} as the class of functions $v(x, t)$ which satisfy (10) that are continuous and uniformly bounded on the infinite strip $Re \times [0, t_0]$. Moreover, we will consider this set of functions $v \in \zeta_{t_0}$ that have the norm $\| v \|_\zeta = \sup_{x \in Re,\, 0 \le t \le t_0} |v(x, t)|$ which is a Banach space of bounded continuous functions where t_0 is left as arbitrary.

If we consider two functions $v_1, v_2 \in \zeta_{t_0}$ and consider $|\overline{A}[v_1] - \overline{A}[v_2]|$, then after some algebra, we obtain

$$|\overline{A}[v_1] - \overline{A}[v_2]| \leq t_0\left(1 + \varepsilon v + \frac{1}{2}\| v_1 \|_\zeta + \frac{1}{2}\| v_2 \|_\zeta\right)\| v_1 - v_2 \|_{\overline{\zeta}}.$$

From this it follows that $\overline{A}$ is a continuous mapping; moreover, $\overline{A}$ is a contraction of the ball $\| v \|_\zeta < R$ if

$$t_0\left(1 + \varepsilon v + \frac{R}{2} + \frac{R}{2}\right) \leq L < 1.$$

If this condition is satisfied, then by the Theory of fixed points one can assure that $\overline{A}$ has a unique fixed point in the ball $\| v \|_\zeta < R.$ Hence, we have obtained that equation (9) has a unique solution for the restrictive value of t_0.

In order to investigate equation (5) we first consider equation (8) in the (x, t) coordinate system. Now, by applying the $(x, t) \to (\bar{x}, \bar{t})$ transformation, we obtain equation (9). However, we have just proven that equation (9) has a unique solution. Thus, we can obtain that equation (5) has a unique solution by taking the uniqueness of equation (9) and inverting the transformation, hence, obtaining the uniqueness of equation (8). Finally, by taking the limit as $\varepsilon \to 0$ we see that equation (8) becomes equation (5), hence, the Theorem 1 is proven.

Proof of Theorem 2. To begin we divide equation (7) through by $a(t)$, then noting that $a(t) = d(t)$ we obtain

$$u^{(0,1)} + Bu^{(1,0)} + Cu^{(1,0)}u - u^{(2,1)} = 0,$$

where $B = \dfrac{b(t)}{a(t)}$ and $C = \dfrac{c(t)}{a(t)}$. Now, following a similar argument as in the proof of Theorem 1 we obtain that (7) can be rewritten as

$$[(1 - \partial_x^2)]u^{(0,1)} = -\partial_x\left[Bu + C\frac{1}{2}u^2\right].$$

Following a similar argument one views the above equation as differential equation for $u^{(0,1)}$; hence, one obtains that

$$u^{(0,1)} = \int_{-\infty}^{+\infty} K(x-\xi)\left(Bu(\xi, t) + C\frac{1}{2}u^2(\xi, t)\right),$$

where the Kernel is defined as $K(x) = \frac{1}{2}(sgnx)e^{-|x|}$.

Now, the above differential equation can easily be rewritten as an integral equation as

$$u(x, y) = g(x) + \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)\left(Bu(\xi, \tau) + C\frac{1}{2}u^2(\xi, \tau)\right) d\xi d\tau. \tag{11}$$

Now, we will write (11) in operator notation as

$$u = \overline{A}[u] = g + \overline{B}[u].$$

Using the same notations as discussed in the proof of Theorem 1, we will consider the class of functions $v(x, t)$ which satisfy (11) that are continuous and uniformly bounded on the infinite strip $R \times [0, t_0]$.

If we consider two functions $v_1, v_2 \in \zeta_{t_0}$ and consider $|\overline{A}[v_1] - \overline{A}[v_2]|$, then after some algebra, we obtain

$$|\overline{A}[v_1] - \overline{A}[v_2]| \le t_0\left(B + C\frac{1}{2}\| v_1 \|_\zeta + C\frac{1}{2}\| v_2 \|_\zeta\right)\| v_1 - v_2 \|_{\overline{\zeta}},$$

where the values of B and C are understood to be bounds of the functions $B(t)$ and $C(t)$. From this it follows that $\overline{A}$ is a continuous mapping; moreover, $\overline{A}$ is a contraction of the ball $\| v \|_\zeta < R$ if

$$t_0\left(B + C\frac{R}{2} + C\frac{R}{2}\right) \le L < 1.$$

If this condition is satisfied, then by the Theory of fixed points one can assure that $\overline{A}$ has a unique fixed point in the ball $\| v \|_\zeta < R$. Hence, we have obtained that equation (11) has a unique solution.

It is noted that in the prior discussion the values of B and C were treated as constants that bounded the respective functions $B(t)$ and $C(t)$. Moreover, if the argument was redeveloped for B and C being either functions of x or x, t, then one could redo the argument but would need to include an additional term on the right hand side to take into consideration the standard product rule. Hence, in the latter case the integral equation (11) would have a slightly different form, but still only contains a u and u^2 appearance of the dependent variable. The actual fixed point argument would not have any major differences except in the final inequality, of course, one would replace the current B and C values by bounds on the functions $B(x, t)$ and $C(x, t)$ and of course it would be expected that these functions would also need to satisfy a Lipschitz type condition. The details of this are not included here as it is a routine modification.

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