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FUZZY MEASURES OF THE INCIDENCE OF RELATIVE POVERTY AND DEPRIVATION: A LONGITUDINAL AND COMPARATIVE PERSPECTIVE

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Abstract

In this paper, we present a methodology for the study of multi-dimensional and longitudinal aspects of poverty and deprivation, and apply this in a multi-country comparative context. The conventional poor/non-poor dichotomy is replaced by defining poverty as a matter of degree, determined by the place of the individual in the income distribution. The same methodology facilitates the inclusion of other dimensions of deprivation into the analysis: by appropriately weighting indicators of deprivation to reflect their dispersion and correlation, we can construct measures of non-monetary deprivation in its various dimensions. An important contribution of the paper is to identify rules for the intersection and union of fuzzy sets appropriate for the study of poverty and deprivation. These rules allows us to meaningfully combine income and the diverse non-income deprivation indices at the micro-level and construct what we have termed 'latent' and 'manifest' indicators of deprivation. Mathematically the same approach is carried over for

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studying the persistence of poverty and deprivation over time. We establish the consistency of the approach when applied to a time sequence of any length. We can thus study longitudinally over time a whole range of indicators of poverty and deprivation, from cross-sectional monetary poverty rates to multi-dimensional 'latent' and 'manifest' indicators of deprivation.

1. Introduction

1.1. A multi-dimensional, longitudinal and comparative perspective

For understanding poverty and social exclusion, it is necessary to consider deprivation simultaneously in its multiple dimensions – low income as well as diverse non-monetary aspects of deprivation. Furthermore, these multiple aspects must be considered longitudinally, identifying the extent to which households and individuals are subject to persistent deprivation. Using fuzzy set representation of individual propensities, this paper presents a methodology for *multi-dimensional* and *longitudinal analysis* of poverty and deprivation in a multi-country *comparative* context.

The necessity of adopting a multi-dimensional approach has been noted, among others, by Kolm [16], Atkinson and Bourguignon [2], Tsui [21], Maasoumi [17] and Sen [20]. In the present work, we go beyond the conventional study of poverty based simply on the poor/non-poor dichotomy defined in relation to some chosen poverty line. Rather, *poverty and multi-dimensional deprivation are treated as matters of degree determined in terms of the individual's position in the distribution of income and other aspects of living condition*. The state of deprivation is thus seen in the form of 'fuzzy sets' to which all members of the population belong but to varying degrees. In this way, we are able to clarify and propose a solution for one of the problems raised in the literature (Atkinson et al. [3], Duclos et al. [12] and especially Atkinson [1]): "…How can different attributes be aggregated? … A distinction may be drown between those who adopt a *union* approach and those who use an *intersection* measure …".

A number of authors have evoked the concepts of fuzzy sets in the analysis of poverty and living conditions, for instance Chiappero Martinetti [11], and Vero and Werquin [22]. In more specific terms, the present contribution represents a continuation and further development in a longitudinal perspective of the work of Cerioli and Zani [8], Cheli and Lemmi [10], Cheli [9], and Betti and Verma [7]. Aspects of this methodology have been applied at cross-sectional level in the Eurostat official publication *Second European Social Report* (Giorgi and Verma [13]).

1.2. Poverty and deprivation as a matter of degree

The basic idea of the fuzzy approach is of treating poverty and deprivation as a matter of degree, replacing the conventional classification of the population into a simple dichotomy. In principle all individuals in a population are subject poverty or deprivation, but to varying degrees. We say that each individual has a certain *propensity* to poverty or deprivation, the population covering the whole range [0, 1]. The conventional approach is a special case of this, with the population dichotomised as $\{0, 1\}$: those with income below a certain threshold are deemed to be poor (i.e., are all assigned a constant propensity $= 1$); others with income at or above that threshold are deemed to be non-poor (i.e., are all assigned a constant propensity $= 0$).

There are several advantages of treating poverty and deprivation as a matter of degree, applicable to all members of the population, rather than as simply a 'yes-no' state.

1. Further insight into the relative income situations of individuals and groups can be obtained by incorporating into the poverty rates a measure of the actual levels of incomes received, particularly at the lower end of the income distribution.

2. Non-monetary deprivation depends on forced non-access to various facilities or possessions determining the basic conditions of life. An individual may have access to some but not to others. Hence non-monetary deprivation is inherently a matter of degree, and some *quantitative approach* such as the present one is essential.

3. The combined analysis, considering income poverty and non-monetary deprivation simultaneously, is greatly facilitated by treating each dimension as a matter of degree. The need to divide the population into numerous discrete groups – as would normally be required in the conventional analysis, especially in the longitudinal context – is avoided.

4. More important is the potential of this approach in studying poverty (or more generally, deprivation in multiple dimensions) in the *longitudinal* context. The conventional approach measures mobility simply in terms of movements across some designated poverty line, and does not reflect the actual magnitude of the changes affecting individuals at all points in the distribution. Consequently, the degree of mobility of persons near to the chosen line tends to be over-emphasised, while that of persons far from that line largely ignored.

5. We can expect the resulting measures to be more precise. The sampling error of a distribution is lower than that of a dichotomy with values concentrated at the two end points. We can also expect the measures to be less sensitive to local irregularities in the income distribution curve, and to the particular choice of the poverty threshold.

1.3. Scope of this paper

In order to illustrate the richness of this approach, we analyse *five types of measures of poverty and deprivation* (proposed in Betti and Verma [7]) in relation to each other: (1) income poverty as conventionally viewed in the form of a poor/non-poor dichotomy; (2) poverty viewed as a propensity to which all individuals are subject to a greater or lesser degree; (3) non-monetary deprivation in its various dimensions ('domains') determined by the lack of access to non-monetary facilities and opportunities; and two measures of income poverty and non-monetary deprivation in combination – (4) 'latent deprivation' representing the presence of either dimension, and (5) 'manifest deprivation' representing the situation of individuals subject to both simultaneously. Then we analyse each of these measures in *four aspects in the time dimensions*: (1) cross-sectional measures (including their averaging over time); (2) the incidence of poverty and deprivation at any time during an interval; (3) the persistence and (4) continuity of the state of poverty/deprivation over time. An important contribution of the paper is to identify rules for the intersection and union of fuzzy sets appropriate for the study of poverty and deprivation. These rules allow us to meaningfully combine income and the diverse non-income deprivation indices at the micro-level. Mathematically the same approach is carried over for studying the persistence of poverty and deprivation over time. We establish the consistency of the approach when applied to a time sequence of any length. While the concern of this paper is primarily *methodological*, some illustrative results based on real, nationally representative and comparable data from EU countries are presented[1].

2. Fuzzy Measures in a Cross-sectional Perspective

In this section, we briefly describe the cross-sectional fuzzy measures proposed by Betti and Verma [7] which are based also on the seminal contributions of Cerioli and Zani [8], Cheli and Lemmi [10] and Betti and Verma [6].

2.1. The conventional income poverty measure ('Head Count Ratio')

Diverse 'conventional' measures of monetary poverty and inequality are well known and are not discussed here. In this paper, we will focus on only the most commonly used indicator, namely the proportion of a population classified as 'poor' in purely relative terms on the following lines. To dichotomise the population into the 'poor' and the 'non-poor' groups, each person j is assigned the equivalised income y_j of the person's household. Persons with equivalised income below a certain threshold or poverty line (say 60% of the median equivalised income) are considered to be poor (assigned a poverty index, say, $H_j = 1$), and the others as non-poor (assigned a poverty index $H_j = 0$). The conventional income poverty rate (the Head Count Ratio, H) is the population average

[1]The highly comparable data available for a number of EU countries is the European Community Household Panel (ECHP).

of this poverty index, appropriately weighted by sample weights (w_j):

$$\overline{H} = \frac{\sum_j w_j \cdot H_j}{\sum_j w_j}.$$

2.2. The propensity to income poverty ('Fuzzy Monetary')

Apart from the various methodological choices involved in the construction of conventional poverty measures, the introduction of fuzzy measures brings in *additional* factors on which choices have to be made. These concern at least two aspects:

° Choice of 'membership functions', meaning a quantitative specification of the propensity to poverty of each person given the level and distribution of income of the population.

° Choice of 'rules' for manipulation of the resulting fuzzy sets, specifically the rules defining complements, intersections, union and aggregation of the sets.

To be meaningful both these choices must meet some basic logical and substantive requirements. It is also desirable that they be useful in the sense of elucidating aspects of the situation not captured (or not captured as adequately) by the conventional approach.

We begin with the issue of choice of the poverty membership function (m.f.). In the conventional head count ratio *H*, the m.f. may be seen as $f(y_j) = 1$ if $f(y_j) < z$, $f(y_j) = 0$ if $y_j \geq z$, where y_j is equivalised income of individual *j*, and *z* is the poverty line. In order to move away from the poor/non-poor dichotomy, Cerioli and Zani [8] proposed the introduction of a transition zone $(z_1 - z_2)$ between the two states, a zone over which the m.f. declines from 1 to 0 linearly:

$$f(y_j) = 1 \quad \text{if } f(y_j) < z_1;$$

$$f(y_j) = \frac{z_2 - y_j}{z_2 - z_1} \quad \text{if } z_1 \leq y_j < z_2;$$

$$f(y_j) = 0 \quad \text{if } y_j \geq z_2.$$

In what has been called the 'Totally Fuzzy and Relative' approach, Cheli and Lemmi [10] defined the m.f. as the distribution function $H(y_i)$ of income, normalised (linearly transformed) so as to equal 1 for the poorest and 0 for the richest person in the population. The mean of m.f. so defined is always 0.5, by definition. In order to make this mean equal to some specified value (such as 0.1) so as to facilitate comparison with the conventional poverty rate, Cheli [9] took the m.f. as normalised distribution function, raised to some power $\alpha \geq 1.$ Increasing the value of this exponent implies giving more weight to the poorer end of the income distribution: empirically, large values of the m.f. would then be concentrated at that end, making the propensity to income poverty sensitive to the *location* of the poorer persons in the income distribution. Beyond that, the choice of the value of α is essentially arbitrary, or at best based on some external consideration: this is unavoidable since any method for the quantification of the extent of poverty is inevitably based on the arbitrary choice of some parameter (Hagenaars [14]). Betti and Verma [6] have used a somewhat refined version of the above formulations in the following form:

$$FM_j = (1 - L_j^{(M)})^{\alpha}$$

$$= \left(\frac{\sum_i w_i y_i \mid y_i > y_j}{\sum_i w_i y_i \mid y_i > y_1} \right)^{\alpha},$$

where $L_j^{(M)}$ represents the value of the Lorenz curve for individual (j). In other terms, $1 - L_j^{(M)}$ represents the *share* of the total equivalised income received by all individuals less poor than the person concerned. It varies from 1 for the poorest, to 0 for the richest individual. $1 - L_j^{(M)}$ can be expected to be a more sensitive indicator of the actual disparities in income, compared to the normalised distribution function $1 - F_j^{(M)}$ which is simply the proportion of individuals less poor than the person concerned.

Betti and Verma [7] have combined the TFR approach of Cheli and Lemmi [10] and the approach of Betti and Verma [6] into the 'Integrated Fuzzy and Relative' approach defined as:

$$FM_j = (1 - F_j^{(M)})^{\alpha}(1 - L_j^{(M)})$$

$$= \left(\frac{\sum_i w_i \mid y_i > y_j}{\sum_i w_i \mid y_i > y_1} \right)^{\alpha} \left(\frac{\sum_i w_i y_i \mid y_i > y_j}{\sum_i w_i y_i \mid y_i > y_1} \right).$$

It may be noted that this measure also has an economic meaning, in that the Fuzzy Monetary (FM) measure as defined above is expressible in terms of the generalised Gini measures. This family of measure (often referred as '*s*-Gini') is a generalisation of the standard Gini coefficient. This measure weights the distance (F-L) between the line of perfect equality and the Lorenz curve by a function of the individual's position in the income distribution, giving more weight to its poorer end. Parameter α is determined so as to impose numerical equality between average FM and the conventional HCR.

2.3. Non-monetary deprivation ('Fuzzy Supplementary')

In addition to the level of monetary income, the standard of living of households and persons can be described by a host of indicators, such as housing conditions, possession of durable goods, the general financial situation, perception of hardship, expectations, norms and values. Quantification and putting together of a large set of non-monetary indicators of living conditions involve a number of steps, models and assumptions.

First, from the large set which may be available, a selection has to be made of indicators which are substantively meaningful and useful. For our analysis using the rich ECHP data, a subset of the available indicators was selected. The most important determining factor in the choice of the set of items for analysis was an assessment based on a detailed examination of variations in frequency distributions across countries and background knowledge of national situations – of the extent to which an item could be meaningfully included in *comparative analysis*. Generally, the result has been to include a majority of so-called 'objective' indicators on non-monetary deprivation, such as the possession of material goods and facilities and physical conditions of life, at the expense of what may be called '*subjective'* indicators such as self-assessment of the general health condition, economic hardship and social isolation, or the expressed degree of satisfaction with various aspects of work and life. These latter types of indicators tend to be more culture-specific and hence less comparable across countries and regions. Secondly, it is useful to identify the underlying dimensions and to group the indicators accordingly. Taking into account the manner in which different indicators cluster together (possibly differently in different national situations) adds to the richness of the analysis; ignoring such dimensionality can in fact result in misleading conclusions. In the present analysis we have used the indicators grouped into five dimensions as proposed by Whelan et al. [23].

Putting together categorical indicators of deprivation for individual items to construct composite indices requires decisions about assigning numerical values to the ordered categories and the weighting and scaling of the measures.

Denoting with $d_{k,j}$ the deprivation score for each indicator k for each individual j, an aggregated measure for each dimension δ is defined as $S_{\delta,j} = \sum_{k\in\delta} W_k \cdot (1 - d_{k,j}) / \sum_{k\in\delta} W_k$.

Note that S is a 'positive' score indicating lack of deprivation; thus it is akin to income in Subsection 2.2. As in the Fuzzy Monetary approach, Betti and Verma [7] proposed a combination of the distribution function F and of the Lorenz curve L of the score $S_{\delta,j}$ as follows:

$$FS_j^{\delta} = (1 - F_j^{(\delta)})^{\alpha}(1 - L_j^{(\delta)}) = \left(\frac{\sum_i w_i \mid S_i > S_j}{\sum_i w_i \mid S_i > S_1}\right)^{\alpha}\left(\frac{\sum_i w_i S_i \mid S_i > S_j}{\sum_i w_i S_i \mid S_i > S_1}\right).$$

Parameter α is determined so as to impose numerical equality between average FS and the conventional HCR.

2.4. Some empirical results

Table 1 compares fuzzy measures of income poverty and of non-monetary deprivation across EU-15 countries. For reasons noted, the measures have been averaged over 8 waves, and are scaled to be identical to each other at level of 15 EU countries (EU-15). Countries with low levels of monetary poverty (in particular Finland and Denmark) indicate a higher level of non-monetary deprivation compared to the national level of monetary poverty. Overall, there is a weak negative correlation between the level of income poverty (FM) and the ratio (FS/FM), though the two measures (FM, FS) are quite similar and equally relative. The table also shows mean deprivation rates separately for the five dimensions of living conditions. Results are not available for some countries because of lack of data, and are based on a rather limited number of items in some others. Overall, the levels differ greatly by dimension[2], with the highest rates for 'basic life-style' and 'environmental' dimensions, and very low rates for the dimension 'housing facilities'. The general pattern is that in countries with the highest levels of income poverty, deprivation in specific dimensions can be particularly acute, even after taking into account their high poverty levels. Examples are the 'environmental' dimension in Italy, the two 'life-style' dimensions in Greece, and especially in Portugal, the dimensions concerning 'secondary life-style', 'housing facilities' and 'housing deterioration'.

[2]For determining FS_{δ} for each dimension we have simply used the parameter α determined for the overall FS as defined in Subsection 2.4.

Table 1. Fuzzy cross-sectional measures of poverty and deprivation

	FM	FS	FS/FM	FSup1	FSup2	FSup3	FSup4	FSup5
0 EU-15	16.0	16.0	1.00					
1 FI	8.9	10.9	1.22	12.5	7.0	2.3	4.7	12.6
2 SE	10.0	10.9	1.08					
3 DK	9.2	11.2	1.22	9.9	8.3	1.2	8.1	11.5
4 NL	11.4	12.0	1.06	10.6	7.1	0.9	10.6	13.6
5 LU	11.6	11.1	0.95					
6 DE	12.4	10.9	0.88					
7 AT	11.8	13.4	1.14	12.0	10.0	3.8	8.7	13.3
8 BE	13.5	15.0	1.11	14.4	9.6	3.4	12.1	17.1
9 FR	14.7	15.8	1.07	16.2	10.9	3.2	13.8	18.7
10 UK	18.3	19.0	1.04	13.5	11.5	1.0	8.9	14.2
11 ES	19.5	18.4	0.95	16.8	15.4	2.0	15.2	22.9
12 IE	16.8	17.8	1.06	17.3	14.5	2.5	9.3	16.4
13 IT	19.7	19.4	0.98	21.2	15.0	2.3	10.1	23.5
14 GR	22.2	22.6	1.02	33.7	20.2	6.1	16.5	21.8
15 PT	22.3	23.9	1.07	18.4	23.7	14.4	27.9	26.6
simple average	14.8	15.5	1.06	16.4	12.8	3.6	12.2	17.7

FM fuzzy measure of monetary poverty rate ('fuzzy monetary')

FS fuzzy measure of overall non-monetary deprivation rate ('fuzzy supplementary')

FSup1-5 fuzzy measure of deprivation in specific areas or dimensions of life

1-basic life style; 2-secondary life-style; 3-housing facilities; 4-housing deterioration; 5-environmental problems

Note. Figures show simple averages of cross-sectional results over 8 ECHP waves.

3. Fuzzy Set Operations Appropriate for the Analysis of Poverty and Deprivation

3.1. Multi-dimensional and longitudinal measures

In the previous section, we have defined propensities to poverty and deprivation in multiple dimensions in the form of fuzzy sets: for monetary poverty, overall non-monetary deprivation, and deprivation in particular aspects of life. In multi-dimensional analysis, it is of interest to know the extent to which deprivation in different dimensions tends to overlap for individuals. Similarly, in longitudinal analysis, it is of interest to know the extent to which the state of poverty or deprivation persists over time for the person concerned. Such analyses require the specification of rules for the manipulation of fuzzy sets, such as defining set complements, intersections, unions and aggregations.

As a concrete example, consider deprivation in two dimensions – the states of income poverty and overall non-monetary deprivation, specified by the individual memberships (FM_j, FS_j), respectively (then of course there are their complements, defined simply as $\overline{FM_j} = 1 - FM_j$, $\overline{F}\overline{S}_j = 1 - FS_j$). Their combined incidence is specified in terms of four fuzzy sets: defined by the presence of both forms of deprivation $(FM_j \cap FS_j)$, presence of only the first $(FM_j \cap \overline{F}\overline{S}_j)$, presence of only the second $(\overline{F}\overline{M}_j \cap FS_j)$, and the absence of both $(\overline{F}\overline{M}_j \cap \overline{F}\overline{S}_j)$. Similarly, we may consider the state of deprivation over two points in time, specified by the individual memberships $(F1_j, F2_j)$, say. The persistence of deprivation is again specified in terms of four fuzzy sets: defined by the presence of deprivation at both times, at only the first time, at only the second time, and its absence at both times. A formal treatment of the two situations is identical. In fact, the same applies to a combination of the two: the longitudinal elements $(F1_j, F2_j)$ may themselves represent a combination of deprivation in multiple dimensions.

In the ordinary 'crisp set' formulation each individual belongs to one and only one of the intersection sets and rules for constructing these sets are straightforward. Fuzzy set operations are a generalisation of the corresponding 'crisp' set operations in the sense that the former reduce to (exactly reproduce) the latter when the fuzzy membership functions, being in the whole range $[0, 1]$, are reduced to a $\{0, 1\}$ dichotomy. There are, however, *more than one ways in which the fuzzy set operations can be formulated*, each representing an equally valid generalisation of the corresponding crisp set operations. The choice among alternative formulations has to be made primarily on substantive grounds: some options are more appropriate (meaningful, useful, illuminating, convenient) than others, depending on the context and objectives of the application. While the rules of fuzzy set operations cannot be discussed fully in this paper, we need to clarify their application *specifically for the study of poverty and deprivation.*

3.2. Intersection, union and complement of fuzzy sets

Let us use a simplified notation: (a, b) for (FM_j, FS_j) or $(F1_j, F2_j)$, the membership functions of two sets for individual j (subscript j can be dropped when not essential); also $s_1 = \min(a, b)$ and $s_2 = \max(a, b)$ for the smaller and the larger of the two values. We also denote by $c(..)$, $i(..)$ and $u(..)$ the basic set operations of complementation, intersection and union, respectively. To be meaningful, consistent and useful, these operations must satisfy some essential, and some additional desirable requirements, briefly as follows (Klir and Yuan [15])[3]:

Fuzzy complement $c(..)$

1. reduction to the crisp set operation $c(a) = 1 - a$ with dichotomous membership $\{0, 1\}$;

2. boundary conditions, $c(0) = 1$, $c(1) = 0$;

3. monotonicity, if $a' \leq a$, then $c(a') \geq c(a)$;

4. $c(a)$ is continuous and involutive, $c(c(a)) = a$.

Fuzzy intersection $i(..)$

1. reduction to the crisp set operation with dichotomous membership $\{0, 1\}$;

2. boundary conditions, $i(a, 1) = a$, $i(a, 0) = 0$;

3. monotonicity, if $a' \leq a$, then $i(a', b) \leq i(a, b)$; if $a' < a$ and $b' < b$, then $i(a', b') < i(a, b)$;

4. cumutativity, $i(a, b) = i(b, a)$; associativity, $i(a, i(b, c)) = i(i(a, b), c)$; and continuity;

5A. $i(a, a) = a$ (idempotency), or 5B. $i(a, a) < a$ (*sub*idempotency).

[3]We have used the notation and terminology from this excellent text throughout this section.

Fuzzy union $u(..)$

1. reduction to the crisp set operation with dichotomous membership $\{0, 1\}$;

2. boundary conditions, $u(a, 0) = a$, $u(a, 1) = 1$;

3. monotonicity, if $a' \leq a$, then $u(a', b) \leq u(a, b)$; if $a' < a$ and $b' < b$, then $u(a', b') < u(a, b)$;

4. cumutativity, associativity, and continuity as above;

5A. $u(a, a) = a$ (idempotency), or 5B. $u(a, a) > a$ (*super*idempotency).

Standard fuzzy set operations

The distinction between conditions 5A and 5B is important in our context. Operations satisfying 1-4 and 5A are termed 'standard' as they have certain special properties and are commonly used. These operations are

Standard fuzzy complement $c_s(..)$

$$c_s(a) = (1 - a) = \overline{a}, \quad \text{say,} \quad \text{and} \quad c_s(b) = (1 - b) = \overline{b},$$

being the degree to which the individual belongs to, for example the 'NON-POOR' set, or alternatively, does not belong to the 'POOR' set.

Standard fuzzy intersection $i_s(..)$

$$i_s(a, b) = \min(a, b) = s_1, \quad \text{say,}$$

being the degree to which the individual is subject to deprivation of both forms (monetary poverty and non-monetary deprivation), or at both times.

Standard fuzzy union $u_s(..)$

$$u_s(a, b) = \max(a, b) = s_2, \quad \text{say,}$$

being the degree to which the individual is subject to deprivation of either (one, the other, or both) of the two forms of deprivation, or at either of the two times.

The Standard operations are commonly used in particular because these are the *only ones* which satisfy the intuitively and substantively desirable Condition 5A (idempotency), namely,

$$i_s(a, a) = \min(a, a) = a \quad \text{and} \quad u_s(a, a) = \max(a, a) = a.$$

Other options

The Standard operations defined above are in fact not the only possible and acceptable generalisations of the corresponding crisp set rules. Other options can be more appropriate depending on the context and objectives of the application. Some important ones are listed in Table 2. They all meet the basic requirement of reproducing the corresponding crisp set operation with dichotomous membership $\{0, 1\}$, and satisfy Conditions 2-4; however, apart from the Standard operation, they satisfy Condition 5 only in the form 5B (i.e., are not idempotent – $i(a, a) < a$, and $u(a, a) > a$). For the Algebraic operations for instance, $i(a, a) = a^2 \le a$, $u(a, a) = 2a - a^2 \ge a$.

There is in fact a whole range of possible options between '*D*' and '*S*' in Table 2 in the choice of the set operations: $i_{\min} \le i(a, b) \le i_{\max}$; $u_{\max} \ge u(a, b) \ge u_{\min}$. An example is provided in the last row, the so-called Yager's parametric family, $w = (0, \infty)$. We obtain form '*D*' for $w = 0$, form '*B*' for $w = 1$, and the Standard form '*S*' for $w \to \infty$.

Table 2. Basic forms of fuzzy set intersections and unions

		Intersection	Union
S	Standard	$i_{\max}(a, b) = \min(a, b) = s_1$	$u_{\min}(a, b) = \max(a, b) = s_2$
A	Algebraic	$i(a, b) = a * b$	$u(a, b) = a + b - a * b$
B	Bounded	$i(a, b) = \max(0, a + b - 1)$	$u(a, b) = \min(1, a + b)$
D	Drastic	$i_{\min} = s_1$ when $s_2 = 1$; $= 0$ otherwise	$u_{\max} = s_2$ when $s_1 = 0$; $= 1$ otherwise
Example of a parametric family covering the whole range			
Y	Yager, $w = (0, \infty)$	$i_w(a, b) = 1 - \min(1, ((1-a)^w + (1-b)^w)^{1/w})$	$u_w(a, b) = \min(1, (a^w + b^w)^{1/w})$

Condition 6

Permissible forms of the two operations, intersection and union, go in pairs: to be consistent, it is necessary to select the two from the same row of Table 2, so as to satisfy the De Morgan laws of set operations $\overline{A \cap B} = \overline{A} \cup \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$, which in the fuzzy case can be written as

$$c[i(a, b)] = u[c(a), c(b)]; \quad c[u(a, b)] = i[c(a), c(b)].$$

Any of the above intersection-union pairs is consistent not only with the standard definition of the complement, $c_s(a) = 1 - a$, but also with any complement which satisfies Conditions 1-4 noted above, such as $c_w(a) = (1 - a^w)^{1/w}$, termed the Yager's class of fuzzy complements by Klir and Yuan [15].

3.3. Additivity requirement

For our application, a most important observation is that with the Standard fuzzy operations, i_s provides the *largest* (the most loose or the weakest) intersection among all the permitted forms (it is for this reason that it has been labelled as $i_{\max}$ in Table 2); all other forms give a smaller, or at least no larger, value for the intersection. By contrast, u_s provides the *smallest* (the most tight or the strongest) union among all the permitted forms (it is for this reason that it has been labelled as $u_{\min}$ in the table); all other forms give a larger, or at least no smaller, value for the union. *It is this factor which makes it inappropriate to use the Standard set operations uniformly throughout in our application to poverty analysis*. If the Standard operations were applied to all the four intersections, (a, b), $(\overline{a}, \overline{b})$, $(a, \overline{b})$, $(\overline{a}, b)$, the sum of membership functions of an individual could be verified to equal $1 + 2 \cdot (s_1 - \max(0, \delta))$, where $s_1 = \min(a, b)$, $s_2 = \max(a, b)$ and $\delta = (a + b - 1)$, i.e., to equal $(1 + 2 \cdot s_1)$ for $\delta \leq 0$, and $1 + 2 \cdot (1 - s_2)$ for $\delta > 0$.

Hence, because of the 'expansive' nature of the Standard intersection, the sum of the resulting membership functions for the four subsets exceeds 1. However, this is in conflict with substantive requirements of our situation in the following sense. In the conventional analysis, the population is similarly divided into four crisp sets (exhaustive and non-overlapping classes) according to joint incidence of monetary/non-monetary deprivation or of deprivation at two points in time (yes-yes, yes-no, no-yes, no-no); and by definition, the proportions in the four groups must sum to 1. This should be true with fuzzy sets as well, since it is precisely these proportions that we wish to estimate and compare with the conventional analysis[4]. This substantive requirement may be specified as 'Condition 7', as follows.

Condition 7

If a set of fuzzy membership functions is to reflect exhaustive and non-overlapping categories of the conventional (crisp) formulation, then at the individual level (or at least averaged over individuals), the fuzzy membership functions should add up to 1.

Now it can be seen that the Algebraic form '*A*', applied to all the four intersections, is the *only one* which meets this condition, as shown in Table 3[5]. But despite this numerical consistency, we *do not* regard the Algebraic form to give results which, for our particular application, are generally acceptable on intuitive or substantive grounds. For instance, for an individual with propensity to monetary poverty $a = 0.5,$ and propensity to non-monetary deprivation $b = 0.4,$ the resulting propensity to be subject to BOTH will be only $0.5 * 0.4 = 0.2$ under this rule, while the propensity to be subject to EITHER would be as high as $1 - (1 - 0.5) * (1 - 0.4) = 0.7.$ By contrast, the Standard operations give much more plausible results (0.4 for BOTH and 0.5 for EITHER). The same pattern can be seen by considering other values of the membership functions

[4]Of course, when the membership functions are dichotomous $\{0, 1\},$ the standard fuzzy set operations do reproduce the corresponding conventional results (since in that case, $s_1 = 0$ and $s_2 = 1$). The issue here is when the membership functions are fuzzy [0, 1].

[5]The SUM will exceed 1 as we move towards the standard operations, and fall below 1 as we move the other way, towards the Bounded operations.

(a, b). Such numerical results appear even more striking when we apply the procedure to the study of persistence or otherwise of deprivation over two points in time.

The following is a possible reason for uniform application of the Algebraic rule failing to give reasonable results in our application. If we take the liberty of viewing the fuzzy propensities as probabilities, then the Algebraic product rule $i(a, b) \to$ joint probability $(a, b) = a * b$ implies zero correlation between the two forms of deprivation, which is clearly at variance with the high positive correlation we expect in the real situation for *similar* states. The rule therefore seems to provide *unrealistically low* estimates for the resulting membership function for the intersection; the Standard rules (S), giving higher overlaps (intersections) are more realistic for (a, b) representing similar states.

By contrast, in relation to *dissimilar* states $\langle(\overline{a}, b), (a, \overline{b})\rangle$ (lack of an overlap between deprivations in two dimensions, or over two points in time), it appears that the Algebraic rule (and hence also the Standard rules) tend to give *unrealistically high* estimates for the resulting membership function for the union. The reasoning similar to the above applies: in real situations, we expect large negative correlations (hence reduced intersections) between *dissimilar* states in the two dimensions of deprivation or over two points in time. In fact, it can be seen by considering some particular numerical values for (a, b) that Bounded union, for instance, gives more realistic results for dissimilar states.

3.4. The Composite set operations

Table 3 shows that the following two options meet the consistency condition, namely that the sum of membership functions for the four resulting sets equals 1:

Algebraic set operations

These meet "Condition 7" on consistency, but their limitations in the context of our present application have already been noted.

Composite set operations

This consists of applying two different types of operations depending on the nature of the fuzzy sets under consideration. As seen in the table, *these two together perfectly meet the consistency condition.*

◦ For sets representing *similar states* – such as the presence (or absence) of both types of deprivation, or of deprivation at both times – the Standard operations (which provide less restrictive intersections than Algebraic operations) are used.

◦ For sets representing *dissimilar states* – such as the presence of one type but the absence of the other type of deprivation, or the presence of deprivation at one time but its absence at the other time – we use the Bounded operations (which provide more restrictive intersections than Algebraic operations).

Table 3. Application of the Algebraic and Composite set operations

Deprivation in two dimensions, or at two points in time	Algebraic set operations	Composite set operations		$a \le b$	$a > b$
1. BOTH (dimensions/times)	a.b	Standard	$\min(a, b)$	s_1	s_1
2. Only one	$a \cdot \bar{b} = a \cdot (1 - b)$	Bounded	$\max(0, a + \bar{b} - 1)$ $= \max(0, a - b)$	0	$s_2 - s_1$
3. Only the other	$\bar{a} \cdot b = b \cdot (1 - a)$	Bounded	$\max(0, \bar{a} + b - 1)$ $= \max(0, b - a)$	$s_2 - s_1$	0
4. NEITHER of the two	$\bar{a} \cdot \bar{b} = (1 - a) \cdot (1 - b)$	Standard	$\min(\bar{a}, \bar{b})$	$1 - s_2$	$1 - s_2$
SUM, $\Sigma(1 - 4)$	1			1	1
5. EITHER (one, the other, or both)	$(a + b) - a \cdot b$	$\max(a, b)$			

Membership function for set "EITHER" using the Composite set operations

(i) as union (a, b)	$= \max(a, b)$, by definition of Standard union
(ii) as complement of set (4)	$= 1 - \min(\bar{a}, \bar{b}) = 1 - (1 - \max(a, b)) = \max(a, b)$
(iii) as aggregation $(1) + (2) + (3)$	$= \min(a, b) + \max(0, b - a) + \max(0, a - b)$ $= \min(a, b) + \lvert a - b \rvert = \max(a, b)$

The Standard complement, $c_s(a) = 1 - a$, is used throughout; $s_1 = \min(a, b)$ and $s_2 = \max(a, b)$.

Of special interest is the last row of Table 3: the set defined by the presence of EITHER of the two states: one, the other or both types of deprivation, or alternatively, the presence of deprivation in at least one of the two times. Membership function of this set may be viewed in any of the three ways which, to be consistent, must be equivalent: (i) as the *union* of the original sets a and b; (ii) as the *complement* of set (4); or (iii) as the *aggregation* of membership functions of sets (1)-(3). (The last two together imply the satisfaction of 'consistency Condition 7'.) The Algebraic operations obviously meet these requirements. They are also met by the Composite operations, as seen at the bottom of the table. For reasons noted, we consider the Composite operations to be a more reasonable choice than uniform application of the Algebraic operations.

A possible more flexible alternative

It may be noted that a weighted combination of the Composite and Algebraic set operations, for instance in the following form, will also meet the consistency requirement:

For sets representing *similar* states	$(1-w)$.(Standard) + w.(Algebraic)
For sets representing *dissimilar* states	$(1-w)$.(Bounded) + w.(Algebraic)

The point of such a scheme would be to accommodate *degrees* of similarity or dissimilarity in a simple way. Parameter w can be thought of as a measure of degree to which different types of states can be distinguished. The choice of weight $w = 0$ gives the Composite scheme defined above, with sharp distinction between similar and dissimilar states. With $w = 1$, we have the Algebraic scheme, applicable when all states are of the same ('neutral') type. With $0 < w < 1$, one may represent intermediate types of situations, incorporate a degree of 'history' or 'memory' in longitudinal analysis, etc. We plan to explore such generalisation in a future study.

3.5. A graphical representation of the fuzzy set operational forms

To elucidate these fuzzy set operational forms, which are central to our methodology, we have developed a graphical representation as shown in Figure 1. The 'universal set' X (i.e., membership $\equiv 1$ for any element of the population of interest) is represented by a rectangle of unit length, and within it is placed the units membership functions $(0 \le a \le 1,$ $0 \le b \le 1)$ on the two subsets. Different placements correspond to different types of fuzzy set operations.

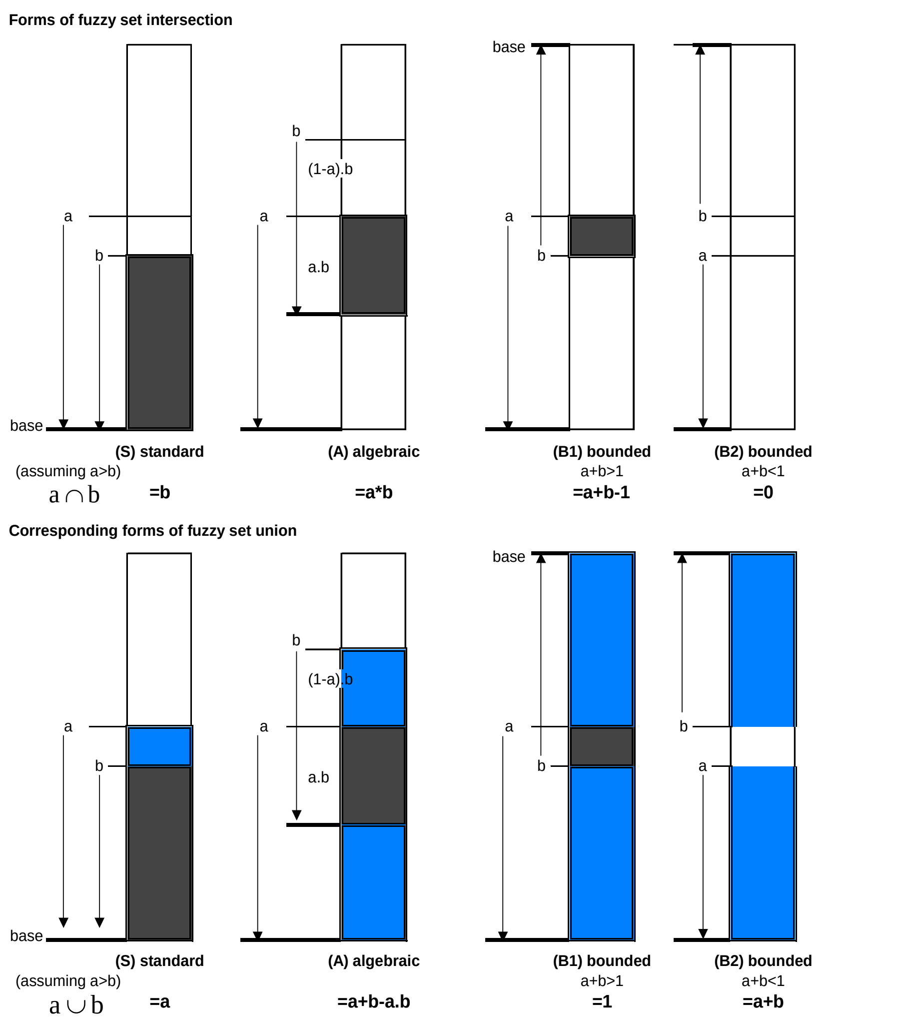

Figure 1. Graphical representation of the fuzzy set operations.

In the Standard form, appropriate for *similar* sets, the two sets (a, b) are placed on the same base, so that the smaller (say b) lies completely within the larger (say a). Consequently, their intersection is maximised, so as to equal the smaller of the subsets. By the same token, their union is minimised, so as to equal the larger of the subsets. The union is represented in the lower figure; it shows separately the amount $(= a - b$ in this case) by which it exceeds the intersection. By placing one set higher than the other within X, the overlap (intersection) is generally reduced, and the underlay (union) increased.

In the Algebraic form, set (b) is placed symmetrically over sets (a) and (non-a), i.e., each of the two receiving a proportionate share of (b), respectively, $a * b$ and $(1 - a) * b$. Hence $a * b$ is the overlap (intersection), while the underlay (union) is $[a + (1 - a) * b] = [a + b - a * b]$.

In the Bounded form, appropriate for *dissimilar* sets, the two sets are placed at the opposite ends of X, thus further reducing their intersection to $(a + b - 1)$ (which is non-zero only if $a + b > 1$); and increasing their union to $(a + b)$, or to 1 if $a + b > 1$. Representation beyond this form (up to the limits $i_{\min}$ and $u_{\max}$ introduced earlier) appears possible, but relevant for our present application.

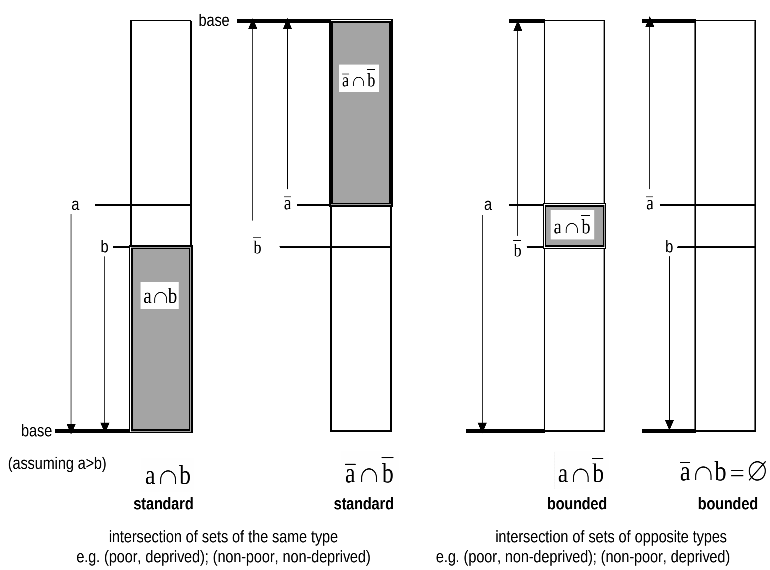

Figure 2. The Composite fuzzy set operations.

Figure 2 shows graphically the Composite set operations we have proposed and used. Each similar state (such as poor/deprived, or non-poor/non-deprived) is treated using the Standard operations. That is, its two constituents are placed on the same base, thus maximizing their intersection and minimizing their union. This is reflective of the positive correlation between similar states in reality. The pair (a, b) is placed at the opposite end to its complement $(\overline{a}, \overline{b})$. Hence sets in a pair representing dissimilar states get placed at opposite ends, resulting in the Bounded form, with appropriately reduced intersection and increased union so as to meet the consistency condition. Note that membership function of one of the two pairs of dissimilar states, $(a, \overline{b})$ and $(\overline{a}, b)$, is always equal to zero. Finally, it may be noted that all these different forms reduce to exactly the same form for corresponding crisp sets with dichotomous $\{0, 1\}$ membership. In this situation, for any unit in the population one or two of the four membership functions equals to 1, i.e., cover entire *X*, and the others equal to 0, so that it makes no difference to the resulting intersections and unions as to where any of the membership functions is placed within *X*.

4. Income Poverty and Non-monetary Deprivation in Combination

4.1. Manifest and latent deprivation

The two measures $-FM_j$ propensity to income poverty, and FS_j the overall non-monetary deprivation propensity – may be combined to construct Composite measures which indicate the extent to which the two aspects of income poverty and non-monetary deprivation overlap for the individual concerned. These measures are as follows.

M_j *manifest deprivation*, representing the propensity to both income poverty and non-monetary deprivation simultaneously. One may think of this as the 'more intense' degree of deprivation.

L_j *latent deprivation*, representing the individual being subject to at least one of the two, income poverty and/or non-monetary deprivation; one may think of this as the 'less intense' degree of deprivation.

Once the propensities to income poverty (FM_j) and non-monetary deprivation (FS_j) have been defined at the individual level (j), the corresponding combined measures are obtained in a straightforward way, using the Composite set operations defined in the previous section. These can then be aggregated to produce the relevant averages or rates for the population. The manifest deprivation propensity of individual j is the intersection (the smaller) of the two measures FM_j and FS_j:

$$M_j = \min(FM_j,\ FS_j).$$

Similarly, the latent deprivation propensity of individual j is the union (the larger) of the two measures FM_j and FS_j:

$$L_j = 1 - \min(\overline{F}\,\overline{M}_j,\ \overline{F}\overline{S}_j) = \max(FM_j,\ FS_j),$$

where $(\overline{F}\,\overline{M}_j,\ \overline{F}\overline{S}_j)$ are the respective complements.

Since the Standard operations provide maximal estimates for both the intersections (min(..)) in the above equations, we have a *maximal* estimate for manifest deprivation, and a *minimal* for latent deprivation. As noted in Section 3, we argue that on substantive grounds, this is a reasonable (indeed desirable) choice for intersections of 'similar' states.

4.2. Multiple dimensions of deprivation

The principle above can be extended to the analysis of prevalence of deprivation in more than two dimensions, such as to the five areas of non-monetary deprivation identified in Section 2, any of these possibly in combination with monetary poverty. Such extension, which is similar to the extension to more than two periods discussed in Section 5, is based on the following results.

With a_i $(i = 1 \text{ to } \Delta)$ as the membership function in deprivation dimension i,

∘ The propensity to deprivation in ALL the dimensions simultaneously $= \min(a_i)$.

◦ The propensity to deprivation in ANY of the dimensions $= \max(a_i)$.

◦ The propensity to deprivation in AT LEAST $(\Delta - \delta + 1)$, $1 \leq \delta \leq \Delta$, of the Δ dimensions $= \min_{\delta}(a_i)$, where $\min_{\delta}$ refers to the δth smallest value in the set (a_i), i.e., the minimum disregarding the $(\delta - 1)$ smallest values.

4.3. Some empirical results

Table 4 shows measures reflecting the degree of overlap between income poverty and non-monetary deprivation at the individual level and the resulting manifest and latent deprivation rates. To see these in a relative perspective – i.e., controlling for the level of poverty in the country – the second panel shows the ratio of the manifest and latent deprivation rates to the mean of these two rates[6].

The pattern is very clear. The (manifest/mean) deprivation ratio is the lowest in countries with the lowest levels of poverty, and the highest in courtiers with the highest levels of poverty: it is for instance ≤ 0.30 for Finland, Sweden and Denmark, and ≥ 0.45 for Portugal, Greece, Italy and Ireland. The pattern is reversed as concerns latent deprivation (though the variation across countries is less marked here): the ratios for the above-mentioned two sets of countries being, respectively, ≥ 1.70 and ≤ 1.55. Consequently, the ratio (manifest/latent) is nearly twice as high for the countries with the highest poverty levels compared to those with the lowest poverty levels.

The implication of these results is important. In countries where levels of poverty/deprivation are already high, deprivations in the income and non-monetary domains are more likely to afflict the same individuals in the population – accentuating the impact of disparities.

[6]This mean is identical by definition to the mean of FM and FS, as $(a + b) = \min(a, b) + \max(a, b)$; in fact, the mean is close to either of the two last mentioned measures since $FS \sim FM$. Cols. FM, FS and FS/FM have been repeated from Table 1 to facilitate comparison.

Table 4. Fuzzy measures of deprivation: monetary, non-monetary, and the two forms in combination

	Fuzzy deprivation rates					Ratios			
	FM	FS	Manifest	Latent	Mean	FS/ FM	Manifest/ Mean	Latent/ Mean	Manifest/ Latent
0 EU-15	16.0	16.0	6.4	25.7	16.0	1.00	0.40	1.60	0.25
1 FI	8.9	10.9	3.0	16.7	9.9	1.22	0.30	1.70	0.18
2 SE	10.0	10.9	3.0	17.9	10.5	1.08	0.29	1.71	0.17
3 DK	9.2	11.2	2.8	17.6	10.2	1.22	0.27	1.73	0.16
4 NL	11.4	12.0	4.5	18.9	11.7	1.06	0.38	1.62	0.24
5 LU	11.6	11.1	4.0	18.7	11.4	0.95	0.35	1.65	0.21
6 DE	12.4	10.9	3.4	19.9	11.7	0.88	0.30	1.70	0.17
7 AT	11.8	13.4	4.0	21.2	12.6	1.14	0.32	1.68	0.19
8 BE	13.5	15.0	5.3	23.1	14.2	1.11	0.37	1.63	0.23
9 FR	14.7	15.8	6.6	23.9	15.2	1.07	0.43	1.57	0.28
10 UK	18.3	19.0	7.8	29.4	18.6	1.04	0.42	1.58	0.27
11 ES	19.5	18.4	8.2	29.7	18.9	0.95	0.43	1.57	0.28
12 IE	16.8	17.8	8.2	26.4	17.3	1.06	0.47	1.53	0.31
13 IT	19.7	19.4	9.0	30.1	19.6	0.98	0.46	1.54	0.30
14 GR	22.2	22.6	10.2	34.6	22.4	1.02	0.45	1.55	0.29
15 PT	22.3	23.9	10.7	35.5	23.1	1.07	0.46	1.54	0.30
Simple average	14.8	15.5	6.1	24.2	15.1	1.06	0.38	1.62	0.24

FM fuzzy measure of monetary poverty rate ('fuzzy monetary'),

FS fuzzy measure of overall non-monetary deprivation rate ('fuzzy supplementary'),

Manifest propensity to both FM and FS deprivation,

Latent propensity to either form of deprivation (FM and/or FS),

Mean mean of (FM, FS) = mean of (Manifest, Latent).

Note. Figures show simple averages of cross-sectional results over 8 ECHP waves.

5. Longitudinal Aspects: Persistence of Poverty and Deprivation

Above we have described five main measures which have been developed and analysed in this paper. In addition to H_j, the conventional income poverty index $\{0, 1\}$, these include the fuzzy measures: FM_j, income poverty; FS_j, non-monetary deprivation; M_j, manifest deprivation, representing the propensity to both income poverty and non-monetary deprivation simultaneously; and L_j, latent deprivation, representing the individual's propensity to being subject to at least one of the two, income poverty and/or non-monetary deprivation. In addition, the propensity to non-monetary deprivation can be analysed separately in its various dimensions, such as the five dimensions (Sup1-Sup5) identified earlier. Then in principle there are also measures corresponding to FS, M and L, etc. in the conventional dichotomous $\{0, 1\}$ form, which we have not reported here.

Any of these diverse measures can be studied in the time dimension: both in the cross-sectional and the longitudinal contexts. The cross-sectional may refer to levels over single periods (say years) or to averages over a number (T) of periods. In the longitudinal dimension, indicators may be designed to capture the experience of poverty and deprivation at any time during a period, or persistently or continuously over the period[7]. A more general analysis of complex fuzzy measures pursued here in general terms. Here we confine the analysis to somewhat simpler but important measures concerning *persistence or otherwise of deprivation over time*, generalising the results of Subsection 4.1.

5.1. Longitudinal analysis over two time periods

The methodology for longitudinal analysis of a deprivation measure over two consecutive time periods is, formally and in statistical terms, the same as that for cross-sectional analysis of two different measures over a single time period described in Section 4. Here instead of considering the fuzzy sets of monetary poverty and non-monetary deprivation, we consider any kind of fuzzy deprivation measured at two consecutive periods, say period (1) and period (2). So if for instance we consider the Fuzzy Monetary measure, we can define $a = FM^{(1)}$ and $b = FM^{(2)}$ as the membership functions of the sets 'POOR at time 1' and 'POOR at time 2', respectively and the corresponding NON-POOR complements, $(1 - a)$ and $(1 - b)$, thus giving four membership functions.

[7]One can also construct, for instance, individual propensities and average rates of exit and of re-entry into the state of poverty and deprivation (individual histories, with or without 'memory'), the distribution of the time spent in such states, or possibly panel regression and other models, etc., using fuzzy generalisation of the traditional approach.

An early attempt to define joint membership functions over two periods was in Cheli [9] for what has been called the *Totally Fuzzy and Relative* (TFR) poverty measures. The four membership functions were taken in the form $g_{k,1} = \min(a, b)$, which corresponds to the Standard fuzzy set operations, but applied so as to meet the marginal constraints (which amount to 'Condition 7' of Section 3 above). The actual procedure involved applying the Standard procedure independently to any of the four sets, and then determining the remaining three from the marginal constraints. In order to choose one of the four alternative solutions (determined by which of the four sets is chosen as the starting point) the author followed Manton et al. [18] in choosing the solution that produced the joint membership function matrix with *minimum entropy*. Subsequently, Betti et al. [4] showed that in fact there are only two possible outcomes: starting from either of the two similar sets, (poor$_{(1)}$--poor$_{(2)}$) or (non-poor$_{(1)}$--non-poor$_{(2)}$), produces the same result; a different result is obtained by starting from either of the two dissimilar sets, (poor$_{(1)}$--non-poor$_{(2)}$) or (non-poor$_{(1)}$--poor$_{(2)}$).

In any case, approaches such as the above can be criticised on two counts. First, they make no distinction between similar and dissimilar states, disregarding the large positive correlation which can be expected between the former and the large negative correlation between the latter. Secondly, using the minimum entropy actually results in a discontinuity, with a sudden shift in the outcome when the poverty membership functions for the two periods pass from being concordant (both greater than or both less than 0.5) to being discordant (one < 0.5 and the other > 0.5). Such a discontinuity is not meaningful in a real situation.

The Composite solution described in Section 3 of course explains, and provides a solution to, this problem. In this the *similar sets*, (poor$_{(1)}$--poor$_{(2)}$) and (non-poor$_{(1)}$--non-poor$_{(2)}$), are treated with the Standard set operations, and *dissimilar sets*, (poor$_{(1)}$--non-poor$_{(2)}$) and (non-poor$_{(1)}$--poor$_{(2)}$), with the Bounded set operations. The procedures described in the previous section for the joint analysis of *cross-sectional* measures of monetary and non-monetary forms of deprivation in fact also apply to *longitudinal analysis over two time periods*. Propensities $F1_j$ and $F2_j$ to deprivation (monetary, non-monetary, or any combination of the two) at two points in time may be combined to construct Composite measures which indicate the extent to which they persist over time for the individual concerned. These measures are as follows.

P_j *persistent deprivation*, representing the intersection of the propensities to deprivation at the two times, $P_j = \min(F1_j, F2_j)$. In the crisp set counterpart, this refers to an individual being in the state of deprivation at both the times.

A_j *anytime deprivation*, representing the union of the propensities to deprivation at the two times, $P_j = \max(F1_j, F2_j)$. In the crisp set counterpart, this refers to an individual being in the state of deprivation at either (one, the other, or at both) of the times.

The two temporal concepts can be seen graphically exactly as discussed in Section 4 for deprivation in two dimensions. As before, the Standard operations provide a *maximal* estimate for persistent deprivation, and a *minimal* for anytime deprivation compared to any other permissible form of the fuzzy set operations.

5.2. Persistence over t > 2 points in time

Three time periods

Consider first three consecutive time points, with an individual's propensities to poverty (or to a more general measure of deprivation) as (a_1, a_2, a_3). Table 5, in a form similar to Table 3, displays the results of the application of Composite fuzzy set intersections. The full picture is represented by membership functions for $2^3 = 8$ fuzzy sets. Set (1) $= (a_1, a_2, a_3)$ represents propensity to be *continuously poor* over the three years; set $(8) = (\overline{a}_1, \overline{a}_2, \overline{a}_3)$ represents the propensity to be *never poor* over that period. Each of these involves a sequence of similar states, and hence is given by the Standard fuzzy set intersections. The propensity to be poor for at least one time is given in three equivalent forms as defined in Table 3. Clearly, as the union of (a_1, a_2, a_3), or as the complement of (8), this equals $\max(a_1, a_2, a_3)$. Consistency of the procedure is confirmed by the fact that the same result is obtained when anytime poverty is seen as the aggregation of membership functions (1) to (7), as demonstrated in the lower panel of Table 5. Note that it is sufficient for the present purpose to identify only $6 = 2 \cdot T$ separate groups of sets[8].

In distinction from continuous poverty, we may define *persistent poverty* as poverty over at least two of the three years. In line with 'maximising the intersection' implied in the application of the Standard operation, we disregard the lowest of the three values (a_1, a_2, a_3). In the same way as noted in Subsection 4.2, *persistent poverty* is given by $\min_2(a_1, a_2, a_3)$, meaning the 2nd smallest value in the set.

Table 5. Membership function for 8 interaction sets for three time periods

		a_3	$\bar{a}_3 = (1 - a_3)$
a_1	a_2	(1) $\min(a_1, a_2, a_3)$	(2) $\max(0, \min(a_1, a_2) + \bar{a}_3 - 1) = \max(0, s_1 - a_3)$
	$\bar{a}_2 = (1 - a_2)$	(3) + (4) $\max(0, a_1 + \bar{a}_2 - 1) = \max(0, a_1 - a_2)$	
$\bar{a}_1 = (1 - a_1)$	a_2	(5) + (6) $\max(0, \bar{a}_1 + a_2 - 1) = \max(0, a_2 - a_1)$	
	$\bar{a}_2 = (1 - a_2)$	(7) $\max(0, \min(\bar{a}_1, \bar{a}_2) + a_3 - 1) = \max(0, a_3 - s_2)$	(8) $\min(\bar{a}_1, \bar{a}_2, \bar{a}_3) = 1 - \max(a_1, a_2, a_3)$

Aggregation of membership function (1) to (7) in the panel above

Term condition	(1)	+ (2)	+ (3) to (6)	+ (7)	=
$a_3 \le s_1$	a_3	$s_1 - a_3$	$s_2 - s_1$	0	$s_2 = \max(a_1, a_2, a_3)$
$s_1 < a_3 \le s_2$	s_1	0	$s_2 - s_1$	0	$s_2 = \max(a_1, a_2, a_3)$
$a_3 > s_2$	s_1	0	$s_2 - s_1$	$a_3 - s_2$	$a_3 = \max(a_1, a_2, a_3)$

Note. $s_1 = \min(a_1, a_2)$; $s_2 = \max(a_1, a_2)$.

[8] For the present purpose it is not necessary to break down the membership functions between sets (3) and (4), or between sets (5) and (6). Note that the Composite procedure implies that in forming the intersections in cells such as (2) or (7) involving different types of states, we first take the intersection of the first two *similar* states using the Standard operation, and then the intersection of the results with the *dissimilar* state which follows using the Bounded operation.

The general case of *T* time periods

Consider a period of T time points. For each time i there are two complementary cross-sectional sets, 'poor' and 'non-poor', with membership functions for any individual as a_i and $(1 - a_i)$, respectively. This gives a total of 2^T longitudinal sets over the period (each representing an intersection of a particular sequence of T cross-sectional sets). In order to establish consistency of the present procedure in a way similar to that for $T = 3$ above, it is sufficient to identify membership for only the $2 \cdot T$ aggregations of the intersection sets as shown in Table 6. In the table, the symbol '+' stands for a 'set of poor' at a particular time i, with membership function a_i; the symbol '–' stands for a 'set of non-poor' at time i, with membership function $(1 - a_i)$; and the symbol '?' stands for a set of either type, meaning that it is not necessary for the present purpose to specify the type of set involved[9].

The table shows two panels. Each row in each of the panels represents an aggregation of sets formed by the intersection of a certain type of sequences of single period sets: starting with a certain number of sets of the *same* type (e.g., all '+'), then with one set of the *different* type (e.g., a '–'), and subsequently with a sequence of sets of *any* type (represented by the symbol '?').

The last row represents two important individual sets: 'continuously poor' $= \min(a_i : i = 1 \text{ to } T)$; and 'never poor' $= \min(\overline{a}_i : i = 1 \text{ to } T) = 1 - \max(a_i : i = 1 \text{ to } T)$. The propensity to be poor for at least one time ('ever poor') is given in three equivalent forms as before. Clearly, as the union of $(a_1, ..., a_T)$, or as the complement of 'never poor', this equals $\max(a_i : i = 1 \text{ to } T)$. To confirm the consistency of our procedure, we establish below that the aggregation of first $(T - 1)$ rows of Table 6

[9]The presence of symbols '?' indicates that the row represents an aggregation of 2^Q sets where Q is the number of these question marks appearing in the row.

equals the complement of ('continuously poor' + 'never poor'), i.e., $= \max(a_i) - \min(a_i)$.

Let $r_i = \min(a_j : j = 1 \text{ to } i)$ be the minimum value of the membership functions for the first i periods, and $R_i = \max(a_j : j = 1 \text{ to } i)$ be the corresponding maximum value. Then for row i, the aggregation of all the sets, it represents, is given, separately for each panel, by the intersection of the first $(i+1)$ *specified* cells as shown at the bottom of the table[10]. Summing over the first $(T-1)$ rows gives:

$$\sum_{i=1}^{T-1}(r_i - r_{i+1}) + \sum_{i=1}^{T-1}(R_{i+1} - R_i) = (r_1 - r_T) + (R_T - R_1)$$

$$= R_T - r_T$$

$$= \max(a_i) - \min(a_i).$$

In conclusion, the result is pleasantly simple. For any number of periods with propensities to poverty (or more general form of deprivation) as (a_i),

- The propensity to continuous, $C_i = \min(a_i)$;
- The propensity to any-time, $A_i = \max(a_i)$.

[10]As in the $T = 3$ case, the following rules are used in computing the intersection of a sequence of cross-sectional sets of the form appearing in rows of Table 6. In accordance with the Composite rules defined in Section 3, we first take the Standard intersection of the sequence of similar states (such as all '+') over i periods, and then take the intersection of the result with the dissimilar state which immediately follows it using the Bounded operation. Each of the remaining $(T-i-1)$ cells marked '?' represents all states, i.e., membership function identically equal to 1, so that it makes no difference to the intersection being considered.

Table 6. Membership function for interaction sets for T time periods

Panel 1														Panel 2											
Time	1	2	3	4	5	6	..		T-3	T-2	T-1	T		1	2	3	4	5	6	..		T-3	T-2	T-1	T
set group																									
1	+	–	?	?	?	?	..	?	?	?	?	?		–	+	?	?	?	?	..	?	?	?	?	?
2	+	+	–	?	?	?	..	?	?	?	?	?		–	–	+	?	?	?	..	?	?	?	?	?
3	+	+	+	–	?	?	..	?	?	?	?	?		–	–	–	+	?	?	..	?	?	?	?	?
4	+	+	+	+	–	?	..	?	?	?	?	?		–	–	–	–	+	?	..	?	?	?	?	?
..	..	..	..	..	..	..	..	..	..	..	..	..		..	..	..	..	..	..	..	..	..	..	..	..
T-3	+	+	+	+	+	+	..	+	+	–	?	?		–	–	–	–	–	–	..	–	–	+	?	?
T-2	+	+	+	+	+	+	..	+	+	+	–	?		–	–	–	–	–	–	..	–	–	–	+	?
T-1	+	+	+	+	+	+	..	+	+	+	+	–		–	–	–	–	–	–	..	–	–	–	–	+
T	+	+	+	+	+	+	..	+	+	+	+	+		–	–	–	–	–	–	..	–	–	–	–	–

Intersection represented by row i

Panel 1	$\max(0,\ r_i + \bar{a}_{i+1} - 1) = \max(0,\ r_i - a_{i+1}) = r_i - r_{i+1}$
Panel 2	$\max(0,\ \bar{r}_i + a_{i+1} - 1) = \max(0,\ a_{i+1} - R_i) = R_{i+1} - R_i$

Notes. We have used the notation $\bar{a}_i = (1 - a_i)$; $\bar{r}_i = \min(\bar{a}_i) = 1 - \max(a_i) = 1 - R_i$.

The last terms in the above expressions follow from the following observation.

Condition	Giving	
$a_{i+1} < r_i$	$r_{i+1} = a_{i+1} < r_i$	$R_{i+1} = R_i > a_{i+1}$
$r_i \le a_{i+1} \le R_i$	$r_{i+1} = r_i \le a_{i+1}$	$R_{i+1} = R_i \ge a_{i+1}$
$a_{i+1} > R_i$	$r_{i+1} = r_i < a_{i+1}$	$R_{i+1} = a_{i+1} > R_i$

Persistent poverty

The argument given above for $T = 3$ for the construction of persistent poverty is directly extended to the general case. We adopt the following definition of persistent poverty for the numerical results presented here. It refers to poverty during *at least a majority* of the T years, i.e., for at least T_p years, where $T_p = \text{int}(T/2) + 1$ (i.e., the smallest integer strictly larger than $T/2$). For instance, for a 4 or 5 year period, persistent refers to poverty for at least 3 years; for $T = 6$ or 7 years, it refers to poverty for at least 4 years, etc. At the individual level, *persistent poverty* is given by $\min_P(a_i)$, meaning the Pth smallest value in the set, where $P = T - T_p + 1$. Continuous and any-time poverty are merely special cases of this with, respectively, $(T_P = T, P = 1)$ and $(T_P = 1, P = T)$.

5.3. Some empirical results

Table 7 analyses the levels of income poverty in the time dimension. The results are based on a balanced panel consisting of individuals present in all the 8 waves of the survey. For the conventional and fuzzy measures separately, four types of rates are shown: (1) the rate of poverty experienced at any time (at least 1 year) during the 8-year period; (2) the average of cross-sectional poverty rates over the period; (3) the persistent poverty rate, meaning poverty for a majority (5 or more out of 8) of the years; and (4) the continuous poverty rate over the entire period. The rates sharply decline from (1) to (4): taking a simple average over courtiers, from 35-37% for any-time poverty to only 3-4% for continuous poverty.

Noteworthy from a methodological point, however, is the difference in the performance of the conventional and the fuzzy approaches, especially concerning the estimated incidence of continuous poverty. It appears that movements in and out of poverty tend to be somewhat over-estimated (and hence the persistent or continuous poverty rates under-estimated) with the conventional approach, presumably because it gives too much weight to even small movements across the poverty line.

Table 8 illustrates the usefulness of the present (fuzzy set) methodology in dealing with the double complexity of *longitudinal* analysis of *multi-dimensional* measures. Latent and manifest deprivation measures are constructed for each time taking into account the degree of overlap between income and non-monetary aspects at the micro level.

These measures are then studied longitudinally taking into account their degree of persistence over time, again at the micro level.

The most intense deprivation is reflected in the last column, M(4), of the table. The rates are under 0.5% in Denmark, Netherlands and Germany, and at the other end 1.5-2.5% in Italy, Greece and Portugal for the continuous experience of income poverty simultaneously with non-monetary deprivation. By contrast, the experience of one or the other form of deprivation at some time during the 8-year period, column L(1), varies between 40% in Denmark to 60% in Portugal.

Table 7. Conventional and fuzzy measures of longitudinal monetary poverty rates

	Conventional head-count ratio (H)				Fuzzy monetary (FM)				Ratio (H/FM)			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
3 DK	28.0	9.1	4.5	0.9	23.9	8.0	4.4	1.0	1.17	1.14	1.02	0.90
4 NL	26.8	8.8	5.6	0.6	26.8	9.8	6.9	1.6	1.00	0.90	0.82	0.38
6 DE	26.6	10.2	6.2	1.6	26.2	10.5	6.9	2.3	1.01	0.97	0.89	0.67
8 BE	32.9	12.6	8.5	2.3	29.4	12.0	8.7	2.8	1.12	1.05	0.98	0.82
9 FR	33.4	13.4	9.3	2.3	30.4	13.0	9.3	3.5	1.10	1.04	1.00	0.66
10 UK	39.3	16.5	11.1	2.5	36.9	16.5	12.2	4.2	1.07	1.00	0.90	0.60
11 ES	46.5	18.8	13.3	2.8	43.7	19.4	14.8	4.3	1.06	0.97	0.90	0.65
12 IE	45.2	19.6	15.0	4.4	38.7	17.3	13.6	4.4	1.17	1.13	1.10	1.00
13 IT	42.6	18.7	13.2	3.0	40.2	19.2	14.5	4.8	1.06	0.97	0.91	0.63
14 GR	47.7	22.3	15.6	4.7	45.4	23.1	17.6	6.5	1.05	0.96	0.89	0.73
15 PT	46.5	21.6	18.5	5.3	44.5	22.5	19.6	8.1	1.04	0.96	0.95	0.66
Simple average	37.8	15.6	11.0	2.8	35.1	15.6	11.7	4.0	1.08	1.00	0.94	0.70

Poverty/deprivation rates:

(1) Anytime: propensity to poverty/deprivation for at least 1 out of the 8 years of ECHP,

(2) Cross-sectional: rate averaged over 8 waves,

(3) Persistent: propensity to poverty/deprivation for at least 5 out of the 8 years,

(4) Continuous: propensity to poverty/deprivation over all the 8 years of ECHP.

Note. The results are for a 'balanced panel', i.e., for the population present in all 8 ECHP waves.

Table 8. Fuzzy supplementary, latent and manifest measures of longitudinal deprivation rates

	Latent (L)				Fuzzy supplementary (FS)				Manifest (M)			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
3 DK	40.2	15.5	9.7	2.1	27.9	9.8	5.0	0.8	8.4	2.2	0.5	0.1
4 NL	43.1	16.7	12.2	4.3	31.4	10.7	7.0	2.0	11.6	3.7	2.4	0.4
6 DE	43.4	17.7	11.5	4.1	32.0	10.0	4.3	1.1	10.2	2.9	1.1	0.2
8 BE	48.7	20.8	15.1	6.0	36.2	13.2	8.2	2.6	11.9	4.3	2.6	0.6
9 FR	45.3	21.7	17.0	7.0	32.8	14.4	10.6	3.5	14.3	5.7	3.8	1.1
10 UK	55.3	27.0	21.2	8.6	41.7	17.3	11.8	3.8	18.1	6.8	4.2	1.1
11 ES	59.3	29.0	23.4	7.8	43.4	17.4	11.6	2.1	21.1	7.7	4.9	0.8
12 IE	52.8	26.2	22.5	8.7	39.5	16.8	13.2	3.7	20.0	7.9	5.8	1.4
13 IT	56.3	29.1	23.3	9.6	42.2	18.4	13.0	4.1	20.3	8.5	5.7	1.5
14 GR	64.8	34.9	28.6	11.4	51.3	22.4	16.1	4.1	25.1	10.7	6.9	1.7
15 PT	61.0	34.8	32.6	15.3	46.1	22.6	20.0	7.5	25.2	10.3	8.1	2.4
Simple average	51.8	24.9	19.7	7.7	38.6	15.7	11.0	3.2	16.9	6.4	4.2	1.0

Manifest propensity to both fuzzy monetary (FM) and non-monetary (FS) deprivation,

Latent propensity to either form of deprivation (FM and/or FS),

Poverty/deprivation rates: (1) Anytime; (2) Cross-sectional; (3) Persistent; (4) Continuous.

Note. The results are for a 'balanced panel', i.e., for the population present in all 8 ECHP waves.

6. Concluding Remarks

In this paper, we have focussed on aggregate (average) measures of relative poverty and deprivation, dealing with their multi-dimensional and longitudinal aspects, and analysing them in a multi-country comparative context. Our main aim has been to clarify, refine and empirically demonstrate some aspects of the application of the fuzzy set approach to this type of analysis.

We conclude by identifying some directions requiring further work. (1) Models and methods for constructing fuzzy indicators of poverty and deprivation rates may be further elaborated and options compared, along with sensitivity analyses. (2) Potential advantage of the fuzzy over the conventional approach needs further and more convincing empirical demonstration, especially in the analysis of poverty of subpopulations such as children, old persons, minorities. (3) The analysis has to be extended to other poverty measures beyond simply poverty rates. In particular, further work is required to extend the approach to more complex longitudinal analyses, such as transitions in and out of the state of poverty/deprivation, possibly taking into account the influence of the individual's 'poverty history'. (4) Sampling errors and design effects for fuzzy measures based on complex sample design should be computed; it is important to demonstrate the extent to which fuzzy measures are more precise than the corresponding conventional measures. (5) Finally, we must mention the potential for integration of the fuzzy methods into the modelling and analysis of poverty dynamics[11]. The list is by no means exhaustive, but it presents a promising agenda.

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