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CONGRUENCES FOR TANGENT AND GENOCCHI NUMBERS

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Abstract

In this note we provide congruences for the numbers defined to be the "odd part" of the tangent numbers. As a result, we obtain congruences for Genocchi numbers whose indexes are powers of two, and also intriguing congruences for alternating power sums of natural numbers.

1. Introduction

Consider the following power series:

$$G(t) = 2t/(e^t + 1) = t + \sum_{n=1}^{\infty} (t^{2n}/(2n)!)G_{2n};$$

$$A(t) = 1 - \tanh(t) = 2/(e^{2t} + 1) = 1 + \sum_{n=1}^{\infty} (t^{2n-1}/(2n-1)!)A_{2n-1}.$$

Clearly $A_{2n+1} = 2^{2n}G_{2n+2}/(n+1)$, where the numbers G_{2n}, $n \geq 1$ are called *Genocchi numbers* (see, e.g., [7]). Now, define the tangent numbers

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$T_{2n-1},\ n \geq 1$ to be the coefficients in the power series

$$tg(t) = \sum_{n=1}^{\infty} (t^{2n-1}/(2n-1)!)T_{2n-1}.$$

Since $tg(x) = -i\tanh(ix)$, we get $T_{2n-1} = (-1)^n A_{2n-1}$, and then

$$T_{2n+1} = (-1)^{n+1}2^{2n}G_{2n+2}/(n+1), \quad n \geq 0.$$

Proposition 1 [4]**.** *Consider the set of n-tuples* $u = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ *such that* $u_1 = 1$ *and* $u_i = u_{i-1}$ *or* $u_i = u_{i-1} + 1$ *for* $i \geq 2$. *Then, the Genocchi numbers are given by* $G_{2n+2} = -\sum (-1)^{u_n}(u_1 u_2 \cdots u_n)^2$.

Remark 1. As a consequence of this result, one concludes that G_{2n+2} is an odd number. In fact, the only odd term in $-\sum(-1)^{u_n}(u_1 u_2 \cdots u_n)^2$ is $(1 \times 1 \times \cdots \times 1)$.

Proposition 2 [10]**.** *Let p be an odd prime and let* λ *be the period-length of the sequence* $(T_n \bmod p)$, *for n odd. Then,*

$$\lambda = \begin{cases} p-1, & \text{if } p = 4k+1; \\ 2(p-1), & \text{if } p = 4k+3, \end{cases}$$

and $T_{n+\lambda} \equiv T_n \pmod p$, *for all odd numbers* $n \geq 1$.

2. Results

Many authors have found interesting congruences for tangent and Genocchi numbers, like Gandhi [5], Carlitz [2], Gessel [6] and Chen [3].

Nevertheless, we shall prove a simple result concerning tangent numbers, which seems to have been so far (at least partially) unnoticed. In fact, items (ii) and (iii) of our theorem have been stated in terms of Bernoulli numbers by Srinivasa Ramanujan (see, e.g., [8]).

Definition. Let $\alpha(n)$ be the exponent of the highest power of two that divides T_{2n+1}, and define $\hat{T}_{2n+1} = T_{2n+1}/2^{\alpha(n)}$.

Theorem. *Let* $n \in \mathbb{Z}_+$. *Then*,

(i) $T_{2n+1}/2^n \equiv 1 \pmod 3$;

(ii) $\hat{T}_{4n+1} \equiv 1 \pmod{30}$;

(iii) $\hat{T}_{8n+3} \equiv 1 \pmod{30}$;

(iv) $\hat{T}_{4n+3} \equiv 17 \pmod{30}$, *if* $n \equiv 1 \pmod 4$.

Moreover, *if* $n = 2^r(1+2t)-1$, *with* $r \geq 2$, $t \geq 0$, *then*

$$\text{(v)}\ \hat{T}_{4n+3} \equiv \begin{cases} 1 \pmod{30}, & \textit{if } r \equiv 0 \pmod 4; \\ 17 \pmod{30}, & \textit{if } r \equiv 1 \pmod 4; \\ 19 \pmod{30}, & \textit{if } r \equiv 2 \pmod 4; \\ 23 \pmod{30}, & \textit{if } r \equiv 3 \pmod 4. \end{cases}$$

Proof. (i) It is well known that the numbers T_{2n+1} are divisible by 2^n (see [9] for an elementary proof). By Proposition 2, $T_{n+4} \equiv T_n \pmod 3$. Thus, since

$$\begin{cases} T_{2n+1} \equiv 1 \pmod 3 \text{ and } 2^n \equiv 1 \pmod 3), & \text{if } n \text{ is even;} \\ T_{2n+1} \equiv 2 \pmod 3 \text{ and } 2^n \equiv 2 \pmod 3), & \text{if } n \text{ is odd,} \end{cases}$$

we conclude that

$$T_{2n+1} = 2^n(3s+1).$$

(ii)-(v) By Proposition 2,

$$\begin{cases} T_{n+4} \equiv T_n \pmod 3; \\ T_{n+4} \equiv T_n \pmod 5. \end{cases}$$

Furthermore, we also have $T_{n+4} \equiv T_n \pmod 2$ for $n \geq 3$. Hence,

$$T_{n+4} \equiv T_n \pmod{30}, \quad n \geq 3.$$

Since $T_3 = 2$, we obtain $T_{4n+3} \equiv 2 \pmod{30}$. Analogously, since $T_5 = 16$, $T_{4n+1} \equiv 16 \pmod{30}$.

(ii) We have that $T_{4n+1} = 2^{4n} | G_{4n+2} |/(2n+1) = 2^{4n}(2c+1)$ for some $c \in \mathbb{Z}_+$. Since $2^{4n} \equiv 16 \pmod{30}, n \geq 1,$ we have $(2c+1) \equiv 1 \pmod{30}$. Thus,

$$T_{4n+1} = 2^{4n}(30s+1).$$

(iii) We have $T_{4n+3} = 2^{4n+1}G_{4(n+1)}/(n+1)$. Then, if n is even, we have $T_{4n+3} = 2^{4n+1}(2c+1)$ for some $c \in \mathbb{Z}_+$. Since $2^{4n+1} \equiv 2 \pmod{30}$, $n \geq 1$, we have $(2c+1) \equiv 1 \pmod{30}$, and so

$$T_{4n+3} = 2^{4n+1}(30s+1), \text{ if } n \text{ is even.}$$

(iv) Let $n \equiv 1 \pmod 4$. Then, $n+1 = 2(2k+1)$, and then $T_{4n+3} = 2^{4n}(2c+1)$, for some $c \in \mathbb{Z}_+$. Since $2^{4n} \equiv 16 \pmod{30}$, $n \geq 1$, we have $(2c+1) \equiv 17 \pmod{30}$, and thus

$$T_{4n+3} = 2^{4n}(30s+17), \text{ if } n \equiv 1 \pmod 4.$$

(v) Let $n \equiv 3 \pmod 4$. Then, we can write $n = 2^r(1+2t)-1$, with $r \geq 2$, $t \geq 0$. Hence, $T_{4n+3} = 2^{4n-r+1}G_{4(n+1)}/(1+2t) = 2^{4n-r+1}(2c+1)$, for some $c \in \mathbb{Z}_+$. We have that

$$2^{4n-r+1} \equiv \begin{cases} 2 \pmod{30}, & \text{if } r \equiv 0 \pmod 4; \\ 16 \pmod{30}, & \text{if } r \equiv 1 \pmod 4; \\ 8 \pmod{30}, & \text{if } r \equiv 2 \pmod 4; \\ 4 \pmod{30}, & \text{if } r \equiv 3 \pmod 4, \end{cases}$$

which implies

$$(2c+1) \equiv \begin{cases} 1 \pmod{30}, & \text{if } r \equiv 0 \pmod 4; \\ 17 \pmod{30}, & \text{if } r \equiv 1 \pmod 4; \\ 19 \pmod{30}, & \text{if } r \equiv 2 \pmod 4; \\ 23 \pmod{30}, & \text{if } r \equiv 3 \pmod 4. \end{cases}$$

Therefore,

$$T_{4n+3} = \begin{cases} 2^{4n-r+1}(30s+1), & \text{if } r \equiv 0 \pmod 4; \\ 2^{4n-r+1}(30s+17), & \text{if } r \equiv 1 \pmod 4; \\ 2^{4n-r+1}(30s+19), & \text{if } r \equiv 2 \pmod 4; \\ 2^{4n-r+1}(30s+23), & \text{if } r \equiv 3 \pmod 4. \end{cases}$$

Remark 2. Chen [3] has obtained a variant of item (i) of our theorem. He proved, using a lemma on regular continued fractions, that

$$\frac{T_{2n+1}}{2^n} \equiv 4^{n-1} \pmod{6}.$$

Corollary 1. *Let* $r \in \mathbb{Z}_+$. *Then,*

$$G_{2^{r+2}} \equiv \begin{cases} 1 \pmod{30}, & \text{if } r \equiv 0 \pmod{4}; \\ 17 \pmod{30}, & \text{if } r \equiv 1 \pmod{4}; \\ 19 \pmod{30}, & \text{if } r \equiv 2 \pmod{4}; \\ 23 \pmod{30}, & \text{if } r \equiv 3 \pmod{4}. \end{cases}$$

Proof. Since $G_4 = 1$ and $G_8 = 17$, the result holds for $r = 0, 1$. For $r \geq 2$, apply the identity $2^{4n+1}G_{4(n+1)} = (n+1)T_{4n+3}$ to the expressions for T_{4n+3} when $n = 2^r - 1$, obtained in the proof of the theorem.

Now we provide two intriguing congruences for alternating power sums of natural numbers, as a corollary of item (i) of our theorem and of a result of Arnol'd [1].

Corollary 2. *Given* $k \in \mathbb{Z}_+$, *there exists* $u(k) \in \mathbb{Z}_+$ *such that*

$$1 + 2^{2k+1} - 3^{2k+1} - 4^{2k+1} + + - \cdots - (4r)^{2k+1} \equiv (-2)^k (3u+1) \pmod{4r+1};$$

$$-1 + 2^{2k+1} + 3^{2k+1} - 4^{2k+1} - + + \cdots + (4r+2)^{2k+1} \equiv (-2)^k (3u+1) \pmod{4r+3},$$

for all $r \in \mathbb{N}$.

Proof. Let the numbers $f(n)$ be defined by $\sum_{n=1}^{\infty} f(n)z^n/n! = 1 + \tanh(z)$. Since $tg(x) = -i\tanh(ix)$, we have $f(2m+1) = (-1)^m T_{2m+1}$. The just mentioned result of Arnol'd says that for each odd number $p > 1$ one has the following:

$$1^n + 2^n - 3^n - 4^n + + - \cdots - (p-1)^n \equiv f(n) \pmod{p = 4r+1};$$

$$1^n - 2^n - 3^n + 4^n + - - \cdots - (p-1)^n \equiv -f(n) \pmod{p = 4r+3}.$$

Hence, Corollary 2 readily follows by applying this result to $n = 2k+1$ and using item (i) of our theorem.

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