Advances in Differential Equations and Control Processes
Volume 3, Number 1, 2009, Pages 1-17
Published Online: July 3, 2009
This paper is available online at http://www.pphmj.com

EXISTENCE OF POSITIVE SOLUTIONS FOR STURM-LIOUVILLE BOUNDARY VALUE PROBLEM ON THE HALF-LINE WITH SINGULARITY AT $t = 0$

ZHONGHAI XU, ZHENGUO FENG and SHENGQUAN LIU

Department of Mathematics
College of Science
Northeast Dianli University
Jilin, 132012, P. R. China

Abstract

In this paper, by considering the following boundary value problem

$$\begin{cases} (p(t)x'(t))' + \lambda\phi(t)f(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t\to 0^+} \dfrac{p(t)x'(t)}{r'(t)} = 0, \\ \alpha \lim\limits_{t\to+\infty} x(t) + \beta \lim\limits_{t\to+\infty} \dfrac{p(t)x'(t)}{r'(t)} = 0, \end{cases} \tag{$*$}$$

where $p \in C([0, +\infty), [0, +\infty))$ with $p > 0$ on $(0, +\infty)$, $r \in C^2(0, +\infty)$ and p is singular at $t = 0$, i.e., $\int_0^{\varepsilon} \frac{dt}{p(t)} = +\infty$, and $\int_{\varepsilon}^{+\infty} \frac{dt}{p(t)} < +\infty$, for $\forall\varepsilon > 0$, we can establish sufficient conditions to guarantee the existence of positive solutions of BVP $(*)$ under proper conditions.

1. Introduction

The history of boundary value problems on infinite intervals started at the end of the nineteenth century with the pioneering work of A. Kneser about monotone solutions and their derivatives on $[0, +\infty)$ for

2000 Mathematics Subject Classification: 34B40.

Keywords and phrases: Sturm-Liouville BVP on the half-line, singularity at $t = 0$, generalized Green function, positive solutions.

Received August 9, 2008

second-order ordinary differential equations. The Kneser-type results were then followed by others until now, such as [8, 9].

Recently, several facts from classical analysis and topology combined with arguments of the modern fixed point theorems have been employed to study the existence of solutions to various types of nonlinear boundary value problems on infinite intervals. For example, the nonlinear alternative theorem used in [2, 10], a wonderful diagonalization process adopted in [2, 6], the fixed point theorems in Fréchet space used in [1, 2, 3, 4], these results were used to study the existence of nontrivial solutions for differential equations on the half-line.

In [5], Lian and Ge considered the following Sturm-Liouville boundary value problem (BVP)

$$\begin{cases} (p(t)x'(t))' + \lambda\phi(t)f(t, x(t)) = 0, & t \in (0, +\infty), \\ \alpha_1 x(0) - \beta_1 \lim\limits_{t \to 0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim\limits_{t \to +\infty} x(t) + \beta_2 \lim\limits_{t \to +\infty} p(t)x'(t) = 0, \end{cases} \tag{**}$$

where λ is a positive parameter, $f : [0, +\infty) \times [0, +\infty) \to R$ and $\phi : (0, +\infty) \to (0, +\infty)$ are continuous functions, $p : C([0, +\infty), [0, +\infty))$ with $p > 0$ on $(0, +\infty)$. Under the condition of $\int_0^{+\infty} \frac{dx}{p(t)} < +\infty$, the authors obtained the existence of positive solutions for the problem (**).

In this paper, under the condition of p is singular at $t = 0$, i.e., $\int_0^{\varepsilon} \frac{dt}{p(t)} = +\infty$, and $\int_{\varepsilon}^{+\infty} \frac{dt}{p(t)} < +\infty$, for $\forall \varepsilon > 0$, by considering the BVP (*) and constructing the generalized Green function, we obtain some existence theorems of positive solutions of BVP (*) by using the fixed point theorems in cone under proper conditions.

As far as we know, the existence of positive solutions of Sturm-Liouville boundary value problem on the half-line usually needs the condition of $\int_0^{+\infty} \frac{dt}{p(t)} < +\infty$, however, the most interesting point of our results is that the condition of $\int_0^{+\infty} \frac{dt}{p(t)} < +\infty$ can be removed.

Remark 1.1. When $\int_0^{+\infty} \frac{dt}{p(t)} < +\infty$, $r(t) = t$, the BVP (*) returns to the BVP (**).

Finally, we give a brief outline of the rest of the paper. In Section 2, we do some preliminary knowledge for the discussion; in Section 3, we prove the existence of the positive solutions.

2. Preliminary

In this paper, we consider the space $X = \{x \in C[0, +\infty) : \lim_{t \to +\infty} x(t)$ exists$\}$ with the supremum norm and the cone $K = \{x \in X : x(t) \geq 0,$ $t \in [0, +\infty)\}$ of X, and study the BVP

$$\begin{cases} (p(t)x'(t))' + \lambda\phi(t)f(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t \to 0^+} \dfrac{p(t)x'(t)}{r'(t)} = 0, \\ \alpha \lim\limits_{t \to +\infty} x(t) + \beta \lim\limits_{t \to +\infty} \dfrac{p(t)x'(t)}{r'(t)} = 0, \end{cases} \tag{2.1}$$

under the following assumptions throughout the paper:

(H1) $p \in C([0, +\infty))$, $p > 0$ on $(0, +\infty)$, $\phi \in C((0, +\infty), (0, +\infty))$, $\phi \in L^1(0, +\infty)$;

(H2) if $f \in C([0, +\infty] \times [0, +\infty), R)$, and $\| x \| \leq L$, then $| f(t, x) | \leq S_L$, for $\forall t \in [0, +\infty)$, where $L > 0$, $S_L > 0$ are constants;

(H3) $r(t)$ satisfies

$$r(t) \in C^2(0, +\infty),\ r(0) = 0,\ \lim_{t \to +\infty} r(t) = +\infty,$$

$$r'(t) > 0 \text{ on } (0, +\infty),\ \lim_{t \to +\infty} r'(t) = 1,\ \lim_{t \to 0^+} r'(t) = 0,$$

and

$$\lim_{t \to 0^+} \frac{\int_0^t \phi(s)ds}{r'(t)} = 0;$$

(H4) $\int_0^{+\infty} \frac{d\tau}{p(r^{-1}(\tau))} < \infty$;

(H5) $\alpha, \beta \geq 0$, and $\alpha > 0$.

Remark 2.1. When $p(t) = t^{3/2}$, $\int_0^{\varepsilon} \frac{dt}{p(t)} = +\infty$, $\int_{\varepsilon}^{+\infty} \frac{dt}{p(t)} < +\infty$ for any $\varepsilon > 0$, if $r(t) = t^2$ on $[0, \varepsilon]$, $r(t) = t$ on $[T, +\infty)$ which has enough smoothness, where $\varepsilon > 0$ is small enough, and $T > 0$ is large enough, then $\int_0^{+\infty} \frac{d\tau}{p(r^{-1}(\tau))} < +\infty$.

Writing

$$\begin{cases} a(t) = 1, \\ b(t) = \beta + \alpha \displaystyle\int_{r(t)}^{+\infty} \frac{d\tau}{p(r^{-1}(\tau))}, \end{cases} \tag{2.2}$$

$$\begin{cases} a(+\infty) = 1, \\ b(0) = \beta + \alpha \displaystyle\int_{0}^{+\infty} \frac{d\tau}{p(r^{-1}(\tau))}, \end{cases} \tag{2.3}$$

and

$$G(t, s) = \begin{cases} \frac{1}{\alpha} a(t)b(s), & 0 \leq t \leq s < +\infty, \\ \frac{1}{\alpha} a(s)b(t), & 0 \leq s \leq t < +\infty, \end{cases} \tag{2.4}$$

$$V_{\phi}(t) = \left(\frac{\int_0^t \phi(\tau) d\tau}{r'(t)} \right)'. \tag{2.5}$$

Lemma 2.1. *Suppose* (H1)-(H5) *hold. For any* $\nu \in L^1(0, +\infty)$ *which satisfies* $\lim_{t \to 0^+} \frac{\int_0^t \nu(s) ds}{r'(t)} = 0$, *the following BVP*

$$\begin{cases} (p(t)x'(t))' + \nu(t) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t\to 0^+} \dfrac{p(t)x'(t)}{r'(t)} = 0, \\ \alpha \lim\limits_{t\to+\infty} x(t) + \beta \lim\limits_{t\to+\infty} \dfrac{p(t)x'(t)}{r'(t)} = 0, \end{cases} \tag{2.6}$$

has a unique solution. Moreover, this unique solution can be expressed in the form

$$x(t) = \int_0^{+\infty} G(t, s) V_\nu(s) ds,$$

where $G(t, s)$ *is defined by* (2.4), $V_\nu(s) = \left(\dfrac{\int_0^s \nu(\tau) d\tau}{r'(s)} \right)'$.

Set

$$w(t) = \int_0^{+\infty} G(t, s) V_\phi(s) ds, \tag{2.7}$$

then $w(t)$ is the unique solution of BVP (2.6) for $\nu(t) \equiv \phi(t)$. Integrating (2.7) by parts, we get

$$w(t) = \frac{\beta}{\alpha} \lim_{s\to+\infty} \frac{\int_0^s \phi(\tau) d\tau}{r'(s)} - \frac{b(t)}{\alpha} \lim_{s\to 0^+} \frac{\int_0^s \phi(\tau) d\tau}{r'(s)} + \int_t^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds,$$

from the condition (H3), we obtain

$$w(t) = \frac{\beta}{\alpha} \lim_{s\to+\infty} \frac{\int_0^s \phi(\tau) d\tau}{r'(s)} + \int_t^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds \geq 0. \tag{2.8}$$

Due to $\phi \in L^1(0, +\infty)$, then $\lim\limits_{t\to+\infty} \dfrac{\int_0^t \phi(\tau) d\tau}{r'(t)} = \int_0^{+\infty} \phi(\tau) d\tau < +\infty$, moreover, $\lim\limits_{t\to 0^+} \dfrac{\int_0^t \phi(\tau) d\tau}{r'(t)} = 0$, then there exists $Q > 0$ such that $\dfrac{\int_0^t \phi(\tau) d\tau}{r'(t)} \leq Q$ on $[0, +\infty)$. Furthermore

$$\int_0^{+\infty} \frac{\int_0^s \phi(\tau)d\tau}{p(s)} ds = \int_0^{+\infty} \frac{\int_0^s \phi(\tau)d\tau}{r'(s)} \frac{r'(s)}{p(s)} ds \le Q\int_0^{+\infty} \frac{d\tau}{p(r^{-1}(\tau))} < +\infty,$$

so

$$\lim_{t\to+\infty} w(t) = \frac{\beta}{\alpha} \lim_{s\to+\infty} \frac{\int_0^s \phi(\tau)d\tau}{r'(s)} = \frac{\beta}{\alpha}\int_0^{+\infty} \phi(\tau)d\tau < +\infty.$$

Then we get:

Lemma 2.2. *Suppose* (H1)-(H5) *hold. Then* $w(t) \in K$.

Next, we introduce the following lemmas:

Lemma 2.3 [5]**.** *Let* X, K *be defined as before. Suppose* $T : X \to X$ *is completely continuous. Define* $\theta : TX \to K$ *by*

$$(\theta y)(t) = \max\{y(t), 0\}, \quad t \in [0, +\infty),$$

where $y \in TX$. *Then* $\theta \circ T : X \to K$ *is also a completely continuous operator.*

Lemma 2.4 [2]**.** *Let* X *be defined as before and* $M \subset X$. *Then* M *is relatively compact in* X *if the following conditions hold*:

(a) *M is uniformly bounded in X*;

(b) *the functions from M are equicontinuous on any compact interval of* $[0, +\infty)$;

(c) *the functions from M are equiconvergent, that is, for any given* $\varepsilon > 0$, *there exists* $T = T(\varepsilon) > 0$ *such that* $|f(t) - f(+\infty)| < \varepsilon$, *for any* $t > T$, $f \in M$.

Lemma 2.5 [11]**.** *Let* P *be a cone of real Banach space* E, $\Omega \subset P$ *be a bounded open set*, $\theta \in \Omega$, *and* $T : \Omega \to P$ *be a completely continuous operator. Suppose* $Tx \neq \lambda x$, $\forall x \in \partial\Omega$, $\lambda \ge 1$. *Then* $\deg_p\{I - T, \Omega, 0\} = 1$.

3. Existence of Solutions

Theorem 3.1. *Suppose* (H1)-(H5) *and the following conditions hold*:

(C1) *let* $x_n, x \in X$, *if* $x_n \to x$, *then* $f(t, x_n) \rightrightarrows f(t, x)$;

(C2) *there exist* $R > M > 0$ *such that*

$$0 < a < b = \frac{R}{\max\limits_{0 \le t < +\infty,\ Mw(t) \le x \le R} \int_0^{+\infty} G(t, s)\widetilde{V}(s, f(s, x(s)))ds},$$

where

$$a = \frac{M}{\min\limits_{0 \le t < +\infty} f(t, Mw(t))}, \ \widetilde{V}(t, f(t, x(t))) = \max\{V(t, f(t, x(t))), 0\}, t \in [0, +\infty),$$

and

$$V(t, f(t, x(t))) = \left(\frac{\int_0^t \phi(s) f(s, x(s))ds}{r'(t)} \right)'.$$

Then when $\lambda \in [a, b)$, *BVP* (2.1) *has at least one positive solution* x *satisfying* $0 < Mw(t) \le x(t) < R$ *on* $[0, +\infty)$.

Proof. Consider the following BVP

$$\begin{cases} (p(t)x'(t))' + \lambda\phi(t) f^*(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t \to 0^+} \dfrac{p(t)x'(t)}{r'(t)} = 0, \\ \alpha \lim\limits_{t \to +\infty} x(t) + \beta \lim\limits_{t \to +\infty} \dfrac{p(t)x'(t)}{r'(t)} = 0, \end{cases} \tag{3.1}$$

where

$$f^*(t, x(t)) = \begin{cases} f(t, x(t)), & x(t) \ge Mw(t), \\ f(t, Mw(t)), & x(t) < Mw(t). \end{cases} \tag{3.2}$$

Define $T : K \to X$ by

$$(Tx)(t) = \lambda \int_0^{+\infty} G(t, s)V(s, f^*(s, x(s)))ds, \quad 0 \le t < +\infty.$$

In view of Lemma 2.1, it is clear that a fixed point of T is also a solution of BVP (3.1).

In the same way as we get (2.8), we can obtain

$$(Tx)(t) = \lambda \frac{\beta}{\alpha} \lim_{s \to +\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x(\tau)) d\tau}{r'(s)} + \lambda \int_t^{+\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x(\tau)) d\tau}{p(s)} ds. \quad (3.3)$$

In view of (H1), (H2) and (H3), for any given $x \in K$, $\lim_{s \to +\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x(\tau)) d\tau}{r'(s)}$ exists.

We claim that $T : K \to X$ is a completely continuous operator. To justify this, the proof is split into three steps:

Step 1. $T : K \to X$ is well defined.

In order to prove $T : K \to X$ is well defined, we will show that $(Tx)(t) \in C[0, +\infty)$ and for $\forall x \in K$, $\lim_{t \to +\infty} (Tx)(t)$ exists.

(1) $(Tx)(t) \in C[0, +\infty)$.

In view of (H1), (H2) and (H3), from (3.2) and (3.3), we know that $(Tx)(t) \in C[0, +\infty)$.

(2) $\lim_{t \to +\infty} (Tx)(t)$ exists.

From (H2), for any given $x \in K$, there exist $L > 0$, $S_L > 0$ such that $\| x \| < L$, $| f(t, x(t)) | \le S_L$ on $[0, +\infty)$, we have

$$\left| \int_0^{+\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x(\tau)) d\tau}{p(s)} ds \right| \le S_L \int_0^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds < +\infty,$$

from (3.3), we know that $\lim_{t \to +\infty} (Tx)(t)$ exists.

Step 2. The operator T is continuous.

In order to prove $T : K \to X$ is continuous, we only prove that $\forall x_n \in K, \| x_n - x \| \to 0 \Rightarrow \| Tx_n - Tx \| \to 0$, i.e., $\forall x_n \in K$, $x_n(t) \rightrightarrows x(t)$ $\Rightarrow Tx_n(t) \rightrightarrows Tx(t)$, as $n \to +\infty$.

Because $x_n \in K$, $x \in K$, $x_n \to x$, there exist $L > 0$ and $S_L > 0$ such that $\max\limits_{n \in N-\{0\}} \{\|x\|, \|x_n\|\} \le L$, $|f^*(\tau, x_n(\tau))| \le S_L$, $|f^*(\tau, x(\tau))| \le S_L$, for $\forall \tau \in [0, +\infty)$.

(1) $|(Tx_n)(+\infty) - (Tx)(+\infty)| \to 0$, $n \to +\infty$.

From (3.3), we know

$$(Tx_n)(+\infty) = \lambda \frac{\beta}{\alpha} \lim_{s \to +\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x_n(\tau)) d\tau}{r'(s)},$$

$$(Tx)(+\infty) = \lambda \frac{\beta}{\alpha} \lim_{s \to +\infty} \frac{\int_0^s \phi(\tau) f^*(\tau, x(\tau)) d\tau}{r'(s)},$$

from (C1), we have

$$\begin{aligned} |(Tx_n)(+\infty) - (Tx)(+\infty)| &\le \lambda \frac{\beta}{\alpha} \lim_{t \to +\infty} \frac{\int_0^t \phi(\tau) |f^*(\tau, x_n(\tau)) - f^*(\tau, x(\tau))| d\tau}{r'(t)} \\ &\le \lambda \frac{\beta}{\alpha} \int_0^{+\infty} \phi(\tau) |f^*(\tau, x_n(\tau)) - f^*(\tau, x(\tau))| d\tau \\ &\to 0, \quad \text{as } n \to +\infty. \end{aligned}$$

Then

$$|(Tx_n)(+\infty) - (Tx)(+\infty)| \to 0, \quad \text{as } n \to +\infty. \tag{3.4}$$

(2) $|(Tx_n)(t) - (Tx_n)(+\infty)| \to 0$, as $t \to +\infty$, for $\forall x_n \in K$.

From (3.3), we have

$$|(Tx_n)(t) - (Tx_n)(+\infty)| \le \lambda S_L \int_t^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds,$$

due to $\int_0^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds < +\infty$, then

$$|(Tx_n)(t) - (Tx_n)(+\infty)| \to 0, \quad \text{as } t \to +\infty, \quad \text{for } x_n \in K. \tag{3.5}$$

(3) In the same way as we get (3.5), we have

$$|(Tx)(t)-(Tx)(+\infty)| \to 0, \quad \text{as } t \to +\infty. \tag{3.6}$$

(4) For any positive number $T_0 < +\infty$, for $\forall t \in [0, T_0]$, we prove that $|(Tx_n)(t)-(Tx)(t)| \to 0$, as $n \to +\infty$.

From (C1) and (3.3), we have

$$|(Tx_n)(t)-(Tx)(t)| \le \lambda \frac{\beta}{\alpha} \lim_{t\to+\infty} \frac{\int_0^t \phi(\tau)|f^*(\tau, x_n(\tau)) - f^*(\tau, x(\tau))|d\tau}{r'(t)}$$

$$+ \lambda \int_t^{+\infty} \frac{\int_0^s \phi(\tau)|f^*(\tau, x_n(\tau)) - f^*(\tau, x(\tau))|d\tau}{p(s)} ds$$

$$\to 0, \quad \text{as } n \to +\infty.$$

Then $\forall t \in [0, T_0]$,

$$|(Tx_n)(t)-(Tx)(t)| \to 0, \quad \text{as } n \to +\infty. \tag{3.7}$$

Combining (3.4), (3.5), (3.6) and (3.7), we can get that the operator T is continuous.

Step 3. T maps a bounded subset $B \subset K$ into a relatively compact set in X.

In order to prove that, we only show that $T(B)$ is uniformly bounded, equicontinuous and equiconvergent.

Let B be a bounded subset of K, there exist $L > 0$ and $S_L > 0$ such that $\forall x \in B$, $\|x\| \le L$ and $|f(\tau, x(\tau))| \le S_L$, for any function $x \in B$.

(1) $T(B)$ is uniformly bounded.

For $\forall x \in B$, from (3.3), we have

$$|(Tx)(t)| \le \lambda S_L \frac{\beta}{\alpha} \lim_{s\to+\infty} \frac{\int_0^s \phi(\tau)d\tau}{r'(s)} + \lambda S_L \int_0^{+\infty} \frac{\int_0^s \phi(\tau)d\tau}{p(s)} ds < +\infty,$$

so $T(B)$ is uniformly bounded.

(2) $T(B)$ is equicontinuous.

We only prove that $\forall \varepsilon > 0, \exists \delta > 0, |t_1 - t_2| < \delta, |(Tx)(t_1) - (Tx)(t_2)| < \varepsilon,$ $\forall x \in B, \forall t_1, t_2 \in [T_1, T_2],$ where $T_1, T_2 \in [0, +\infty)$.

For $x \in B$, from (3.3), when $t_1 \to t_2$, we have

$$|(Tx)(t_1) - (Tx)(t_2)|$$

$$= \lambda \left| \int_{t_1}^{t_2} \frac{\int_0^s \phi(\tau) f^*(\tau, x_n(\tau)) d\tau}{p(s)} ds \right| \le \lambda S_L \left| \int_{t_1}^{t_2} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds \right| \to 0,$$

due to $\int_0^{+\infty} \frac{\int_0^s \phi(\tau) d\tau}{p(s)} ds < +\infty,$ we get $|(Tx)(t_1) - (Tx)(t_2)| \to 0,$ as $t_1 \to t_2$.

(3) $T(B)$ is equiconvergent.

We only prove that $\forall \varepsilon > 0, \exists T_0 > 0, t > T_0 \Rightarrow |(Tx)(t) - (Tx)(+\infty)| < \varepsilon,$ $\forall x \in B,$ where $T_0 \in [0, +\infty)$.

From (3.3), we can get that

$$|(Tx)(t) - (Tx)(+\infty)| = \lambda \int_t^{+\infty} \frac{\int_0^s \phi(\tau) | f^*(\tau, x(\tau)) | d\tau}{p(s)} ds$$

$$\le \lambda S_L \int_t^{+\infty} \frac{\int_s^{+\infty} \phi(\tau) d\tau}{p(s)} ds,$$

due to $\int_0^{+\infty} \frac{\int_s^{+\infty} \phi(\tau) d\tau}{p(s)} ds < +\infty,$ we get $|(Tx)(t) - (Tx)(+\infty)| \to 0,$ as $t \to +\infty,$ for $\forall x \in B$.

Combining Steps 1 and 2 with Step 3, we know that T is a completely continuous operator.

For the operator $\theta : X \to K$ defined by $(\theta y)(t) = \max\{y(t), 0\}$, Lemma 2.3 implies that $\theta \circ T : K \to K$ is also completely continuous.

Set $\Omega = \{x \in K : \| x \| < R\}$ and $\Delta_x = \{t \in [0, +\infty) : V(t, f^*(t, x(t))) \geq 0\}$. Then $\forall x \in \partial\Omega$, we have

$$(\theta \circ T)x(t) = \max\left\{\lambda \int_0^{+\infty} G(t, s)V(s, f^*(s, x(s)))ds, 0\right\}$$

$$\leq \lambda \int_{\Delta_x} G(t, s)V(s, f^*(s, x(s)))ds$$

$$< b \max_{0 \leq t < +\infty,\ 0 \leq x(t) \leq R} \int_{\Delta_x} G(t, s)V(s, f^*(s, x(s)))ds$$

$$= b \max_{0 \leq t < +\infty,\ Mw(t) \leq x(t) \leq R} \int_0^{+\infty} G(t, s)\widetilde{V}(s, f(s, x(s)))ds = R,$$

which concludes that

$$\deg_K\{I - \theta \circ T, \Omega, 0\} = 1,$$

where $\deg_K$ stands for the degree in cone K. Then $\theta \circ T$ has a fixed point $x \in \Omega$. Obviously, $\| x \| < R$.

Now, for this x, we claim that

$$(Tx)(t) \geq Mw(t), \quad t \in [0, +\infty).$$

Otherwise, $\sup\limits_{0 \leq t < +\infty} \{Mw(t) - (Tx)(t)\} > 0$.

Case 1. If $t_0 \in (0, +\infty)$ such that

$$Mw(t_0) - (Tx)(t_0) = \sup_{t \in [0, +\infty)} \{Mw(t) - (Tx)(t)\} > 0,$$

we have $Mw'(t_0) - (Tx)'(t_0) = 0$.

Furthermore, if there exists $t_1 \in [0, t_0)$ such that

$$Mw(t_1) - (Tx)(t_1) = 0 \quad \text{and} \quad Mw(t) - (Tx)(t) > 0, \quad t \in (t_1, t_0],$$

then for any $t \in (t_1, t_0]$, we have

$$Mw'(t) - (Tx)'(t) = \frac{1}{p(t)} \int_{t_0}^{t} (p(s)(Mw'(s) - (Tx)'(s)))' ds$$

$$= \frac{1}{p(t)} \int_{t_0}^{t} \phi(s)(\lambda f^*(s, x(s)) - M) ds$$

$$= \frac{1}{p(t)} \int_{t_0}^{t} \phi(s)(\lambda f(s, Mw(s)) - M) ds \leq 0,$$

and so $Mw(t_1) - (Tx)(t_1) \geq Mw(t_0) - (Tx)(t_0) > 0$, which is a contradiction. Then we have $Mw(t) - (Tx)(t) > 0$ on $t \in [0, t_0]$. Similarly, we can show that $Mw(t) - (Tx)(t) > 0$ on $t \in [t_0, +\infty]$.

So,

$$Mw(t_0) - (Tx)(t_0)$$

$$= \frac{\beta}{\alpha} \lim_{s \to +\infty} \frac{\int_0^s \phi(\tau)[M - \lambda f^*(\tau, x(\tau))] d\tau}{r'(s)} + \int_{t_0}^{+\infty} \frac{\int_0^s \phi(\tau)[M - \lambda f^*(\tau, x(\tau))] d\tau}{p(s)} ds$$

$$= \frac{\beta}{\alpha} \lim_{s \to +\infty} \frac{\int_0^s \phi(\tau)[M - \lambda f(\tau, Mw(\tau))] d\tau}{r'(s)} + \int_{t_0}^{+\infty} \frac{\int_0^s \phi(\tau)[M - \lambda f(\tau, Mw(\tau))] d\tau}{p(s)} ds \leq 0,$$

which is a contradiction to Case 1.

Case 2. $\lim_{t \to +\infty} Mw(t) - (Tx)(t) = \sup_{0 \leq t < +\infty} \{Mw(t) - (Tx)(t)\} > 0.$

For this case, we can get immediately that there exists $T > 0$ such that

$$Mw(t) - (Tx)(t) > 0, \quad Mw'(t) - (Tx)'(t) \geq 0, \tag{3.8}$$

for $\forall t \geq T$, where $Mw'(t) - (Tx)'(t) \not\equiv 0$, otherwise Case 2 returns to Case 1.

Notice that both $Mw(t)$ and $(Tx)(t)$ satisfy the boundary value conditions of (2.1) and we have

$$\alpha \lim_{t\to+\infty}(Mw(t)-(Tx)(t))+\beta\lim_{t\to+\infty}\frac{p(t)(Mw'(t)-(Tx)'(t))}{r'(t)}=0,$$

due to (3.8) and the properties of $r(t)$, we get that $\lim_{t\to+\infty}\frac{p(t)(Mw'(t)-(Tx)'(t))}{r'(t)}=0.$

Because $r'(t)\to 1,$ when $t\to+\infty,$ $\lim_{t\to+\infty}p(t)(Mw'(t)-(Tx)'(t))=0.$

Furthermore, if there exists $t_1\in[0,+\infty)$ such that

$$Mw(t_1)-(Tx)(t_1)=0 \quad\text{and}\quad Mw(t)-(Tx)(t)>0, \quad t\in(t_1,+\infty),$$

then $\forall t\in(t_1,+\infty),$ we have

$$\begin{aligned}Mw'(t)-(Tx)'(t)&=-\frac{1}{p(t)}\int_t^{+\infty}(p(s)[Mw'(s)-(Tx)'(s)])'ds\\&=\frac{1}{p(t)}\int_t^{+\infty}\phi(s)[M-\lambda f^*(s,x(s))]ds\\&=\frac{1}{p(t)}\int_t^{+\infty}\phi(s)[M-\lambda f(s,Mw(s))]ds\le 0,\end{aligned}$$

and so $Mw(t_1)-(Tx)(t_1)\ge\lim_{t\to+\infty}(Mw(t)-(Tx)(t))>0,$ which is a contradiction. Then we have

$$Mw(t)-(Tx)(t)>0, \quad t\in[0,+\infty),$$

and so

$$\begin{aligned}\lim_{t\to+\infty}(Mw(t)-(Tx)(t))&=\frac{\beta}{\alpha}\lim_{s\to+\infty}\frac{\int_0^s\phi(\tau)[M-\lambda f^*(\tau,x(\tau))]d\tau}{r'(s)}\\&=\frac{\beta}{\alpha}\lim_{s\to+\infty}\frac{\int_0^s\phi(\tau)[M-\lambda f(\tau,Mw(\tau))]d\tau}{r'(s)}\le 0,\end{aligned}$$

which is a contradiction to Case 2.

Case 3. If $t_0 = 0$ such that

$$Mw(t_0) - (Ty)(t_0) = \sup_{0 \le t < +\infty} \{Mw(t) - (Ty)(t)\} > 0,$$

then $\exists \delta > 0$, when $t \in [0, \delta]$, we have

$$Mw(t) - (Tx)(t) > 0, \quad Mw'(t) - (Tx)'(t) \le 0,$$

where $Mw'(t) - (Tx)'(t) \not\equiv 0$ on $[0, \delta]$, otherwise Case 3 returns to Case 1.

From (2.8) and (3.3), we have

$$w'(t) = -\frac{\int_0^t \phi(\tau) d\tau}{p(t)} ds,$$

$$(Tx)'(t) = -\frac{\lambda \int_0^t \phi(\tau) f^*(\tau, x(\tau)) d\tau}{p(t)},$$

then $\forall t \in (0, \delta]$,

$$Mw'(t) - (Tx)'(t) = -\frac{1}{p(t)} \int_0^t \phi(s) [M - \lambda f(s, Mw(s))] ds > \text{or} \equiv 0,$$

which is a contradiction to Case 3.

Above all, we get that $(Tx)(t) \ge Mw(t)$ on $[0, +\infty)$. Then $(\theta \circ T)x = Tx = x$ and x is a positive solution of BVP (2.1) with $Mw(t) \le x(t) < R$. □

Corollary 3.1. *Suppose* (H1)-(H5) *and* (C1) *hold. Further suppose* $f(t, 0) \ge 0$. *If there exists* $R > 0$ *such that*

$$b = \frac{R}{\max\limits_{0 \le t < \infty, 0 \le x \le R} \int_0^{+\infty} G(t, s) \widetilde{V}(s, f(s, x(s))) ds} > 0,$$

where $V(t, f(t, x(t)))$, $\widetilde{V}(t, f(t, x(t)))$ *are defined as in Theorem* 3.1.

Then when $0 < \lambda < b$, *BVP* (2.1) *has at least one nonnegative solution* $x(t)$ *satisfying* $0 \le \| x \| < R$.

Remark 3.1. If $\phi(t)f(t, 0) \not\equiv 0$ in Corollary 3.1, x is a positive solution with $0 < \| x \| < R$ to corresponding BVP.

Corollary 3.2. *Suppose* (H1)-(H5) *and* (C1) *hold. If there exists* $M > 0$ *such that*

$$a = \frac{M}{\min_{0 \le t < +\infty} f(t, Mw(t))} > 0$$

and

$$\lim_{x \to +\infty} \frac{\max_{0 \le t < +\infty} \int_0^{+\infty} G(t, s)\widetilde{V}(s, f(s, x(s)))ds}{x} = 0,$$

where $V(t, f(t, x)), \widetilde{V}(t, f(t, x))$ *are defined as in Theorem* 3.1. *Then when* $\lambda \ge a$, *BVP* (2.1) *has at least a positive solution x with*

$$0 < Mw(t) \le x(t), \quad \| x \| < \infty.$$

References

[1] R. P. Agarwal and D. O'Regan, Fixed point theory for self maps between Fréchet spaces, J. Math. Anal. Appl. 256 (2001), 498-512.

[2] R. P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic, 2001.

[3] M. Frigon, Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations, Bull. Belg. Math. Soc. Simon Stevin 9 (2002), 23-37.

[4] M. Frigon and D. O'Regan, Fixed points of cone-compressing and cone-extending operators in Fréchet spaces, Bull. London Math. Soc. 35 (2003), 672-680.

[5] Hairong Lian and Weigao Ge, Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line, J. Math. Anal. Appl. 321 (2006), 781-792.

[6] R. Ma, Existence of positive solutions for second-order boundary value problems on infinite intervals, Appl. Math. Lett. 16 (2003), 33-39.

[7] D. O'Regan, Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1994.

[8] Oleg Palumbíny, On existence of Kneser solutions of a certain class of n-th order nonlinear differential equations, Math. Bohem. 123 (1998), 49-65.

[9] Irena Rachunkova, On a Kneser problem for a system of nonlinear ordinary differential equations, Czechoslovak Math. J. 31 (1981), 114-126.

[10] B. Yan, Boundary value problems on the half-line with impulses and infinite delay, J. Math. Anal. Appl. 259 (2001), 94-114.

[11] C. Zhong, X. Fan and W. Chen, The Introduction of Nonlinear Functional Analysis, Lanzhou University Press, P. R. China, 1998.