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SOME CONNECTIONS BETWEEN THE LOG SOBOLEV TYPE INEQUALITIES

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Abstract

As a limiting case of the theorem, various types of logarithmic Sobolev inequalities have been obtained with different techniques and were related to other analytic inequalities. Here, from regular logarithmic Sobolev inequality, we try to demonstrate the connection with the log Sobolev inequality which was proved by Beckner.

1. Introduction

As a part of the study of relations between a function and its partial derivatives, many results have been introduced with functions in the Sobolev space which consist of all functions on R^n whose derivatives up to and including higher order belong to $L_p(R^n)$.

As a useful tool in a variety of problems in this type of study, Sobolev theorem gives a simple way how restrictions on the size of partial derivatives control corresponding restrictions on the related functions [5]. Also, as a limiting case of the Sobolev theorem, the logarithmic Sobolev inequality was studied. It was first

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observed by Weissler [6] and in [2], Gross formulated the logarithmic Sobolev inequality

$$\int_{R^n} |G|^2 \ln |G| \, d\mu \leq \int_{R^n} |\nabla G|^2 \, d\mu$$

as an infinite dimensional estimate with respect to Gaussian measure $d\mu$. And the linearized form on the two dimensional sphere with sharp constant played an important role in some geometric analysis and nonlinear PDE problems. Similarly, various types of logarithmic Sobolev inequalities, as a limiting case of the theorem, have been obtained with different techniques and were related to other analytic inequalities such as Nash inequality and the Hardy's inequality. Here, from regular logarithmic Sobolev inequality, our result demonstrates the connection with the log Sobolev inequality which was proved by Beckner.

2. Preliminary Result

Essential part of the theory in the Sobolev space is detailed in the following Sobolev theorem:

Theorem 1 [Sobolev]**.** *If* $p < \frac{n}{k}$, *then*

$$L_k^p(R^n) \subset L^q(R^n), \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}. \tag{1}$$

For smooth F with $1 < p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, the above embedding (1) can be observed as the inequality with the sharp constant in the following (see [1]):

$$(\| F \|_{L^q(R^n)})^p \leq C_p \int_{R^n} |\nabla F|^p \, dx, \tag{2}$$

where

$$C_p = \frac{1}{n}\left(\frac{p-1}{n-p}\right)^{p-1} \frac{\Gamma(n)\Gamma(1+m/2)}{\pi^{n/2}\Gamma(n/p)\Gamma(1+n/p')}.$$

Beckner used asymptotic structural argument to show that the Sobolev imbedding inequality (2) implies the log Sobolev inequality

$$\int_{R^n} |F|^p \ln |F| \, dx \leq \frac{m}{p^2} \ln\left(C_p \int_{R^n} |\nabla F|^p \, dx\right). \tag{3}$$

By using the product structure on R^n, the concaveness of logarithmic function and asymptotics for sharp Sobolev embedding, he was able to derive the logarithmic Sobolev inequality (3) with more explicit form. In the proof, he used the product structure of Euclidean space and the following lemma about functional inequalities on L^p over a manifold M to get the log Sobolev inequality as the infinite dimensional limit of the inequality (2).

Lemma 1. *Consider an inequality of the form*

$$(\| f \|_{L^q(M)})^p \leq A_q \int_M \Phi(f)\, dm,$$

where $1/q = 1/p - 1/Q$ *for* $Q > p$ *and* Φ *is homogeneous of degree* p. *If this inequality holds and* $f \in L^p(M)$ *with* $\| f \|_p = 1$, *then*

$$\int_M | f |^p \ln | f | dm \leq \frac{Q}{p^2} \ln\left(A_q \int_M \Phi(f)\, dm \right). \tag{4}$$

When $p = 2$, the inequality (3)

$$\int_{R^n} | f |^2 \ln | f | dx \leq \frac{n}{4} \ln\left(\frac{2}{\pi e n} \int_{R^n} | \nabla f |^2 dx \right) \tag{5}$$

holds and the constant $\frac{2}{\pi e n}$ is optimal. But, for other *p*, the constant C_p is non-optimal. So, the sharp constant played a fundamental role in the study of these types of inequalities. Nash's inequality

$$\left(\int_{R^n} | f |^2 dx \right)^{1+2/n} \leq B_n \int_{R^n} | \nabla f |^2 dx \left(\int_{R^n} | f | dx \right)^{4/n}$$

which is equivalent to the Poincaré inequality, was reproved by Beckner by using the inequality (3).

3. Main Result

From (3), by letting

$$F(x) = \prod f(x_k),$$

$\| f \|_p = 1$, $x_k \in R^n$ and $m = nl$, Beckner obtained the inequality

$$\int_{R^n} | f |^p \ln | f | dx \le \frac{n}{2p} \left(\frac{p^2}{2\pi en} \int_{R^n} | \nabla f |^2 | f |^{p-2} dx \right) \tag{6}$$

and gave a connection with the logarithmic Sobolev inequality (5) by letting $f = g^{2/p}$ (see [1]). On the reverse, by direct calculation using the inequality (5), we obtain the following inequality which was also mentioned by Beckner.

Theorem 2. *For* $f \in L_p(R^n)$, $\| f \|_p = 1$, *the logarithmic Sobolev inequality*

$$\int_{R^n} \ln | f | | f |^p dx \le \frac{n}{p^2} \ln \left(A_p \int_{R^n} | \nabla f |^p dx \right)$$

holds with the constant given by

$$A_p = \left(\frac{p}{\sqrt{2\pi en}} \right)^p.$$

Lemma 2. *For* $f \in L_p(R^n)$ *with* $\| f \|_p = 1$,

$$\int_{R^n} | \nabla f |^2 | f |^{p-2} dx \le \left(\int_{R^n} | \nabla f |^p \right)^{\frac{2}{p}}. \tag{7}$$

Proof. Suppose that *f* is such that $f \in L_p(R^n)$. Then

$$\int_{R^n} | \nabla f |^2 | f |^{p-2} dx \le \left(\int_{R^n} | \nabla f |^p dx \right)^{\frac{2}{p}} \left(\int_{R^n} | f |^p dx \right)^{\frac{p-2}{p}} \tag{8}$$

by using Hölder's inequality and since $\| f \|_p = 1$, it makes the proof here complete.

Proof of Theorem. By Lemma 2, we have

$$\int_{R^n} | \nabla f^{\frac{p}{2}} |^2 dx \le \left(\int_{R^n} | \nabla f |^p dx \right)^{\frac{2}{p}}$$

for $\| f \|_p = 1$. Thus, by letting $f = g^{\frac{p}{2}}$, the logarithmic Sobolev inequality (5) can be reduced to

$$\frac{p}{2}\int_{R^n} |g|^p \ln|g|\,dx \le \ln\left(\frac{2}{\pi e n}\left(\frac{p}{2}\right)^2\left(\int_{R^n} |\nabla g|^p\,dx\right)^{\frac{2}{p}}\right).$$

So, we have

$$\int_{R^n} |g|^p \ln|g|\,dx \le \frac{n}{p^2}\ln\left(\left(\frac{p}{\sqrt{2\pi e n}}\right)^p \int_{R^n} |\nabla g|^p\,dx\right)$$

and the desired inequality is attained. □

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