Far East Journal of Applied Mathematics
Volume 35, Number 3, 2009, Pages 315-319
Published Online: June 25, 2009
This paper is available online at http://www.pphmj.com

ON THE ETA-PRODUCT OF THE ELLIPTIC ROOT SYSTEM $A_{28}^{(1,1)}$

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Abstract

We obtain the Dirichlet series of the Fourier expansion of the elliptic eta-product $\eta_{A_{28}^{(1,1)}}(4\tau) = \eta(4\tau)^{30}$ of type $A_{28}^{(1,1)}$.

1. Introduction

In 1985, Saito [3] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belonging to a positive semi-definite quadratic form I whose radical has rank two. Therefore 2-extended affine root systems are also called *elliptic root systems*. In the cases of 1-codimensional elliptic root systems, Saito [4] described elliptic eta-products and their Fourier coefficients at ∞. In the previous articles [5, 6, 7], we examined the elliptic eta-product of type $A_l^{(1,1)}$ $(l \geq 1)$, and more concretely the cases of types $A_{10}^{(1,1)}$ and $A_{20}^{(1,1)}$. In this article, we obtain the Dirichlet series of $\eta(4\tau)^{30}$ of type $A_{28}^{(1,1)}$ according to the theory of Hecke operators due to van Lint [8], Rankin [2] and Rangachari [1].

2000 Mathematics Subject Classification: 20F55.

Keywords and phrases: elliptic root system, eta-product, Euler product.

Received March 24, 2009

2. Elliptic Eta-product of Type $A_{28}^{(1,1)}$

Dedekind's η-function, defined by the infinite product $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$ (= the upper half of the complex plane) is a modular form of weight $\frac{1}{2}$. The elliptic eta-product of type $A_{28}^{(1,1)}$ is given by [5]; $\eta_{A_{28}^{(1,1)}}(4\tau) = \eta(4\tau)^{30}$, which is a cusp form of weight $k = 15$ and level $N = 16$. Therefore, $\eta(4\tau)^{30} \in S_{15}(\Gamma_0(16), \epsilon)$, and the space $S_{15}(\Gamma_0(16), \epsilon)$ is 3-dimensional (see [1]). From the result of [1], $F(\tau) = E_6^2 \eta^6 + \alpha E_6 \eta^{18} + \beta \eta^{30} = \sum_{n=1}^{\infty} r(n) q^{\frac{n}{4}}$ is a normalized eigenfunction of the Hecke operators T_n for some values α and β. Here E_6 is Eisenstein series given by $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}$. We have

$$
\begin{aligned}
\eta^{30} &= \sum_{n=1}^{\infty} a(n) q^{\frac{n}{24}} \\
&= \sum_{n=1}^{\infty} a(6 + 24n) q^{\frac{6+24n}{24}} \\
&= q^{\frac{5}{4}} - 30 q^{\frac{9}{4}} + 405 q^{\frac{13}{4}} - 3190 q^{\frac{17}{4}} + 15660 q^{\frac{21}{4}} \\
&\quad - 45036 q^{\frac{25}{4}} + 40745^{\frac{29}{4}} + \cdots, \\
E_6 \eta^{18} &= \sum_{n=1}^{\infty} b(n) q^{\frac{n}{24}} \\
&= \sum_{n=1}^{\infty} b(-6 + 24n) q^{\frac{-6+24n}{24}} \\
&= q^{\frac{3}{4}} - 522 q^{\frac{7}{4}} - 7425 q^{\frac{11}{4}} + 107850 q^{\frac{15}{4}} - 306675 q^{\frac{19}{4}} \\
&\quad - 490158 q^{\frac{23}{4}} + 1743858 q^{\frac{27}{4}} + \cdots,
\end{aligned}
$$

$$E_6^2\eta^6 = \sum_{n=1}^{\infty} c(n) q^{\frac{n}{24}}$$

$$= \sum_{n=1}^{\infty} c(-18+24n) q^{\frac{-18+24n}{24}}$$

$$= q^{\frac{1}{4}} - 1014 q^{\frac{5}{4}} + 226809 q^{\frac{9}{4}} + 15185530 q^{\frac{13}{4}}$$

$$+ 302379810 q^{\frac{17}{4}} + 2378315520 q^{\frac{21}{4}} + \cdots.$$

We recall the result [8]. For Hecke operator T_p, we set $T_p = (-1)^{\frac{k(p-1)}{2}} \times p^{-1} T(p)$, then for $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{24}}$, we have $f(\tau)|T(p) = \sum_{n=0}^{\infty} \{p^k a(n/p) + (-1)^{\frac{k(p-1)}{2}} pa(np)\} q^{\frac{n}{24}}$, $(a(x) = 0$ if x is not an integer). From this, we obtain the following:

Lemma 2.1.

$$\eta^{30} | T(p) = \frac{p^{15} + pa(30p^2) - pc(30p)a(6p)}{a(30p)} \eta^{30}$$

$$+ pa(6p) E_6^2 \eta^6 \quad (p \equiv 1, 5 \pmod{12}),$$

$$\eta^{30} | T(p) = -pa(18p) E_6 \eta^{18} \quad (p \equiv 7, 11 \pmod{12}),$$

$$E_6 \eta^{18} | T(p) = pb(18p) E_6 \eta^{18} \quad (p \equiv 1, 5),$$

$$E_6 \eta^{18} | T(p) = \frac{p^{15} - pb(18p^2) + pc(18p)b(6p)}{a(18p)} \eta^{30}$$

$$- pb(6p) E_6^2 \eta^6 \quad (p \equiv 7, 11),$$

$$E_6^2 \eta^6 | T(p) = \frac{p^{15} + pc(6p^2) - p(c(6p))^2}{a(6p)} \eta^{30} + pc(6p) E_6^2 \eta^6 \quad (p \equiv 1, 5),$$

$$E_6^2 \eta^6 | T(p) = \frac{p^{15} - pc(6p^2)}{b(6p)} E_6 \eta^{18} \quad (p \equiv 7, 11).$$

Proof. It is easily proved from the formula for $T(p)$ and the expressions of η^{30}, $E_6\eta^{18}$ and $E_6^2\eta^6$. □

From the fact that $E_6^2\eta^6 + \alpha E_6\eta^{18} + \beta\eta^{30}$ is an eigenfunction for Hecke operators and Lemma 2.1, for $p \equiv 1, 5 \pmod{12}$, we see that $\beta^2 + 43008\beta - 16533393408 = 0$, that is, $\beta = 108864, -151872$. If $\beta = 108864$, then $\alpha = \pm 48\sqrt{-3398}$, and its eigenvalue is $-\alpha pb(18p)$. If $\beta = -151872$, then $\alpha = 0$, and its eigenvalue is $pc(6p) + \beta pa(6p)$. Further, from the action of $T(p)$ $(p \equiv 7, 11)$, we see that if $\beta = 108864$, $\alpha = \pm\sqrt{-3398}$, then its eigenvalue is $-\alpha pb(6p)$, if $\beta = -151872$, $\alpha = 0$, then its eigenvalue is 0. We choose $\alpha = 48\sqrt{-3398}$, $\beta = 108864$, $\widetilde{\beta} = -151872$, and set $E_6^2 + \alpha E_6\eta^{18} + \beta\eta^{30} = \sum a_1(n)q^{\frac{n}{4}}$, $E_6^2 - \alpha E_6\eta^{18} + \beta\eta^{30} = \sum a_2(n)q^{\frac{n}{4}}$, $E_6^2\eta^6 + \widetilde{\beta}\eta^{30} = \sum a_3(n)q^{\frac{n}{4}}$. Then we obtain the following.

Proposition 2.2. *We have*

$$2(\beta - \widetilde{\beta})\eta^{30}(\tau) = 521472\eta^{30}(\tau) = \sum (a_1(n) + a_2(n) - 2a_3(n))q^{\frac{n}{4}} := \sum e(n)q^{\frac{n}{4}},$$

and its Dirichlet series is given as follows:

$$\sum e(n) \cdot n^{-s} = \sum (a_1(n) + a_2(n) - 2a_3(n))n^{-s},$$

where

$$\sum a_1(n) \cdot n^{-s} = \prod_{p \equiv 1, 5 \pmod{12}} (1 - b(18p)p^{-s} + p^{14-2s})^{-1}$$
$$\times \prod_{p \equiv 7, 11 \pmod{12}} (1 - \alpha b(6p)p^{-s} - p^{14-2s})^{-1},$$

$$\sum a_2(n) \cdot n^{-s} = \prod_{p \equiv 1, 5 \pmod{12}} (1 - b(18p)p^{-s} + p^{14-2s})^{-1}$$
$$\times \prod_{p \equiv 7, 11 \pmod{12}} (1 + \alpha b(6p)p^{-s} - p^{14-2s})^{-1},$$

$$\sum a_3(n) \cdot n^{-s} = \prod_{p \equiv 1, 5 \pmod{12}} (1 - (c(6p) + \widetilde{\beta} a(6p))\, p^{-s} + p^{14-2s})^{-1}$$

$$\times \prod_{p \equiv 7, 11 \pmod{12}} (1 - p^{14-2s})^{-1},$$

(*where p is prime number*).

Proof. It is easily proved from the following result [8]. If $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{24}}$ is eigenfunction for $T(p)$ with eigenvalue c, then Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ is given by

$$\varphi(s) = \left(\sum_{n \not\equiv 0 \pmod{p}} a(n) n^{-s} \right) (1 - (-1)^{\frac{k(p-1)}{2}} c p^{-s-1} + (-1)^{\frac{k(p-1)}{2}} p^{k-1-2s})^{-1}. \quad \square$$

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