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CHARACTERIZATIONS OF ANTI FUZZY POSITIVE IMPLICATIVE IDEALS IN BCK-ALGEBRAS

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Abstract

The aim of this paper is to give some characterizations of anti fuzzy positive implicative ideals in *BCK*-algebras. Also, we give an example to show that an anti fuzzy ideal may not be an anti fuzzy positive implicative ideal.

1. Introduction

The concept of a fuzzy set, which was introduced in [12], is applied to other algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. In 1991, Xi [11] applied the concept of fuzzy sets to *BCK*-algebras introduced by Imai and Iséki [5]. *BCK*-algebras generalize, on the one hand, the notion of the algebra of sets with the set subtraction as the only fundamental non-nullary operation and, on the other hand, the notion of the implication algebra (see [5]). In [1], Biswas introduced the concept of anti fuzzy subgroups of groups. Modifying his idea, in [4], Hong and Jun applied the idea to *BCK*-algebras. They introduced the notion of anti fuzzy ideals of *BCK*-algebras, lower level cuts of a fuzzy set, lower level ideal, they also fuzzified lower level cuts and proved some results on these. In this paper, we introduce the notion of anti fuzzy positive

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implicative ideals of *BCK*-algebras, and investigate some related properties, we give an example to show that an anti fuzzy ideal may not be an anti fuzzy positive implicative ideal. We show that in a positive implicative *BCK*-algebra, a fuzzy subset is an anti fuzzy ideal if and only if it is an anti fuzzy positive implicative ideal. We prove that a fuzzy subset of a *BCK*-algebra is a fuzzy positive implicative ideal if and only if the complement of this fuzzy subset is an anti fuzzy positive implicative ideal. We also prove that if a fuzzy subset is an anti fuzzy positive implicative ideal then so is the fuzzifications of its lower level cuts.

2. Preliminaries

Definition 2.1 [6]**.** An algebra $(X, *, 0)$ of type (2, 0) is called a *BCK-algebra* if it satisfies the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0,$

(ii) $(x * (x * y)) * y = 0,$

(iii) $x * x = 0,$

(iv) $0 * x = 0,$

(v) $x * y = 0$ and $y * x = 0$ imply $x = y.$

We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0.$

Proposition 2.2 [6]**.** *In any BCK-algebra X, the following hold for all* $x, y, z \in X$:

(i) $(x * y) * z = (x * z) * y,$

(ii) $x * y \leq x,$

(iii) $x * 0 = x,$

(iv) $(x * z) * (y * z) \leq x * y,$

(v) $x * (x * (x * y)) = x * y,$

(vi) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x.$

A *BCK*-algebra is said to be *positive implicative* if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$ (see [6, 10]).

Definition 2.3 [9]**.** A non-empty subset I of a BCK-algebra X is called an *ideal of X* if it satisfies

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.4 [9]**.** A non-empty subset I of a BCK-algebra X is called a *positive implicative ideal of X* if it satisfies (I_1) and (I_3) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

Definition 2.5 [12]**.** Let S be a non-empty set. A fuzzy subset A of S is a function $A : S \to [0, 1]$.

Let A be a fuzzy subset of S. Then for $t \in [0, 1]$, the t-level cut of A is the set $A_t = \{x \in S \mid A(x) \geq t\}$, and the complement of A, denoted by A^C, is the fuzzy subset of S given by $A^c(x) = 1 - A(x)$ for all $x \in S$ (see [2, 3, 7]).

Definition 2.6 [11]**.** A fuzzy subset A of a BCK-algebra is called a *fuzzy subalgebra of X* if

$$A(x * y) \geq \min\{A(x), A(y)\} \quad \text{for all } x, y \in X.$$

Definition 2.7 [11]**.** Let X be a BCK-algebra. A fuzzy subset A of X is called a *fuzzy ideal of X* if

(F_1) $A(0) \geq A(x)$,

(F_2) $A(x) \geq \min\{A(x * y), A(y)\}$,

for all $x, y \in X$.

Definition 2.8 [8]**.** A fuzzy subset A of a BCK-algebra X is called a *fuzzy positive implicative ideal of X* if it satisfies (F_1) and (F_3) $A(x * z) \geq \min\{A((x * y) * z), A(y * z)\}$ for all $x, y, z \in X$.

Definition 2.9 [4]**.** A fuzzy subset A of a BCK-algebra X is called an *anti fuzzy subalgebra of X* if

$$A(x * y) \leq \max\{A(x), A(y)\} \quad \text{for all } x, y \in X.$$

Definition 2.10 [4]**.** A fuzzy subset A of a *BCK*-algebra X is called an *anti fuzzy ideal of X* if

(A_1) $A(0) \leq A(x)$,

(A_2) $A(x) \leq \max\{A(x * y),\ A(y)\}$,

for all $x,\ y \in X$.

Proposition 2.11 [4]**.** *Every anti fuzzy ideal of a BCK-algebra X is an anti fuzzy subalgebra of X.*

Definition 2.12 [4]**.** Let A be a fuzzy subset of a *BCK*-algebra. Then for $t \in [0, 1]$ the lower t-level cut of A is the set

$$A^t = \{x \in X \mid A(x) \leq t\}.$$

Definition 2.13 [4]**.** Let A be a fuzzy subset of a *BCK*-algebra. The fuzzification of A^t, $t \in [0, 1]$, is the *fuzzy subset* μ_{A^t} *of X* defined by

$$\mu_{A^t} = \begin{cases} A(x) & \text{if } x \in A^t, \\ 0 & \text{otherwise.} \end{cases}$$

3. Anti Fuzzy Positive Implicative Ideal

Definition 3.1. A fuzzy subset A of a *BCK*-algebra X is called an *anti fuzzy positive implicative ideal of X* if it satisfies (A_1) and (A_3) $A(x * z) \leq \max\{A((x * y) * z),\ A(y * z)\}$ for all $x,\ y,\ z \in X$.

Example 3.2. Let $X = \{0,\ a,\ b\}$ be a *BCK*-algebra with Cayley table as follows:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Let $t_0,\ t_1 \in [0, 1]$ be such that $t_0 < t_1$. Define $A : X \to [0, 1]$ by $A(0) = A(a) = t_0$ and $A(b) = t_1$. Routine calculations give that A is an anti fuzzy positive implicative ideal.

Proposition 3.3. *Every anti fuzzy positive implicative ideal of a BCK-algebra X is order preserving.*

Proof. Let A be an anti fuzzy positive implicative ideal of a BCK-algebra X and $x, y \in X$ be such that $x \leq y$. Then for all $z \in X$, we have

$$A(x * z) \leq \max\{A((x * y) * z), A(y * z)\}$$

$$= \max\{A(0 * z), A(y * z)\}$$

$$= \max\{A(0), A(y * z)\}.$$

Putting $z = 0$

$$A(x) \leq \max\{A(0), A(y)\}$$

$$= A(y).$$

And so X is order preserving.

Proposition 3.4. *Every anti fuzzy positive implicative ideal of a BCK-algebra X is an anti fuzzy ideal.*

Proof. Let A be an anti fuzzy positive implicative ideal of a BCK-algebra X, so for all $x, y, z \in X$:

$$A(x * z) \leq \max\{A((x * y) * z), A(y * z)\}.$$

Putting $z = 0$

$$A(x) \leq \max\{A(x * y), A(y)\}.$$

Therefore, X is an anti fuzzy ideal.

Combining Propositions 2.11 and 3.4 yields the following result.

Proposition 3.5. *Every anti fuzzy positive implicative ideal of a BCK-algebra X is an anti fuzzy subalgebra of X.*

Remark. An anti fuzzy ideal (subalgebra) of a BCK-algebra X may not be an anti fuzzy positive implicative ideal of X as shown in the following example:

Example 3.6. Let $X = \{0, a, b, c\}$ be a BCK-algebra with Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 < t_1 < t_2$. Define $A : X \to [0, 1]$ by $A(0) = t_0$, $A(a) = A(b) = t_1$ and $A(c) = t_2$. Routine calculations give that A is an anti fuzzy ideal (subalgebra) of X, but not an anti fuzzy implicative ideal of X because

$$A(b * a) = A(a) = t_1 > \max\{A((b * a) * a), A(a * a)\}$$

$$= \max\{A(0), A(0)\} = A(0) = t_0.$$

Proposition 3.7. *If X is a positive implicative BCK-algebra, then every anti fuzzy ideal of X is an anti fuzzy positive implicative ideal of X.*

Proof. Let A be an anti fuzzy ideal of a positive implicative *BCK*-algebra X, so for all $x, y \in X$:

$$A(x) \leq \max\{A(x * y), A(y)\}.$$

By replacing x by $x * z$ and y by $y * z$ we get that:

$$A(x * z) \leq \max\{A((x * z) * (y * z)), A(y * z)\}.$$

Since X is a positive implicative *BCK*-algebra, $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$. Hence

$$A(x * z) \leq \max\{A((x * y) * z), A(y * z)\}.$$

This shows that A is an anti fuzzy positive implicative ideal of X.

By applying Propositions 3.4 and 3.7, we have

Theorem 3.8. *If X is a positive implicative BCK-algebra, then a fuzzy subset A of X is an anti fuzzy ideal of X if and only if it is an anti fuzzy positive implicative ideal of X.*

Proposition 3.9. *A fuzzy subset A of a BCK-algebra X is a fuzzy positive implicative ideal of X if and only if its complement A^c is an anti fuzzy positive implicative ideal of X.*

Proof. Let A be a fuzzy positive implicative ideal of a *BCK*-algebra *X*, and $x, y, z \in X$. Then

$$A^c(0) = 1 - A(0) \leq 1 - A(x) = A^c(x)$$

and

$$A^c(x * z) = 1 - A(x * z) \leq 1 - \min\{A((x * y) * z),\ A(y * z)\}$$

$$= 1 - \min\{1 - A^c((x * y) * z), 1 - A^c(y * z)\}$$

$$= \max\{A^c((x * y) * z),\ A^c(y * z)\}.$$

So, A^c is an anti fuzzy positive implicative ideal of *X*. The converse also can be proved similarly.

Theorem 3.10. *Let A be an anti fuzzy positive implicative ideal of a BCK-algebra X. Then the set*

$$X_A = \{x \in X \mid A(x) = A(0)\}$$

is a positive implicative ideal of X.

Proof. Clearly $0 \in X_A$. Let $x, y, z \in X_A$ be such that $(x * y) * z \in X_A$ and $y * z \in X_A$. Then $A((x * y) * z) = A(y * z) = A(0)$. It follows that

$$A(x * z) \leq \max\{A((x * y) * z),\ A(y * z)\}$$

$$= \max\{A(0),\ A(0)\} = A(0).$$

Since A is an anti fuzzy positive implicative ideal of *X*, $A(x * z) = A(0)$ and hence $x * z \in X_A$.

Theorem 3.11. *Let A be a fuzzy subset of a BCK-algebra X. Then A is an anti fuzzy positive implicative ideal of X if and only if for each* $t \in [0, 1]$, $t \geq A(0)$, *the lower t-level cut* A^t *is a positive implicative ideal of X.*

Proof. Let A be an anti fuzzy positive implicative ideal of *X* and $t \in [0, 1]$ with $t \geq A(0)$. Clearly $0 \in A^t$. Let $x, y, z \in X$ be such that $(x * y) * z \in A^t$ and $y * z \in A^t$. Then $A((x * y) * z) \leq t$, $A(y * z) \leq t$, hence

$$A(x * z) \leq \max\{A((x * y) * z),\ A(y * z)\} \leq t.$$

And so $x * z \in A^t$. Hence A^t is a positive implicative ideal of *X*.

Conversely, let A^t be a positive implicative ideal of X, we first show $A(0) \leq A(x)$ for all $x \in X$. If not, then there exists $x_0 \in X$ such that $A(0) > A(x_0)$. Putting $t_0 = \frac{1}{2}\{A(0) + A(x_0)\}$; then $0 \leq A(x_0) < t_0 < A(0) \leq 1$. Hence $x_0 \in A^{t_0}$, so that $A^{t_0} \neq \emptyset$. But A^{t_0} is a positive implicative ideal of X. Thus $0 \in A^{t_0}$ or $A(0) \leq t_0$ a contradiction. Hence $A(0) \leq A(x)$ for all $x \in X$. Now we prove that $A(x * z) \leq \max\{A((x * y) * z), A(y * z)\}$ for all $x, y, z \in X$. If not, then there exist $x_0, y_0, z_0 \in X$ such that

$$A(x_0 * z_0) > \max\{A((x_0 * y_0) * z_0), A(y_0 * z_0)\}.$$

Taking $s_0 = \frac{1}{2}\{A(x_0 * z_0) + \max\{A((x_0 * y_0) * z_0), A(y_0 * z_0)\}\}$; then $s_0 \leq A(x_0 * z_0)$ and

$$0 \leq \max\{A((x_0 * y_0) * z_0), A(y_0 * z_0)\} < s_0 \leq 1.$$

Thus we have $s_0 > A((x_0 * y_0) * z_0)$ and $s_0 > A(y_0 * z_0)$. Which imply that $(x_0 * y_0) * z_0 \in A^{s_0}$ and $y_0 * z_0 \in A^{s_0}$. But A^{s_0} is a positive implicative ideal of X. Thus $x_0 * z_0 \in A^{s_0}$ or $A(x_0 * z_0) \leq S_0$. This is a contradiction, ending the proof.

Theorem 3.12. *If A is an anti fuzzy positive implicative ideal of a BCK-algebra X. Then* μ_{A^t} *is also an anti fuzzy positive implicative ideal of X, where* $t \in [0, 1]$, $t \geq A(0)$.

Proof. From Theorem 3.11, it is sufficient to show that $(\mu_{A^t})^s$ is a positive implicative ideal of X, where $s \in [0, 1]$ and $s \geq \mu_{A^t}(0)$. Clearly, $0 \in (\mu_{A^t})^s$. Let $x, y, z \in X$ be such that $(x * y) * z \in (\mu_{A^t})^s$ and $y * z \in (\mu_{A^t})^s$, thus $\mu_{A^t}((x * y) * z) \leq s$ and $\mu_{A^t}(y * z) \leq s$. We claim that $x * z \in (\mu_{A^t})^s$ or $\mu_{A^t}(x * z) \leq s$. If $(x * y) * z \in A^t$ and $y * z \in A^t$, then $x * z \in A^t$ because A^t is a positive implicative ideal of X.

Hence

$$\mu_{A^t}(x * z) = A(x * z)$$
$$\leq \max\{A((x * y) * z),\ A(y * z)\}$$
$$= \max\{\mu_{A^t}((x * y) * z),\ \mu_{A^t}(y * z)\} \leq s,$$

and so $x * z \in (\mu_{A^t})^s$. If $(x * y) * z \notin A^t$ or $y * z \notin A^t$, then $\mu_{A^t}((x * y) * z) = 0$ or $\mu_{A^t}(y * z) = 0$, then clearly $\mu_{A^t}(x * z) \leq s$, and so $x * z \in (\mu_{A^t})^s$. Therefore, $(\mu_{A^t})^s$ is a positive implicative ideal of X.

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