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OSCILLATION THEOREMS FOR SECOND ORDER FORCED NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENT

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Abstract

In this paper, we concern with the oscillation of the forced second order nonlinear differential equations with delayed argument in the form

$$(r(t)x'(t))' + p(t)f(x(\tau(t))) + \sum_{i=1}^{n} q_i(t)|x|^{\lambda_i} sgn x = e(t),$$

where $r(t)$, $p(t)$, $q_i(t)$, $e(t)$ are continuous functions defined on $[0, \infty)$, $r(t)$ is positive, $r'(t) \geq 0$ and differentiable, $\lambda_1 > \cdots > \lambda_m > 1 > \lambda_{m+1} > \cdots > \lambda_n > 0$ $(n > m \geq 1)$. Our methodology is somewhat different from that of previous authors.

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1. Introduction

In this paper, we study the oscillatory behavior of the forced nonlinear functional differential equation

$$(r(t)x'(t))' + p(t)f(x(\tau(t))) + \sum_{i=1}^{n} q_i(t)|x|^{\lambda_i} sgnx = e(t), \tag{1.1}$$

where $r(t)$, $p(t)$, $q_i(t)$, $e(t)$ are continuous functions defined on $[0, \infty)$, $r(t)$ is positive, $r'(t) \geq 0$ and differentiable, $\lambda_1 > \cdots > \lambda_m > 1 > \lambda_{m+1} > \cdots > \lambda_n > 0$ $(n > m \geq 1)$. As usual, a solution of equation (1.1) is called *oscillatory* if it is defined on some ray $[T, \infty)$ with $T \geq 0$ and has unbounded set of zeros. Equation (1.1) is called *oscillatory* if all its solutions on some ray are oscillatory.

Very recently, Sun et al. [2, 3] obtained some new oscillation criteria for the equations in the form

$$(r(t)x'(t))' + p(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\lambda_i} sgnx = 0, \tag{1.2}$$

and

$$(r(t)x'(t))' + p(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\lambda_i} sgnx = e(t), \tag{1.3}$$

where $\lambda_1 > \cdots > \lambda_m > 1 > \lambda_{m+1} > \cdots > \lambda_n > 0$ $(n > m \geq 1)$. They also established oscillation theorems when $n > 1$.

The purpose of our paper is to further their investigation for equation (1.1), including the paper of Sun and Wong [2]. By using the similar method of Wong [4], we obtain some new oscillation criteria for equation (1.1). Our methodology is somewhat different from that of previous authors. We believe that our approach is simpler and more general than a recent result of Sun and Wong [2].

2. Main Results

We will need the following lemmas that have been proved in [2]:

Lemma 2.1 [2]**.** *Let* λ_i, $i = 1, 2, \ldots, n$, *be n-tuple satisfying* $\lambda_1 > \cdots > \lambda_m > 1 > \lambda_{m+1} > \cdots > \lambda_n > 0$. *Then there exists an n-tuple* $(k_1, k_2, \ldots, k_n)$ *satisfying*

$$\sum_{i=1}^{n} \lambda_i k_i = 1, \tag{a}$$

which also satisfies either

$$\sum_{i=1}^{n} k_i < 1, \quad 0 < k_i < 1 \tag{b}$$

or

$$\sum_{i=1}^{n} k_i = 1, \quad 0 < k_i < 1. \tag{c}$$

Theorem 2.1. *Let* $f(x) \in C(R, R)$, $\frac{f(x)}{x} \geq M > 0$, $x \neq 0$. *Suppose that for any* $T \geq 0$, *there exist constants* a_1, b_1, a_2, b_2 *such that* $T \leq a_1 < b_1 \leq a_2 < b_2$, *and*

$$\begin{cases} q_i(t) \geq 0, \ t \in [\tau(a_1), b_1] \cup [\tau(a_2), b_2], \quad i = 1, \dots, n, \\ e(t) \leq 0, \ t \in [\tau(a_1), b_1], \\ e(t) \geq 0, \ t \in [\tau(a_2), b_2]. \end{cases} \tag{2.1}$$

Let $D(a_i, b_i) = \{u \in C^1[a_i, b_i] : u^{\nu+1} > 0, \ \nu > 0$ *is a constant* $t \in (a_i, b_i)$, *and* $u(a_i) = u(b_i) = 0\}$ *for* $i = 1, 2$. *Assume that there exists a positive nondecreasing function* $\rho \in C^1([t_0, \infty), \mathbb{R})$ *such that, for some* $H \in D(a_i, b_i)$ *and for some* $\theta \geq 1$,

$$\int_{a_i}^{b_i} \left[H^{\nu+1}(t)\rho(t)R(t) - \frac{\theta\rho(t)r(t)H^{\nu-1}(t)A^2(t)}{4} \right] dt > 0, \tag{2.2}$$

for $i = 1, 2$, *then equation* (1.1) *is oscillatory*, *where*

$$R(t) = Mp(t)\frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} + a_0 |e(t)|^{k_0} \prod_{i=1}^{n} q_i^{k_i}(t),$$

$$A(t) = H(t)\frac{\rho'(t)}{\rho(t)} + (\nu + 1)H'(t),$$

$a_0 = \prod_{i=0}^{n} k_i^{-k_i}$, *and* $k_0, k_1, \dots, k_n$ *are positive constants satisfying* (a) *and* (b) *of Lemma* 2.1.

Proof. Assume to the contrary that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0,\ x(\tau(t)) > 0,$ when $t \geq t_0 > 0,$ for some t_0 depending on the solution $x(t)$. When $x(t)$ is eventually negative, the proof follows the same argument using the interval $[\tau(a_2), b_2]$ instead of $[\tau(a_1), b_1]$. Define

$$w(t) = \rho(t)\frac{r(t)x'(t)}{x(t)}, \quad t \geq t_0. \tag{2.3}$$

It follows from equation (1.1) that $w(t)$ satisfies the following differential equality:

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\frac{p(t)f(x(\tau(t)))}{x(t)} - \frac{w^2(t)}{\rho(t)r(t)} + \rho(t)\frac{e(t)}{x(t)} - \rho(t)\sum_{i=1}^{n} q_i(t)x^{\lambda_i - 1}(t), \tag{2.4}$$

and by the condition $f(x)/x \geq M > 0,$ we have

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - M\rho(t)p(t)\frac{x(\tau(t))}{x(t)} - \frac{w^2(t)}{\rho(t)r(t)} + \rho(t)\frac{e(t)}{x(t)} - \rho(t)\sum_{i=1}^{n} q_i(t)x^{\lambda_i - 1}(t). \tag{2.5}$$

By assumption, we can choose $a_1, b_1 \geq t_0,$ such that $b_1 \geq \tau(a_1),$ $\tau^2(a_1) = \tau(\tau(a_1)) \geq t_0,$ $q_i(t) \geq 0,$ for $t \in [\tau(a_1), b_1],$ and $e(t) \leq 0,$ for $t \in [\tau(a_1), b_1]$ and $i = 1, 2, ..., n.$ Recall the arithmetic-geometric mean inequality see [1]

$$\sum_{i=1}^{n} k_i u_i \geq \prod_{i=1}^{n} u_i^{k_i}, \quad u_i \geq 0, \tag{2.6}$$

where $k_0 = 1 - \sum_{i=1}^{n} k_i$ and $k_i > 0,$ $i = 1, 2, ..., n,$ are chosen according to given $\lambda_1, \lambda_2, ..., \lambda_n > 0$ as in Lemma 2.1 satisfying (a) and (b). Now return to (2.5) and identify $u_0 = k_0^{-1}|e(t)|x^{-1}(t)$ and $u_i = k_i^{-1}q_i(t)x^{\lambda_i - 1}(t)$ in (2.6) to obtain

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - M\rho(t)p(t)\frac{x(\tau(t))}{x(t)} - \frac{w^2(t)}{\rho(t)r(t)} - \rho(t)k_0^{-k_0}|e(t)|^{k_0}\prod_{i=1}^{n} k_i^{-k_i}q_i^{k_i}(t). \tag{2.7}$$

From equation (1.1), we can easily obtain that $x''(t) \leq 0,$ for $t \in [\tau(a_1), b_1].$ Therefore, we have that for $t \in [\tau(a_1), b_1],$

$$x(t) - x(\tau(a_1)) = x'(s)(t - \tau(a_1)) \geq x'(t)(t - \tau(a_1)), \tag{2.8}$$

where $s \in [\tau(a_1), b_1]$. Noting that $x(t) > 0$ for $t \geq \tau(a_1)$, we get by (2.8) that

$$x(t) \geq x'(t)(t - \tau(a_1)), \quad t \in [\tau(a_1), b_1],$$

i.e.,

$$\frac{x'(t)}{x(t)} \leq \frac{1}{t - \tau(a_1)}, \quad t \in [\tau(a_1), b_1]. \tag{2.9}$$

Integrating (2.9) from $\tau(t)$ to $t > a_1$, we obtain

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)}, \quad t \in (a_1, b_1]. \tag{2.10}$$

By using (2.10) in (2.7), we have that for $t \in (a_1, b_1]$,

$$\begin{aligned} w'(t) &\leq \frac{\rho'(t)}{\rho(t)} w(t) - M\rho(t)\, p(t) \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)} - \frac{w^2(t)}{\rho(t) r(t)} \\ &\quad - \rho(t) k_0^{-k_0} |\, e(t)|^{k_0} \prod_{i=1}^{n} k_i^{-k_i} q_i^{k_i}(t) \\ &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) R(t) - \frac{w^2(t)}{\rho(t) r(t)}. \end{aligned} \tag{2.11}$$

Multiplying both sides of (2.11) by $H^{\nu+1}(t)$ as given in the hypothesis of Theorem 2.1 and integrating (2.11) from a_1 to b_1, we obtain

$$\begin{aligned} &\int_{a_1}^{b_1} H^{\nu+1}(t) \rho(t) R(t)\, dt \\ &\leq \int_{a_1}^{b_1} H^{\nu+1}(t) \frac{\rho'(t)}{\rho(t)} w(t)\, dt - \int_{a_1}^{b_1} H^{\nu+1}(t) w'(t)\, dt - \int_{a_1}^{b_1} H^{\nu+1}(t) \frac{w^2(t)}{\rho(t) r(t)}\, dt. \end{aligned} \tag{2.12}$$

Using the integration by parts formula, we have

$$\begin{aligned} \int_{a_1}^{b_1} H^{\nu+1}(t) w'(t)\, dt &= H^{\nu+1}(t) w(t)\Big|_{a_1}^{b_1} - \int_{a_1}^{b_1} (\nu + 1) H^{\nu}(t) H'(t) w(t)\, dt \\ &= -\int_{a_1}^{b_1} (\nu + 1) H^{\nu}(t) H'(t) w(t)\, dt, \end{aligned} \tag{2.13}$$

where $H(a_1) = H(b_1) = 0.$ Substituting (2.13) into (2.12), we obtain

$$\int_{a_1}^{b_1} H^{\nu+1}(t)\rho(t)R(t)dt \le \int_{a_1}^{b_1} H^{\nu+1}(t)\frac{\rho'(t)}{\rho(t)}w(t)dt + \int_{a_1}^{b_1}(\nu+1)H^{\nu}(t)H'(t)w(t)dt$$

$$-\int_{a_1}^{b_1} H^{\nu+1}(t)\frac{w^2(t)}{\rho(t)r(t)}dt$$

$$= \int_{a_1}^{b_1} A(t)H^{\nu}(t)w(t)dt - \int_{a_1}^{b_1} H^{\nu+1}(t)\frac{w^2(t)}{\rho(t)r(t)}dt. \tag{2.14}$$

Then

$$\int_{a_1}^{b_1} H^{\nu+1}(t)\rho(t)R(t)dt$$

$$\le -\int_{a_1}^{b_1}\left[-A(t)H^{\nu}(t)w(t) + H^{\nu+1}(t)\frac{w^2(t)}{\rho(t)r(t)}\right]dt$$

$$= -\int_{a_1}^{b_1}\left[\sqrt{\frac{H^{\nu+1}(t)}{\theta\rho(t)r(t)}}w(t) - \left(\sqrt{\frac{\theta\rho(t)r(t)}{4H^{\nu+1}(t)}}H^{\nu}(t)A(t)\right)\right]^2 dt$$

$$+\int_{a_1}^{b_1}\left[\sqrt{\frac{\theta\rho(t)r(t)}{4H^{\nu+1}(t)}}H^{\nu}(t)A(t)\right]^2 dt - \int_{a_1}^{b_1}\frac{(\theta-1)H^{\nu+1}(t)}{\theta\rho(t)r(t)}w^2(t)dt. \tag{2.15}$$

From the hypothesis of Theorem 2.1 and (2.15), we have

$$\int_{a_1}^{b_1}\left[H^{\nu+1}(t)\rho(t)R(t) - \left(\sqrt{\frac{\theta\rho(t)r(t)}{4H^{\nu+1}(t)}}H^{\nu}(t)A(t)\right)^2\right]dt$$

$$= \int_{a_i}^{b_i}\left[H^{\nu+1}(t)\rho(t)R(t) - \frac{\theta\rho(t)r(t)H^{\nu-1}(t)A^2(t)}{4}\right]dt \le 0,$$

which contradicts (2.2). This completes the proof of Theorem 2.1.

Remark 1. We note that it suffices to satisfy (2.2) in Theorem 2.1 for some $\theta \ge 1,$ which ensures a certain flexibility in applications. Clearly, if (2.2) is satisfied for some $\theta_0 \ge 1,$ it shall also hold for any $\theta_1 > \theta_0.$

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