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ON ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF $x'' = -t^{\alpha\lambda-2}x^{1+\alpha}$ WITH $\alpha < 0$ AND $-1 < \lambda < 0$

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Abstract

We consider the Emden-Fowler type differential equation of the form denoted in the title. This is carried out under initial conditions $x(T) = A$, $x'(T) = B$ $(0 < T < \infty, 0 < A < \infty, -\infty < B < \infty)$ and $x(0) = a$, $x'(0) = b$ $(0 < a < \infty, -\infty \le b \le \infty)$. Asymptotic behavior of the positive solution is shown for arbitrarily fixed T, A and every B in the first initial condition, and for arbitrarily fixed a and every b in the second initial condition. Actually this is achieved from getting analytical expressions of the solution valid in the neighborhoods of both ends of its domain.

1. Introduction

In our previous papers [14], [20], [21], we treated a second order nonlinear differential equation

$$x'' = -t^{\alpha\lambda-2}x^{1+\alpha} \quad (' = d/dt) \tag{1.1}$$

in a domain

$$0 < t < \infty, \quad 0 < x < \infty.$$

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Here α, λ are parameters which were supposed $\alpha > 0$, $\lambda > 0$ in [14], $\alpha < 0$, $\lambda < -1$ or $\lambda > 0$ in [20], and $\alpha < 0$, $\lambda = 0, -1$ in [21]. Concretely speaking, given an initial condition

$$x(T) = A, \quad x'(T) = B \tag{1.2}$$

where

$$0 < T < \infty, \quad 0 < A < \infty, \quad -\infty < B < \infty,$$

we discussed asymptotic behavior of the solution of the initial value problem (1.1) and (1.2). The main aim of this paper is to continue this work in a remaining case

$$\alpha < 0, \quad -1 < \lambda < 0.$$

Therefore we suppose this throughout this paper.

Now (1.1) is worth considering, because this is related to various fields: mathematical physics, variational problems, partial differential equations and so forth (cf. [1], [15]). Furthermore differential equations containing (1.1) have been treated in many papers (cf. [5], [7], [8]) which chiefly discussed the solution continuable to ∞. However these papers did not consider such an initial value problem as above.

Our discussion will be carried out as follows: In Section 2, we shall state our theorems. For proving these, we shall transform (1.1) into a first order rational differential equation, denote this as a two dimensional autonomous system and consider these in Sections 3, 4. The proofs of our theorems will be completed in Section 5. Finally in Section 6 we shall obtain theorems of the case $T = 0$ from the theorems of Section 2.

Finally, notice that if $\alpha = -1$, then we get a general solution

$$x = -1/(\lambda + 1)\lambda t^{\lambda} + Ct + D$$

of (1.1), C and D being arbitrary constants. However our conclusions will contain this.

2. Statement of Our Main Theorems

First, fix T such that $0 < T < \infty$. Next, fix A with

$$0 < A < \psi(T) \tag{2.1}$$

where

$$\psi(t) = \{-\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}$$

is a particular solution of (1.1). Then if $x(t)$ denotes the solution of the initial value problem (1.1) and (1.2), we conclude the following:

Theorem 1. *There exists a real number* B_1 *such that if* $B = B_1$, *then* $x(t)$ *is defined for* $0 < t < \infty$ *and has the following representations*: *In the neighborhood of* $t = 0$, *we get*

$$x(t) = a\left\{1 + \sum_{m=1}^{\infty} t^{\alpha\lambda m} p_m(\log t)\right\} \tag{2.2}$$

in the case $1/\alpha\lambda \in \boldsymbol{N}$, *and*

$$x(t) = a\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha\lambda m+n}\right) \tag{2.3}$$

in the case $1/\alpha\lambda \notin \boldsymbol{N}$. *Here a is a positive constant,* p_m *are polynomials with* $\deg p_m \leq [\alpha\lambda m]$ ([ ] *denotes Gaussian symbol*) *and* x_{mn} *are constants with* $x_{0n} = 0$. *In the neighborhood of* $t = \infty$, *we have*

$$x(t) = \{-\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}\left\{1 + \sum_{n=1}^{\infty} x_n t^{(\mu_1/\alpha)n}\right\} \tag{2.4}$$

where x_n *are constants and*

$$\mu_1 = \alpha\{2\lambda + 1 - \sqrt{(2\lambda + 1)^2 + 4\alpha\lambda(\lambda + 1)}\}/2.$$

Moreover we obtain the following:

Theorem 2. *There exists a real number* B_2 *greater than* B_1 *such that if* $B = B_2$, *then* $x(t)$ *is defined for* $0 < t < \infty$ *and has the following representations*: *In the neighborhood of* $t = 0$, *we obtain*

$$x(t) = \{-\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}\left\{1 + \sum_{n=1}^{\infty} x_n t^{(\mu_2/\alpha)n}\right\} \tag{2.5}$$

where x_n *are constants and*

$$\mu_2 = \alpha\{2\lambda + 1 + \sqrt{(2\lambda + 1)^2 + 4\alpha\lambda(\lambda + 1)}\}/2.$$

In the neighborhood of $t = \infty$, *we get*

$$x(t) = Kt\left\{1 + \sum_{m=1}^{\infty} t^{\alpha(\lambda+1)m} p_m(\log t)\right\} \tag{2.6}$$

in the case $-1/\alpha(\lambda+1) \in \boldsymbol{N}$, *and*

$$x(t) = Kt\left\{1 + \sum_{m+n>0} x_{mn} t^{\alpha(\lambda+1)m-n}\right\} \tag{2.7}$$

in the case $-1/\alpha(\lambda+1) \notin \boldsymbol{N}$. *Here* K *is a positive constant,* p_m *are polynomials with* $\deg p_m \leq [-\alpha(\lambda+1)m]$ *and* x_{mn} *are constants with* $x_{0n} = 0$.

Now, suppose that ω_+ and ω_- denote positive finite numbers in theorems below. Then if $B \neq B_1, B_2$, we have the following three theorems:

Theorem 3. *If* $B < B_1$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$ *and has the following representations*:

In the neighborhood of $t = 0$, *we get* (2.2) *and* (2.3), *and in the neighborhood of* $t = \omega_+$, *we have the following*: *If* $-2 < \alpha < 0$, *then we obtain*

$$x(t) = K(\omega_+ - t)\left\{1 + \sum_{j+k+l>0} x_{jkl}(\omega_+ - t)^j (\omega_+ - t)^{-(\alpha/2)k} (\omega_+ - t)^{((\alpha+2)/2)l}\right\}, \tag{2.8}$$

$K(> 0)$ *and* x_{jkl} *being constants, if* $\alpha = -2$,

$$x(t) = \{-\lambda(\lambda+1)\}^{-1/2} t^{-\lambda} U^{1-G(U,C)} e^{CG(U,C)} \tag{2.9}$$

where

$$U \sim -\sqrt{-2\lambda(\lambda+1)} \log \frac{t}{\omega_+} \text{ as } t \to \omega_+,$$

$$G(U, C) = \frac{1}{2}(C - \log U)^{-1} \log(C - \log U)$$
$$+ \sum_{l+m+n\geq 2} g_{lmn}\{U(C - \log U)^2\}^l (C - \log U)^{-m/2}$$
$$\{(C - \log U)^{-1} \log(C - \log U)\}^n$$

C and g_{lmn} being constants, if $\alpha < -4$, $-4 < \alpha < -2$,

$$x(t) = \left\{-\frac{2(\alpha+2)}{\alpha^2 \omega_+^{\alpha\lambda-2}}\right\}^{1/\alpha} (\omega_+ - t)^{-2/\alpha}$$

$$\left\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m (\omega_+ - t)^{(2+4/\alpha)n}\right\}, \tag{2.10}$$

x_{mn} being constants, and if $\alpha = -4$,

$$x(t) = \sqrt{\frac{2}{\omega_+^{2\lambda+1}}}(\omega_+ - t)^{1/2}$$

$$\left\{1 + \sum_{m>0} (\omega_+ - t)^m p_m(\log(\omega_+ - t))\right\}, \tag{2.11}$$

p_m being polynomials with $\deg p_m \leq m$.

Notice that we denote $f(t) \sim g(t)$ as $t \to \tau$ for some τ, if

$$\lim_{t\to\tau} f(t)/g(t) = 1.$$

Theorem 4. *If $B_1 < B < B_2$, then $x(t)$ is defined for $0 < t < \infty$. Moreover $x(t)$ is represented as* (2.2) *and* (2.3) *in the neighborhood of $t = 0$, and as* (2.6) *and* (2.7) *in the neighborhood of $t = \infty$.*

Theorem 5. *If $B > B_2$, then $x(t)$ is defined for $\omega_- < t < \infty$ and has the following representations*:

In the neighborhood of $t = \omega_-$, we get the following: *If $-2 < \alpha < 0$, then we have*

$$x(t) = K(t-\omega_-)\left\{1 + \sum_{j+k+l>0} x_{jkl}(t-\omega_-)^j (t-\omega_-)^{-(\alpha/2)k} (t-\omega_-)^{((\alpha+2)/2)l}\right\}, \tag{2.12}$$

$K(>0)$ and x_{jkl} being constants, if $\alpha = -2$, then

$$x(t) = \{-\lambda(\lambda+1)\}^{-1/2} t^{-\lambda} U^{1-G(U,\,C)} e^{CG(U,\,C)} \tag{2.13}$$

where

$$U \sim \sqrt{-2\lambda(\lambda+1)}\log\frac{t}{\omega_-} \quad as\ t \to \omega_-$$

and $G(U, C)$ *has the same form as in* (2.9), *if* $\alpha < -4, -4 < \alpha < -2,$ *then*

$$x(t) = \left\{-\frac{2(\alpha+2)}{\alpha^2\omega_-^{\alpha\lambda-2}}\right\}^{1/\alpha}(t-\omega_-)^{-2/\alpha}$$

$$\left\{1 + \sum_{m+n>0} x_{mn}(t-\omega_-)^m(t-\omega_-)^{(2+4/\alpha)n}\right\}, \tag{2.14}$$

x_{mn} *being constants, and if* $\alpha = -4,$ *then*

$$x(t) = \sqrt{\frac{2}{\omega_-^{2\lambda+1}}}(t-\omega_-)^{1/2}$$

$$\left\{1 + \sum_{m>0}(t-\omega_-)^m p_m(\log(t-\omega_-))\right\}, \tag{2.15}$$

p_m *being polynomials with* $\deg p_m \le m.$

In the neighborhood of $t = \infty,$ *we obtain* (2.6) *and* (2.7).

Now, fix A with

$$A = \psi(T) \tag{2.16}$$

instead of (2.1). Then we get the following:

Theorem 6. *If* $B < -\lambda A/T,$ $B = -\lambda A/T,$ *and* $B > -\lambda A/T,$ *then the conclusion of Theorem* 3, $x(t) = \psi(t),$ *and the conclusion of Theorem* 5 *follow respectively*.

Finally, fix A with

$$A > \psi(T)$$

instead of (2.16). Then we have the following two theorems:

Theorem 7. *There exist real numbers* B_3 *and* B_4 *with* $B_3 < B_4$ *such that the following statements are valid*:

If $B = B_3$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$ *and we get* (2.5) *in the neighborhood of* $t = 0$ *and* (2.8) *through* (2.11) *in the neighborhood of* $t = \omega_+$. *If* $B = B_4$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ *and we have* (2.12) *through* (2.15) *in the neighborhood of* $t = \omega_-$ *and* (2.4) *in the neighborhood of* $t = \infty$.

Theorem 8. *If* $B < B_3$ *and* $B > B_4$, *then the conclusions of Theorems* 3 *and* 5 *follow respectively*. *Moreover if* $B_3 < B < B_4$, *then* $x(t)$ *is defined for* $\omega_- < t < \omega_+$ *and we get* (2.12) *through* (2.15) *in the neighborhood of* $t = \omega_-$ *and* (2.8) *through* (2.11) *in the neighborhood of* $t = \omega_+$.

3. The Reduction of (1.1) and Consideration of the Reduced Equation

In this section, we use a transformation

$$y = \psi(t)^{-\alpha} x^{\alpha} \ \text{(namely } x = \psi(t) y^{1/\alpha}\text{)}, \quad z = ty' \tag{3.1}$$

and reduce (1.1) into the first order rational differential equation

$$\begin{aligned} dz/dy = \{&(\alpha - 1)z^2 + \alpha(2\lambda + 1)yz \\ &+ \alpha^2\lambda(\lambda + 1)y^3 - \alpha^2\lambda(\lambda + 1)y^2\}/\alpha yz. \end{aligned} \tag{3.2}$$

The transformation (3.1) has been already used in [14] and the transformation of this kind appeared originally in [9]. Using a parameter *s*, we write (3.2) a two dimensional autonomous system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz, \\ \frac{dz}{ds} &= (\alpha - 1)z^2 + \alpha(2\lambda + 1)yz + \alpha^2\lambda(\lambda + 1)y^3 - \alpha^2\lambda(\lambda + 1)y^2. \end{aligned} \tag{3.3}$$

(3.2) and (3.3) have been got also in [9], [10], and [14]. Notice that we always get $y > 0$ from (3.1), the critical points of (3.3) are points (0, 0) and (1, 0) in the *yz* plane, and orbits of (3.3) are solutions of (3.2). (1, 0) is a saddle point and therefore from the discussion of Section 4 in [9], (3.3) has orbits represented as

$$z = \frac{\mu_1}{\alpha}(y - 1) + \cdots \tag{3.4}$$

$$z = \frac{\mu_2}{\alpha}(y - 1) + \cdots \tag{3.5}$$

in the neighborhood of $y = 1$. Here $\mu_2 < 0 < \mu_1$. Moreover due to the same discussion we obtain solutions of (1.1) represented as (2.4) and (2.5) from (3.4) and (3.5) respectively.

Here, let $z = z_1(y)$ and $z = z_2(y)$ be orbits of (3.3) represented as (3.4) and (3.5) lying in a region $0 < y < 1$ respectively. Then we shall examine asymptotic behavior of $z = z_1(y)$ and $z = z_2(y)$ as these leave (1, 0). For this, we conclude the following:

Lemma 3.1. *If* $z = z(y)$ *is an orbit of* (3.3), *then* $z(y)$ *is bounded as y tends to a nonnegative number*.

Proof. Suppose the contrary. Then there exist a nonnegative number c and a sequence $\{y_n\}$ such that

$$z(y_n) \to \pm\infty \text{ as } y_n \to c. \tag{3.6}$$

Therefore if we put $\zeta(y) = 1/z(y)$, then we get

$$\zeta(y_n) \to 0 \text{ as } y_n \to c$$

and from (3.2), $\zeta = \zeta(y)$ satisfies

$$\begin{aligned} d\zeta/dy = -\{&(\alpha - 1)\zeta + \alpha(2\lambda + 1)y\zeta^2 \\ &+ \alpha^2\lambda(\lambda + 1)y^3\zeta^3 - \alpha^2\lambda(\lambda + 1)y^2\zeta^3\}/\alpha y. \end{aligned} \tag{3.7}$$

If $c \neq 0$, then we conclude a contradiction $\zeta(y) \equiv 0$, for the righthand side is holomorphic at $(y, \zeta) = (c, 0)$. Furthermore if $c = 0$, then from (3.7) we have a Briot-Bouquet differential equation

$$y\frac{d\zeta}{dy} = -\frac{\alpha - 1}{\alpha}\zeta - (2\lambda + 1)y\zeta^2 - \alpha\lambda(\lambda + 1)y^3\zeta^3 + \alpha\lambda(\lambda + 1)y^2\zeta^3.$$

Here we obtain $-(\alpha - 1)/\alpha < 0$, for $\alpha < 0$. Therefore from Lemma 2.5 of [15] we get a contradiction $\zeta(y) \equiv 0$ again. Thus the proof is complete.

Now we conclude the following:

Lemma 3.2. *For* $0 < y < 1$, *we get*

$$z_1(y) > 0, \quad z_2(y) < 0.$$

Moreover in the yz plane $(y, z_1(y))$ *tends to the origin as* $s \to \infty$ *and* $(y, z_2(y))$, *as* $s \to -\infty$.

Proof. On a segment $0 < y < 1$, $z = 0$, we get from (3.3)

$$\frac{dz}{ds} = \alpha^2 \lambda(\lambda + 1) y^2 (y - 1) > 0. \tag{3.8}$$

Moreover from (3.3) we have

$$\frac{dy}{ds} < 0 \text{ if } z > 0, \quad \frac{dy}{ds} > 0 \text{ if } z < 0.$$

Therefore an orbit $z = z_1(y)$ leaves (1, 0) as s increases and from (3.8) cannot pass the y axis. Hence we obtain $z_1(y) > 0$. Similarly we get $z_2(y) < 0$.

Furthermore

$$y \equiv 0, \quad z = -\frac{1}{(\alpha - 1)s + C} \text{ (C is a constant)}$$

is a solution of (3.3) and thus the z axis consists of orbits of (3.3) and the origin. Therefore orbits of (3.3) lying in a region $y > 0$ cannot pass the z axis. Hence from Lemma 3.1 and Poincaré-Bendixon's theorem we conclude

$$(y, z_1(y)) \to (0, 0) \text{ as } s \to \infty,$$

$$(y, z_2(y)) \to (0, 0) \text{ as } s \to -\infty.$$

The orbits have the following property as $y \to 0$:

Lemma 3.3. *If* $z = z(y)$ *is an orbit of* (3.3) *continuable to* $y = 0$, *then we obtain*

$$\lim_{y \to 0} z(y) = 0, \tag{3.9}$$

and

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda, \ \alpha(\lambda + 1).$$

Proof. From the reason why $(y, z_1(y))$ tends to (0, 0), we conclude (3.9). Hence owing to the same reasoning as of Lemma 1 of [16], we get

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda, \ \alpha(\lambda + 1), \ \pm\infty.$$

However if we have

$$\lim_{y \to 0} \frac{z(y)}{y} = \pm\infty,$$

then we put $w = yz^{-1}$ in (3.2) and obtain

$$y\frac{dw}{dy} = \frac{1}{\alpha} w - (2\lambda + 1) w^2 + \alpha\lambda(\lambda + 1) w^3 - \alpha\lambda(\lambda + 1) yw^3. \tag{3.10}$$

Therefore $w = yz(y)^{-1}$ is a solution of (3.10) such that $\lim_{y \to 0} w = 0$, and since $1/\alpha < 0$, we get a contradiction $w \equiv 0$ from Lemma 2.5 of [15]. This completes the proof.

From the orbits tending to the origin we have the following:

Lemma 3.4. *If* $z = z(y)$ *is the orbit such that*

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda, \tag{3.11}$$

then this is represented as

$$z(y) = \alpha\lambda y \left[1 + \sum_{m+n>0} z_{mn} y^m \{ y^{1/\alpha\lambda} (h \log y + C) \}^n \right] \tag{3.12}$$

where z_{mn}, h *and* C *are constants and* $h = 0$ *unless* $1/\alpha\lambda$ *is an integer. Moreover from* (3.12) *we obtain a solution of* (1.1) *represented as* (2.2) *and* (2.3) *in the neighborhood of* $t = 0$.

Proof. We follow the line of obtaining (3.20) of [14]. Put $v = y^{-1}z - \alpha\lambda$ in (3.2). Then we get

$$y\frac{dv}{dy} = (\lambda + 1) y + \frac{1}{\alpha\lambda} v + \cdots$$

where $\cdots$ denotes terms whose degrees are greater than the degree of the previous term. Therefore since $1/\alpha\lambda > 0$, we have

$$v = \sum_{m+n>0} v_{mn} y^m \{ y^{1/\alpha\lambda} (h \log y + C) \}^n$$

where v_{mn} are constants. From this we obtain (3.12) where $z_{mn} = v_{mn}/\alpha\lambda$.

Moreover applying (3.1) to (3.12) we get a differential equation

$$ty' = \alpha\lambda y\left[1 + \sum_{m+n>0} z_{mn} y^m \{y^{1/\alpha\lambda}(h \log y + C)\}^n\right]. \tag{3.13}$$

Solving this we have

$$y = \Gamma t^{\alpha\lambda}\left[1 + \sum_{m+n>0} y_{mn} t^{\alpha\lambda m}\{t(\hat{h} \log t + \hat{C})\}^n\right]$$

where Γ, y_{mn}, $\hat{h}$, and $\hat{C}$ are constants and $\hat{h} = \alpha\lambda h$, $\hat{C} = h \log \Gamma + C$. Therefore we obtain

$$t \to 0 \text{ as } y \to 0. \tag{3.14}$$

Furthermore using (3.1) again we obtain a solution of (1.1) represented as

$$x(t) = a\left[1 + \sum_{m+n>0} \tilde{y}_{mn} t^{\alpha\lambda m}\{t(\hat{h} \log t + \hat{C})\}^n\right]$$

where

$$a = \{-\lambda(\lambda + 1)\}^{1/\alpha}\Gamma^{1/\alpha}$$

and $\tilde{y}_{mn}$ are constants.

If $1/\alpha\lambda \in \boldsymbol{N}$, then we get

$$x(t) = a\left\{1 + \sum_{m+n>0} t^{\alpha\lambda(m+n/\alpha\lambda)} P_{mn}(\log t)\right\}$$

where P_{mn} are polynomials with $\deg P_{mn} \le n$. Hence if we put

$$k = m + \frac{n}{\alpha\lambda}, \quad p_k = P_{mn},$$

and denote k as m, then we have (2.2). Moreover if $1/\alpha\lambda \notin \boldsymbol{N}$, then since $h = 0$ we obtain (2.3). Here it is necessary to show $x_{0n} = 0$. Substitute (2.3) into (1.1). Then we get

$$\begin{aligned}&\sum_{m+n>0} (\alpha\lambda m + n)(\alpha\lambda m + n - 1) x_{mn} t^{\alpha\lambda m + n} \\ &= -a^\alpha t^{\alpha\lambda}\left\{1 + \sum_{m+n>0} Q_{mn}(x_{MN}) t^{\alpha\lambda m + n}\right\}\end{aligned} \tag{3.15}$$

where $Q_{mn}(x_{MN})$ are polynomials of x_{MN} with $M \le m$, $N \le n$. Hence from the righthand side of (3.15), every term of the lefthand side contains $t^{\alpha\lambda}$ and we obtain $x_{0n} = 0$.

Owing to (3.14), representations (2.2) and (2.3) are valid in the neighborhood of $t = 0$. This completes the proof.

Similarly we conclude the following:

Lemma 3.5. *If* $z = z(y)$ *is the orbit such that*

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha(\lambda + 1), \tag{3.16}$$

then this is represented as

$$z(y) = \alpha(\lambda + 1) y \left[1 + \sum_{m+n>0} z_{mn} y^m \{ y^{-1/\alpha(\lambda+1)} (h \log y + C) \}^n \right] \tag{3.17}$$

where z_{mn}, h *and* C *are the same as in Lemma* 3.4. *Furthermore from* (3.17) *we get a solution of* (1.1) *represented as* (2.6) *and* (2.7) *in the neighborhood of* $t = 0$.

Since it suffices to follow the proof of Lemma 3.4, we omit the proof.

4. Solutions of (3.2) Continuable to $y = \infty$

Let us consider (3.2) in the neighborhood of $y = \infty$. For this we put $y = 1/\eta$ in (3.2) and get

$$\frac{dz}{d\eta} = -\frac{F(\eta, z)}{\alpha \eta^4 z} \tag{4.1}$$

where

$$F(\eta, z) = (\alpha - 1)\eta^3 z^2 + \alpha(2\lambda + 1)\eta^2 z + \alpha^2 \lambda(\lambda + 1) - \alpha^2 \lambda(\lambda + 1)\eta.$$

Now, suppose that a solution $z = z(\eta)$ of (4.1) is bounded as $y \to \infty$ namely $\eta \to 0$. Then from

$$\frac{d\eta}{dz} = -\frac{\alpha \eta^4 z}{F(\eta, z)} \tag{4.2}$$

we have a contradiction $\eta \equiv 0$, for the righthand side of (4.2) is holomorphic at $(\eta, z) = (0, c)$ where c is an arbitrary finite number. Therefore we obtain

$$z \to \pm\infty \text{ as } \eta \to 0. \tag{4.3}$$

So we put $z = 1/\zeta$ in (4.1) and get

$$d\zeta/d\eta = \{-\alpha^2\lambda(\lambda+1)\eta\zeta^3 + \alpha(2\lambda+1)\eta^2\zeta^2$$

$$- (1-\alpha)\eta^3\zeta + \alpha^2\lambda(\lambda+1)\zeta^3\}/\alpha\eta^4.$$

Moreover if we put

$$w = \eta^{-3/2}\zeta, \quad \xi = \eta^{1/2}, \tag{4.4}$$

then we have

$$\xi \frac{dw}{d\xi} = G(\xi, w) \tag{4.5}$$

where

$$G(\xi, w) = -\frac{\alpha+2}{\alpha} w + 2(2\lambda+1)\xi w^2$$

$$+ 2\alpha\lambda(\lambda+1)w^3 - 2\alpha\lambda(\lambda+1)\xi^2 w^3.$$

For considering (3.2) in the neighborhood of $y = \infty$, it suffices to treat (4.5) in the neighborhood of $\xi = 0$. If $\xi = 0$ and the righthand side of (4.5) vanishes, then we obtain

$$w = 0 \text{ if } -2 \le \alpha < 0, \quad w = 0, \pm\rho \text{ if } \alpha < -2$$

where

$$\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha+2}{\lambda(\lambda+1)}}.$$

Now, let γ be a cluster point of a solution of (4.5) as $\xi \to 0$. Then we get the following:

Lemma 4.1. γ *is the limit point and*

$$\gamma = 0, \pm\rho.$$

Proof. Suppose $\gamma \neq 0, \pm\rho, \pm\infty.$ Then from (4.5) we have

$$\frac{d\xi}{dw} = \frac{\xi}{G(\xi, w)}$$

whose righthand side is holomorphic at $(\xi, w) = (0, \gamma)$. Therefore we conclude a contradiction $\xi \equiv 0,$ which implies

$$\gamma = 0, \pm\rho, \pm\infty.$$

However if $\gamma = \pm\infty,$ then we put $\theta = 1/w$ in (4.5) and obtain

$$\frac{d\xi}{d\theta} = \frac{\alpha\xi\theta}{(\alpha + 2)\theta^2 - 2\alpha(2\lambda + 1)\xi\theta - 2\alpha^2\lambda(\lambda + 1) + 2\alpha^2\lambda(\lambda + 1)\xi^2}.$$

This implies a contradiction $\xi \equiv 0,$ for the righthand side of this is holomorphic at $(\xi, \theta) = (0, 0)$. Now the proof is complete.

In cases where $\gamma = 0, \pm\rho,$ we obtain the representations of the solutions of (1.1) denoted in the above theorems as follows:

Lemma 4.2. *If* $\gamma = 0,$ *then we get*

$$-2 \leq \alpha < 0.$$

Moreover if $-2 < \alpha < 0,$ *then we have* (2.8) *and* (2.12). *If* $\alpha = -2,$ *then we obtain* (2.9) *and* (2.13).

Proof. If $-2 < \alpha < 0,$ then from (4.5) we get

$$w = \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n$$

where w_{mn} are constants with $w_{01} = 1$ and $w_{m0} = 0,$ and C is a constant, since $-(\alpha + 2)/\alpha > 0$ and w divides the righthand side of (4.5). Therefore from (4.4) and (3.1) we have

$$\xi^3 \sum_{m+n>0} w_{mn}C^n\xi^{m-((\alpha+2)/\alpha)n} ty' = 1.$$

On the other hand, we obtain

$$y' = -\eta^{-2}\eta' = -2\xi^{-3}\xi'.$$

Hence we get

$$\sum_{m+n>0} w_{mn} C^n \xi^{m-((\alpha+2)/\alpha)n} \xi' = -\frac{1}{2t}$$

and integrating both sides,

$$\sum_{m+n>0} \tilde{w}_{mn} C^n \xi^{m-((\alpha+2)/\alpha)n+1} = -\frac{1}{2}\log t + D \tag{4.6}$$

where D is a constant and $\tilde{w}_{mn}$ are constants with

$$\tilde{w}_{01} = -\frac{\alpha}{2}, \quad \tilde{w}_{m0} = 0.$$

Putting $D = (\log\tau)/2$ here, we have $t \to \tau$ as $\xi \to 0$. Moreover from (4.6) we derive

$$\xi\left\{1 + \sum_{m+n>0} a_{mn} \xi^{m-((\alpha+2)/\alpha)n}\right\} = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-\alpha/2}.$$

Hence using Smith's lemma – cf. Lemma 1 of [10], we obtain

$$\xi = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-\alpha/2}\left\{1 + \sum_{m+n>0} b_{mn}\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-(\alpha/2)m}\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{((\alpha+2)/2)n}\right\}$$

and from $\xi = \eta^{1/2} = y^{-1/2}$,

$$y = \xi^{-2} = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{\alpha}\left\{1 + \sum_{m+n>0} c_{mn}\left(\log\frac{t}{\tau}\right)^{-(\alpha/2)m}\left(\log\frac{t}{\tau}\right)^{((\alpha+2)/2)n}\right\}. \tag{4.7}$$

Here, notice (4.3). Then if $z \to \infty$ as $\eta \to 0$ (namely $\xi \to 0$ and $y \to \infty$), we get $y' > 0$ for sufficiently large y. Therefore τ is the right end of the domain of y, for $t \to \tau$ as $\xi \to 0$. So we denote $\tau = \omega_+$. From the same reason we denote $\tau = \omega_-$, if $z \to -\infty$ as $\eta \to 0$.

Moreover we expand $\log t/\tau$ as

$$\log \frac{t}{\omega_+} = -\frac{\omega_+ - t}{\omega_+} - \frac{1}{2}\left(\frac{\omega_+ - t}{\omega_+}\right)^2 - \frac{1}{3}\left(\frac{\omega_+ - t}{\omega_+}\right)^3 - \cdots,$$

$$\log \frac{t}{\omega_-} = \frac{t - \omega_-}{\omega_-} - \frac{1}{2}\left(\frac{t - \omega_-}{\omega_-}\right)^2 + \frac{1}{3}\left(\frac{t - \omega_-}{\omega_-}\right)^3 - \cdots$$

and using (3.1) for (4.7) we have (2.8) and (2.12).

If $\alpha = -2$, then from (4.5) we obtain

$$\xi \frac{dw}{d\xi} = 2(2\lambda + 1)\xi w^2 - 4\lambda(\lambda + 1)w^3 + 4\lambda(\lambda + 1)\xi^2 w^3$$

and from the theory of [3] – cf. formulas (16) and (24) of this paper,

$$w = \pm\{-8\lambda(\lambda + 1)(C - \log \xi)\}^{-1/2}\left[1 + \sum_{0<2j+k<2(N+1)} w_{jk}\xi^j \right.$$

$$\left. \{-8\lambda(\lambda + 1)(C - \log \xi)\}^{-k/2} + \Omega_N\right]$$

$$|\Omega_N| \le K_N |\log \xi|^{-N}$$

where K_N is a constant. Since this has the form similar to the solution of (2.8) of [17], we adopt the discussion done in the proof of Corollary 2.6 of this paper and get (2.8) and (2.12) of the same paper. This was carried out also in [20].

Finally if $\alpha < -2$ then applying Lemma 2.5 of [15] to (4.5), we have a contradiction $w \equiv 0$, for $-(\alpha + 2)/\alpha < 0$. Hence we obtain $-2 \le \alpha < 0$, which completes the proof.

Lemma 4.3. *Suppose that* $\alpha < -2$ *and* $\gamma = \pm\rho$. *Then if* $\alpha < -4$, $-4 < \alpha < -2$, *we get representations* (2.10) *and* (2.14), *and if* $\alpha = -4$, *representations* (2.11) *and* (2.15).

Proof. Put $\theta = w - \gamma$ in (4.5). Then this is the same transformation as in Section 3 of [10] and it suffices to follow the discussion done there. From the same reason as in the proof of Lemma 4.2 we have the representations in the neighborhood of $t = \omega_+$, if $z \to \infty$ as $\eta \to 0$ and those in the neighborhood of $t = \omega_-$, if $z \to -\infty$ as $\eta \to 0$. Now the proof is complete.

5. Proof of the Theorems of Section 2

First, let us review some conclusions stated in Sections 3 and 4. The critical point (1, 0) of (3.3) is a saddle point and (3.3) has four orbits reaching (1, 0). From (3.8) the orbit of (3.3) passes the y axis at most once. Owing to Lemmas 3.1 and 3.3, the orbits tend to the origin as y decreases, unless these tend to (1, 0). Moreover as y increases, from Lemma 3.1 and (4.3) the orbits not tending to (1, 0) is continuable to ∞ and tend to $\pm\infty$. Therefore the phase portrait of (3.3) is as in Figure. Here $z = z_3(y)$ and $z = z_4(y)$ respectively denote the orbits represented as (3.5) and (3.4) in $y > 1$.

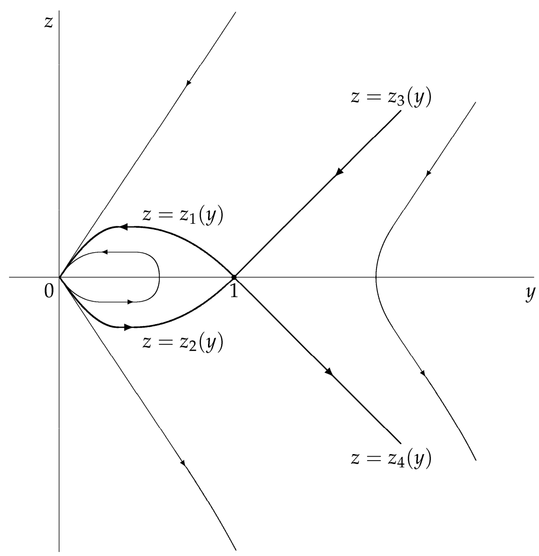

Figure. The phase portrait of (3.3).

Now, let us consider (1.1) under the initial condition (1.2). From (3.1) we get

$$z = \alpha y\left(\lambda + t\,\frac{x'}{x}\right).$$

Let (y_0, z_0) be (y, z) at $t = T$. Then we have

$$y_0 = \psi(T)^{-\alpha} A^{\alpha}, \quad z_0 = \alpha y_0\left(\lambda + \frac{TB}{A}\right).$$

Throughout this section, fix T and A. Then y_0 is fixed and z_0 is a decreasing function of B. Hence (y_0, z_0) descends a line $y = y_0$ as B increases. Moreover from a solution of (1.1) satisfying (1.2) we obtain a solution of (3.2) satisfying an initial condition

$$z(y_0) = z_0 \tag{5.1}$$

and an orbit of (3.3) passing (y_0, z_0). Conversely from the solution of (3.2) with (5.1) or the orbits of (3.3) passing (y_0, z_0) we get the solution of (1.1) with (1.2).

Furthermore, notice that if $z_0 = 0$, then we have $B = -\lambda A/T$, and if $0 < y_0 < 1$, $y_0 = 1$, $y_0 > 0$, then we respectively obtain $A < \psi(T)$, $A = \psi(T)$, $A > \psi(T)$.

Here the following lemma is required:

Lemma 5.1. *If* $x = x(t)$ *is a solution of* (1.1) *whose domain is an interval* (ω_-, ω_+) *and an orbit* (y, z) *of* (3.3) *is defined from* (3.1) *and* $x = x(t)$, *then we get*

$$\lim_{t\to\omega_\pm} (y, z) = (0, 0), (1, 0), (\infty, \pm\infty).$$

Proof. The discussion of the proof of Lemma 2 of [16] implies that (y, z) does not accumulate to a regular point in $y > 0$. Moreover from Lemma 3.1, z is bounded as y tends to a nonnegative number, and from (4.3) we have $z \to \pm\infty$ as $y \to \infty$. This completes the proof.

Let us consider the case $A < \psi(T)$ now. Then we obtain $0 < y_0 < 1$. So as (y_0, z_0), take an intersection of the line $y = y_0$ and the orbit $z = z_1(y)$, and suppose $B = B_1$ then. Moreover, define (y, z) from applying (3.1) to the solution $x = x(t)$ of (1.1) and (1.2). Then (y, z) is situated on $z = z_1(y)$ and from Lemma 5.1 and $z_1(y) = ty' > 0$ we get

$$y \to 0 \text{ as } t \to \omega_-, \quad y \to 1 \text{ as } t \to \omega_+$$

where (ω_-, ω_+) denotes the domain of $x(t)$ again. Therefore from (3.4) we have (2.4), and from Lemmas 3.3 and 3.4, (2.2) and (2.3). This completes the proof of Theorem 1.

Next as (y_0, z_0), take an intersection of the line $y = y_0$ and the orbit $z = z_2(y)$, and assume $B = B_2$. Since z_0 is decreasing in B, we obtain $B_1 < B_2$. Now if we define (y, z) as above, then the same discussion implies (2.5), (2.6), and (2.7), which completes the proof of Theorem 2.

Here, let $z(y, B)$ be a solution of (3.2) and (5.1). Then if $B < B_1$, we get $z(y, B) > z_1(y)$. Hence from Lemma 5.1 and Figure, we have

$$y \to 0 \text{ as } t \to \omega_-, \quad y \to \infty \text{ as } t \to \omega_+.$$

Therefore the same discussion as in the case $B = B_1$ implies (2.2) and (2.3), and from Lemmas 4.2 and 4.3 we obtain (2.8), (2.9), (2.10), and (2.11). Now the proof of Theorem 3 is complete.

In the same way, if $B_1 < B < B_2$ then we get

$$y \to 0, \quad \frac{z(y, B)}{y} \to \alpha\lambda \text{ as } t \to \omega_-$$

$$y \to 0, \quad \frac{z(y, B)}{y} \to \alpha(\lambda + 1) \text{ as } t \to \omega_+$$

and conclude Theorem 4 from Lemmas 3.4 and 3.5. Here, notice that $z(y, B)$ is not a single-valued function of *y*. Furthermore in the same way, if $B > B_2$ then we have

$$y \to \infty \text{ as } t \to \omega_-,$$

$$y \to 0, \quad \frac{z(y, B)}{y} \to \alpha(\lambda + 1) \text{ as } t \to \omega_+$$

which imply Theorem 5.

For proving Theorem 6, fix *T* and *A* so that $A = \psi(T)$. Then we obtain $y_0 = 1$. If $z_0 = 0$, namely $B = -\lambda A/T$, then (y, z) defined as above satisfies $y \equiv 1, z \equiv 0$, for $(y_0, z_0) = (1, 0)$ is the critical point of (3.3). That is, we get $x(t) = \psi(t)$. If $B < -\lambda A/T$, then we have $z(y, B) > 0$, for z_0 is decreasing in *B*. Therefore from Figure $z = z(y, B)$ is the orbit lying above $z = z_1(y)$, if $0 < y < 1$. Hence the discussion of the case $B < B_1$ follows. Similarly if $B > -\lambda A/T$, then the discussion of the case $B > B_2$ follows. This concludes Theorem 6.

Finally for the proof of Theorems 7 and 8, fix T and A such that $A > \psi(T)$. Then we obtain $y_0 > 1$. As (y_0, z_0), take the intersection of the line $y = y_0$ and the orbit $z = z_3(y)$, and put $B = B_3$. Then if $B = B_3$, (y, z) satisfies

$$y \to 1 \text{ as } t \to \omega_-, \quad y \to \infty \text{ as } t \to \omega_+.$$

Similarly as (y_0, z_0), take the intersection of the line $y = y_0$ and the orbit $z = z_4(y)$, and put $B = B_4$. Then if $B = B_4$, (y, z) satisfies

$$y \to \infty \text{ as } t \to \omega_-, \quad y \to 1 \text{ as } t \to \omega_+.$$

Therefore from the above discussion we conclude Theorem 7.

Furthermore if $B < B_3$ and $B > B_4$, then in $0 < y < 1$, $z(y, B)$ is the orbit lying above $z = z_1(y)$ and below $z = z_2(y)$ respectively. Hence the cases $B < B_3$ and $B > B_4$ are the same as the cases $B < B_1$ and $B > B_2$ respectively. Moreover if $B_3 < B < B_4$, then we get

$$y \to \infty, \quad z(y, B) < 0 \text{ as } t \to \omega_-$$

$$y \to \infty, \quad z(y, B) > 0 \text{ as } t \to \omega_+.$$

Here, notice that $z(y, B)$ is not a single-valued function of y. Therefore as above we conclude Theorem 8.

6. The Initial Value Problem of the Case $T = 0$

In this section we consider (1.1) under the initial condition

$$x(0) = a(> 0), \quad x'(0) = b \quad (-\infty \le b \le \infty) \tag{6.1}$$

instead of (1.2). The solution of the initial value problem (1.1) and (6.1) is as follows:

Corollary 1. *Suppose* $1/\alpha\lambda \in \boldsymbol{N}$. *Then if* $\alpha\lambda = 1$ *and* $b = \infty$, *there exist infinitely many solutions, if* $\alpha\lambda \ne 1$ *and* $b = \infty$, *there exists the unique solution, and if* $b \ne \infty$, *there exists no solution*.

Proof. From theorems of Section 2, only (2.2) and (2.3) are solutions of (1.1) continuable to $t = 0$ and satisfying $x(0) > 0$. Since $1/\alpha\lambda \in \boldsymbol{N}$ now, only (2.2) is required. Substitute (2.2) into (1.1). Then we get

$$\ddot{p}_m(\log t) + (2\alpha\lambda m - 1)\,\dot{p}_m(\log t) + (\alpha\lambda m - 1)\alpha\lambda m p_m(\log t)$$

$$= -a^{\alpha} P_{m-1}(p_k) \quad (m = 1, 2, \ldots) \tag{6.2}$$

where $\cdot = d/d\log t$, $P_m(p_k)$ are polynomials of p_k with $k \le m$, and we adopt a convention $P_0(p_k) = 1$.

If $m = 1$, then from (6.2) we obtain

$$\ddot{p}_1(\log t) + (2\alpha\lambda - 1)\,\dot{p}_1(\log t) + (\alpha\lambda - 1)\alpha\lambda p_1(\log t) = -a^{\alpha}. \tag{6.3}$$

Therefore if $\alpha\lambda = 1$, then since $\deg p_1 \le [\alpha\lambda] = 1$, we get

$$p_1(\log t) = -a^{\alpha}\log t + C$$

where C is an arbitrary constant. If $\alpha\lambda \ne 1$ then since $p_1(\log t)$ is a polynomial of $\log t$, we have from (6.3)

$$p_1(\log t) = -\frac{a^{\alpha}}{(\alpha\lambda - 1)\alpha\lambda}.$$

Next if $m \ge 2$, then from (6.2) we obtain

$$p_m(\log t) = a^{\alpha}\left\{e^{-\alpha\lambda m\log t}\int P_{m-1}(p_k)e^{\alpha\lambda m\log s}d\log s\right.$$

$$\left. - e^{-(\alpha\lambda m-1)\log t}\int P_{m-1}(p_k)e^{(\alpha\lambda m-1)\log s}d\log s\right\},$$

for $p_m(\log t)$ are polynomials of $\log t$. Namely $p_m(\log t)$ are uniquely determined from $p_1(\log t)$.

From the above discussion, if $\alpha\lambda = 1$ then we get infinitely many solutions of (1.1) and (6.1) represented as

$$x(t) = a\left\{1 + t(C - a^{\alpha}\log t) + \sum_{m=2}^{\infty} t^m p_m(\log t)\right\}, \tag{6.4}$$

for (6.4) contains an arbitrary constant C. Moreover we have

$$x'(t) = a\left[C - a^{\alpha}\log t - a^{\alpha} + \sum_{m=2}^{\infty} t^{m-1}\{mp_m(\log t) + \dot{p}_m(\log t)\}\right] \to \infty \text{ as } t \to +0.$$

Therefore the first conclusion of this corollary follows. If $1/\alpha\lambda \in \boldsymbol{N}$, $\alpha\lambda \neq 1$, then we obtain $0 < \alpha\lambda < 1$ and the unique solution of (1.1) and (6.1) represented as

$$x(t) = a\left\{1 - \frac{a^{\alpha}}{(\alpha\lambda - 1)\alpha\lambda}t^{\alpha\lambda} + \sum_{m=2}^{\infty} t^{\alpha\lambda m} p_m(\log t)\right\}. \tag{6.5}$$

From (6.5) we get

$$x'(t) = a\left[-\frac{a^{\alpha}}{\alpha\lambda - 1}t^{\alpha\lambda - 1} + \sum_{m=2}^{\infty} t^{\alpha\lambda m - 1}\{\alpha\lambda m p_m(\log t) + \dot{p}_m(\log t)\}\right] \to \infty \text{ as } t \to +0.$$

Hence the second conclusion follows. Since only (6.4) and (6.5) can become the solutions of (1.1) and (6.1), we get the third conclusion. Now the proof is complete.

The following corollary also states existence and nonexistence of the solution of (1.1) and (6.1):

Corollary 2. *Suppose* $1/\alpha\lambda \notin \boldsymbol{N}$. *If* $0 < \alpha\lambda < 1$, *then there exists a unique solution for* $b = \infty$ *and no solution for* $b \neq \infty$. *Moreover if* $\alpha\lambda > 1$, *then there exists a unique solution for* $b = 0$ *and no solution for* $b \neq 0$.

Proof. Here since $1/\alpha\lambda \notin \boldsymbol{N}$, we need only (2.3). Substituting (2.3) into (1.1), we have (3.15) again. Thus comparing the coefficients of $t^{\alpha\lambda m + n}$, we obtain

$$x_{10} = -\frac{a^{\alpha}}{\alpha\lambda(\alpha\lambda - 1)},$$

$$x_{mn} = -\frac{a^{\alpha}Q_{m-1n}(x_{MN})}{(\alpha\lambda m + n)(\alpha\lambda m + n - 1)} \quad (m \geq 1,\ n \geq 1).$$

This implies that x_{mn} are uniquely determined and the solution of (1.1) and (6.1) exists uniquely, if this exists. Moreover we get

$$x'(t) = a\left\{\alpha\lambda x_{10}t^{\alpha\lambda - 1} + \sum_{m+n>1}(\alpha\lambda m + n)x_{mn}t^{\alpha\lambda m + n - 1}\right\} \sim a\alpha\lambda x_{10}t^{\alpha\lambda - 1} \text{ as } t \to +0.$$

Namely as $t \to +0$ we have

$$x'(t) \to \infty \text{ if } 0 < \alpha\lambda < 1, \quad x'(t) \to 0 \text{ if } \alpha\lambda > 1.$$

Therefore in the case $b = \infty$ and $0 < \alpha\lambda < 1$, and in the case $b = 0$ and $\alpha\lambda > 1$, the solution exists, and in the other cases the solution does not exist. Now the proof is complete.

In the case $0 < T < \infty$, we did not obtain a solution of (1.1) and (1.2) with $B = \pm\infty$ from every orbit of (3.3). Therefore it is not necessary to consider the case $B = \pm\infty$.

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