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OSCILLATION OF SECOND ORDER NEUTRAL DIFFERENCE EQUATIONS WITH DAMPING TERM

E. THANDAPANI*, **K. THANGAVELU and E. CHANDRASEKARAN**

*Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chennai 600 005, India

Department of Mathematics
Presidency College
Chennai 600 005, India

Abstract

In this paper, we establish sufficient conditions for the almost oscillation of all solutions of second order neutral difference equations with damping term via comparison technique. Examples are provided to illustrate the results.

1. Introduction

Consider the second order nonlinear neutral delay difference equations with damping term of the form

$$\triangle(a_n\triangle(x_n + cx_{n-k})) + p_n\phi(\triangle x_n, \triangle x_{n-k}) + q_n f(x_{n+1-l}) = 0, \quad n \geq n_0, \tag{1}$$

and

$$\triangle^2(x_n + cx_{n-k}) + q_n f(x_{n-l}) g(\triangle x_{n-m}) = 0, \quad n \geq n_0, \tag{2}$$

where $\triangle$ is the forward difference operator defined by $\triangle x_n = x_{n+1} - x_n$, $\triangle^2 x_n =$

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$\triangle(\triangle x_n)$, $\{a_n\}$ is a positive sequence, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, k, l and m are nonnegative integers, c is a real number, $\phi : \mathbb{R}^2 \to \mathbb{R}$, f and $g : \mathbb{R} \to \mathbb{R}$ are continuous with f is nondecreasing and $uf(u) > 0$ and $g(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, l\}$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfies equation (1) for all $n \geq n_0$. The solution of equation (2) can be defined similarly. A nontrivial solution $\{x_n\}$ of equation (1) or (2) is said to be *oscillatory* if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. It is said to be *almost oscillatory* if $\{x_n\}$ is oscillatory or $\{\triangle x_n\}$ is oscillatory for all $n \geq n_0$.

The oscillation, nonoscillation and asymptotic behaviors of solutions of equation (1) or (2) when either $c = 0$ and $m = 0$ or $p_n = 0$ have been considered by many authors, see for example [1-8, 10-13, 15], and the references cited therein. Following this trend, in this paper, we establish sufficient conditions for the almost oscillation of all solutions of equations (1) and (2).

The plan of the paper is as follows. In Section 2, we present sufficient conditions for the almost oscillation of equation (1) and in Section 3, we establish similar results for equation (2). Examples are provided in Section 4 to illustrate the results.

2. Almost Oscillation of Equation (1)

In this section, we establish sufficient conditions for the almost oscillation of equation (1) when the function ϕ satisfies anyone of the following conditions:

$$\phi(\triangle x_n, \triangle x_{n-k}) = \triangle x_n, \quad n \geq n_0, \tag{3}$$

or

$$\phi(\triangle x_n, \triangle x_{n-k}) = \triangle x_{n-k}, \quad n \geq n_0, \tag{4}$$

or

$$\phi(\triangle x_n, \triangle x_{n-k}) = \triangle x_n + c\triangle x_{n-k}, \quad n \geq n_0. \tag{5}$$

We begin with the following theorem.

Theorem 1. *With respect to difference equation* (1) *assume condition* (3) *holds*. *Further assume that*

$$0 < c < 1, \tag{6}$$

$$-f(xy) \geq f(xy) \geq f(x) f(y) \ \textit{for} \ xy > 0, \tag{7}$$

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \prod_{s=n_0}^{n-1} \left(1 - \frac{p_s}{a_s}\right) = \infty. \tag{8}$$

If the delay difference equation

$$\triangle(a_n \triangle z_n) + q_n f(1-c) f(z_{n+1-l}) = 0, \quad n \geq n_0 \tag{9}$$

is oscillatory, *then all solutions of equation* (1) *are almost oscillatory*.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1), say $x_n > 0$, $x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$. There are two possibilities to consider: (I) $\triangle x_n > 0$ eventually, and (II) $\triangle x_n < 0$ eventually.

Case (I). Assume that $\triangle x_n > 0$ eventually. Then equation (1) leads to

$$\triangle(a_n \triangle(x_n + cx_{n-k})) + q_n f(x_{n+1-l}) \leq 0. \tag{10}$$

Set

$$z_n = x_n + cx_{n-k}. \tag{11}$$

Then inequality (10) takes the form

$$\triangle(a_n \triangle z_n) + q_n f(x_{n+1-l}) \leq 0 \tag{12}$$

eventually, and clearly $\triangle z_n > 0$ eventually. From (11), we have

$$x_n \geq (1-c) z_n. \tag{13}$$

Using (13) in (12) and then applying condition (7), we obtain

$$\triangle(a_n \triangle z_n) + q_n f(1-c) f(z_{n+1-c}) \leq 0$$

eventually. But in view of a result in [14], it follows from the last inequality that equation (9) has an eventually positive solution, which is a contradiction.

Case (II). Assume that $\triangle x_n < 0$ eventually. Then from equation (1), we have

$$\triangle(a_n \triangle z_n) + p_n \triangle x_n = -q_n f(x_{n+1-l})$$

or

$$\triangle(a_n \triangle z_n) + p_n \triangle x_n < 0, \quad n \geq n_1 \geq n_0 + l. \tag{14}$$

Since $\triangle z_n = \triangle x_n + c\triangle x_{n-k}$, we have $\triangle z_n < \triangle x_n < 0$ and from (14), we obtain

$$\triangle(a_n \triangle z_n) + p_n \triangle z_n < 0, \quad n \geq n_1.$$

Let $u_n = -a_n \triangle z_n$. Then we have

$$\triangle u_n + \frac{p_n}{a_n} u_n \geq 0, \quad n \geq n_1.$$

Summing the last inequality from n_1 to $n-1$, we have

$$u_n \geq u_{n_1} \prod_{s=n_1}^{n-1} \left(1 - \frac{p_s}{a_s}\right)$$

or

$$\triangle z_n \leq -\frac{u_{n_1}}{a_n} \prod_{s=n_1}^{n-1} \left(1 - \frac{p_s}{a_s}\right).$$

Again summing the last inequality from n_1 to $n-1$, we have

$$\triangle z_n \leq \triangle z_{n_1} - u_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \prod_{t=n_1}^{s-1} \left(1 - \frac{p_t}{a_t}\right).$$

However condition (8) leads to $z_n \to -\infty$ as $n \to \infty$, a contradiction. The proof for the case $\{x_n\}$ eventually negative is similar. This completes the proof of the theorem.

Theorem 2. *Let* $c > 1$, k *be a negative integer and conditions* (7) *and* (8) *hold. If the delay difference equation*

$$\triangle(a_n \triangle z_n) + f\left(\frac{c-1}{c^2}\right) q_n f(z_{n+1+k-l}) = 0 \tag{15}$$

is oscillatory, then equation (1) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1). We consider two Cases (I) and (II) as in Theorem 2.

Case (I). Assume that $\triangle x_n > 0$ eventually. Then as in the proof of Theorem 1, we obtain the inequality (12). Since k is negative and $c > 1$, we have from (11) obtained that $x_n \geq \left(\frac{c-1}{c^2}\right) z_{n+k}$. The rest of the proof is similar to that of Case (I) of Theorem 1. The proof of Case (II) is similar to that of Theorem 1. The proof is now complete.

Next, we establish an easily verifiable condition for the almost oscillation of equation (1).

Theorem 3. *Let* $0 < c < 1$, $a_n - p_n > 0$ *for all* $n \geq n_0$, *and condition* (8) *holds. If*

$$\sum_{n=n_0}^{\infty} q_n = \infty, \tag{16}$$

then every solution of equation (1) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1). We consider two Cases (I) and (II) as in Theorem 1.

Case (I). Assume that $\triangle x_n > 0$ eventually. Then as in the proof of Theorem 1, we obtain the inequality

$$\triangle(a_n \triangle z_n) + q_n f((1-c) z_{n+1-l}) \leq 0$$

for all $n \geq n_1$. Since $z_n > 0$ and $\triangle z_n > 0$, there exists a constant $d > 0$ such that $z_{n+1-l} \geq d$ for all $n \geq n_2 \geq n_1 + l$. Hence

$$\triangle(a_n \triangle z_n) + q_n f(d(1-c)) \leq 0, \quad n \geq n_2.$$

Summing the last inequality from n_2 to n, we obtain

$$a_{n+1} \triangle z_{n+1} \leq a_{n_2} \triangle z_{n_2} - f(d(1-c)) \sum_{s=n_2}^{n} q_s.$$

Now, from (16), it follows that $a_n \triangle z_n \to -\infty$ as $n \to \infty$, a contradiction. The proof of Case (II) is similar to that of Case (II) of Theorem 1. The proof is now complete.

Remark 1. In a similar way, we find that Theorems 1-3 are applicable to neutral difference equation (1) when condition (4) is satisfied. In fact, if $\{x_n\}$ is an eventually positive solution of equation (1), we see that there are no changes in the proof when $\triangle x_n > 0$ eventually, while for the case when $\triangle x_n < 0$ eventually, we observe that $\triangle z_n \leq \triangle x_{n-k}$ eventually, and the rest of the proof in this case is the same.

Next, we establish oscillation criteria for the neutral difference equation (1) when condition (5) is satisfied.

Theorem 4. *Let conditions of Theorem* 1 *or* 2 *or* 3 *be satisfied*. *Then every solution of equation* (1) *is almost oscillatory*.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1). Define z_n as in (11), and obtain

$$\triangle(a_n \triangle z_n) + p_n \triangle z_n + q_n f(x_{n+1-l}) = 0.$$

Since $\triangle z_n > 0$ eventually, we have

$$\triangle(a_n \triangle z_n) + q_n f(x_{n+1-l}) \leq 0$$

eventually. The rest of the proof is similar to that of Theorem 1 or 3 when $0 < c < 1$ and Theorem 2 when $c > 1.$ The proof is now complete.

3. Almost Oscillation of Equation (2)

In this section, we consider the neutral difference equation (2) subject to the following conditions:

(i) $g(u)$ is nonincreasing on $\mathbb{R}^+$ and nondecreasing on $\mathbb{R}^-$;

(ii) for any constant $M > 0,$ there exists a nonnegative sequence $\alpha(n)$ such that $-f(-Mnu) \geq f(Mnu) \geq \alpha(Mn) f(u),$ for $u > 0$ and $n \geq n_0$;

(iii) $f(u)g(u) \geq u^{\gamma},$ where γ is a ratio of odd positive integers.

Theorem 5. *If* $0 < c < 1,$ $l \geq m > 0,$ *and there exists a constant* $\theta,$ $0 < \theta < 1$ *such that the delay difference equation*

$$\triangle z_n + \alpha(\theta n) q_n z_{n-m}^{\gamma} = 0 \tag{17}$$

is oscillatory, then every solution of equation (2) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (2). We consider two cases as in Theorem 1.

Case (I). Suppose $\triangle x_n > 0$ eventually. Define z_n as in (11). Then we have

$$\triangle^2 z_n + q_n f(x_{n-l}) g(\triangle x_{n-m}) = 0 \tag{18}$$

and $\triangle z_n > 0$ eventually. As in the proof of Theorem 3, we obtain $x_n \geq (1-c) z_n$ eventually and $\triangle z_n \geq \triangle x_n$ eventually. From (18), we obtain

$$\triangle^2 z_n + q_n f((1-c) z_{n-l}) g(\triangle z_{n-m}) \leq 0 \tag{19}$$

eventually. Since $\triangle^2 z_n \leq 0$, $\triangle z_n > 0$ and $z_n > 0$ eventually, there exists a constant β, $0 < \beta < 1$ and a sufficiently large $n_1 \geq n_0$ such that $z_{n-l} \geq \beta n \triangle z_{n-l}$ for $n \geq n_1$ or $z_{n-l} \geq \beta n \triangle z_{n-m}$ for $n \geq n_1$ since $l \geq m$. Thus

$$\triangle w_n + q_n f(\beta(1-c) n w_{n-m}) g(w_{n-m}) \leq 0,$$

where $w_n = \triangle z_n$, and hence we find

$$\triangle w_n + q_n \alpha(\theta n) f(w_{n-m}) g(w_{n-m}) \leq 0$$

or

$$\triangle w_n + q_n \alpha(\theta n) w_{n-m}^{\gamma} \leq 0, \quad n \geq n_1,$$

where $\theta = \beta(1-c)$. But, in view of Lemma 1 of [9], we see that from the last inequality that the equation

$$\triangle w_n + q_n \alpha(\theta n) w_{n-m}^{\gamma} = 0,$$

has an eventually positive solution, which is a contradiction.

Case (II). Suppose $\triangle x_n < 0$ eventually. Then $\triangle z_n < 0$, which contradicts $\triangle z_n > 0$ eventually. This completes the proof of the theorem.

Next, we assume $c > 1$ and k is a negative integer in equation (2). Then we have the following result.

Theorem 6. *If* $c > 1$, k *is a negative integer with* $l - k \geq m$, *and for every constant* $\theta > 0$, *equation* (18) *is oscillatory, then every solution of equation* (2) *is almost oscillatory*.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (2). Proceeding as in Theorem 5, we obtain (18) and $\triangle z_n > 0$ eventually, and conclude that Case (II), that is, $\triangle z_n < 0$ eventually is impossible. Next, from (11), we find $x_n \geq \left(\frac{c-1}{c^2}\right)\triangle z_{n+k}$ and $\triangle z_n \geq \triangle x_n$ eventually. The rest of the proof is similar to that of Theorem 5 and hence the details are omitted.

Remark 2. For the oscillatory behavior of equation (17) one can refer [1, 9], and the references cited therein.

4. Examples

In this section, we present some examples to illustrate the results

Example 1. Consider the neutral difference equation

$$\triangle^2\left(x_n + \frac{1}{2}x_{n-1}\right) + \frac{1}{n}x_n + \frac{2}{(n+1)^2(n+3)}x_{n+1} = 0, \quad n \geq 2. \tag{20}$$

It is easy to check that all the hypotheses of Theorem 1 (Theorem 3) are satisfied except (condition (16)) that on the oscillatory behavior of the equation

$$\triangle^2 z_n + \frac{2}{(n+1)^2(n+3)}z_{n+1} = 0, \quad n \geq 2. \tag{21}$$

Equation (21) has a nonoscillatory solution $\{x_n\} = \left\{\frac{n}{n+1}\right\}$.

Example 2. The neutral difference equation

$$\triangle^2(x_n + 2x_{n+1}) + \frac{2}{n+2}\triangle x_n + \frac{4}{n(n+2)(n+3)}x_{n+1} = 0, \quad n \geq 1 \tag{22}$$

has a nonoscillatory solution $\{x_n\} = \left\{\frac{1}{n}\right\}$. All conditions of Theorem 2 are satisfied except that the oscillatory behavior of the equation

$$\triangle^2 z_n + \frac{1}{n(n+2)(n+3)} z_n = 0, \quad n \geq 1. \tag{23}$$

Example 3. Consider the neutral difference equation

$$\triangle^2\left(x_n + \frac{1}{2}x_{n-k}\right) + q_n x_{n-l} \exp(x_{n-l}^2 - (\triangle x_{n-l})^2) = 0, \quad n \geq n_0, \tag{24}$$

where k and l are nonnegative integers and $\{q_n\}$ is a nonnegative real sequence for all $n \geq n_0$. Here we take $f(u) = ue^{u^2}$ and $g(u) = e^{-u^2}$. Now, for every θ, $0 < \theta < 1$ and all large $n > \frac{1}{\theta}$, we have

$$f(n\theta u) = \theta nue^{\theta^2 n^2 u^2} \geq \theta nue^{u^2}, \quad \alpha(\theta n) = \theta n \tag{25}$$

and $f(u)g(u) = u$. Thus all the conditions of Theorem 4 are satisfied if the equation

$$\triangle z_n + \theta n q_n z_{n-l} = 0 \tag{26}$$

is oscillatory, that is, if

$$\liminf_{n\to\infty} \sum_{s=n-l}^{n-1} sq_s > \frac{1}{\theta}\left(\frac{l}{l+1}\right)^{l+1} \tag{27}$$

(see [1]), and hence we conclude that all solutions of equation (22) are almost oscillatory.

Example 4. Consider the neutral difference equation

$$\triangle^2(x_n + cx_{n-k}) + q_n x_{n-l-k}^{\gamma}\left(\frac{1 + x_{n-l-k}^2}{1 + (\triangle x_{n-l})^2}\right) = 0, \tag{28}$$

where $0 < c < 1$, k, l are nonnegative integers, γ is a ratio of odd positive integers, and $\{q_n\}$ is a nonnegative real sequence. Here we take $f(u) = u^{\gamma}(1 + u^2)$ and $g(u) = \frac{1}{1 + u^2}$. Now for every constant $M > 0$ and all large $n > \frac{1}{M}$, we observe that

$$f(Mnu) \geq (Mn)^{\gamma} u^{\gamma}(1 + u^2), \quad \alpha(Mn) = (Mn)^{\gamma} \tag{29}$$

and hence $f(u)g(u) = u^{\gamma}$. It is easy to check that all conditions of Theorem 4 are satisfied provided the equation

$$\triangle z_n + (Mn)^{\gamma} q_n z_{n-l}^{\gamma} = 0 \tag{30}$$

is oscillatory, then we can conclude that all solutions of equation (28) are almost oscillatory. Clearly, equation (30) is oscillatory if

$$\sum_{n=n_0}^{\infty} n^{\gamma} q_n = \infty, \quad 0 < \gamma < 1$$

or

$$\liminf_{n\to\infty} \sum_{s=n-l}^{n-1} sq_s > \frac{1}{M}\left(\frac{l}{l+1}\right)^{l+1}, \quad \gamma = 1,$$

or there exists a $\lambda > \frac{1}{l}\log\gamma$ such that

$$\liminf_{n\to\infty} n^{\gamma} q_n \exp(-e^{\lambda n}) > 0, \quad \gamma > 1.$$

Remark 3. (1) If we let $c = 0$ in Theorem 3, one can easily prove that all solutions of equation (1) are oscillatory (see [10]). Therefore, we conclude that the disruption in the oscillatory property is due to the presence of neutral term.

(2) It would be interesting to obtain results similar to those presented here for the complete oscillation of equations (1) and (2).

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