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AN INTERMEDIATE VALUE THEOREM FOR SEQUENCES WITH TERMS IN A FINITE SET

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Abstract

We prove an intermediate value theorem of an arithmetical flavor involving the consecutive averages $\{\bar{x}_n\}_{n\geq 1}$ of sequences with terms in a given finite set $\{a_1, ..., a_r\}$. For every such set we completely characterize the numbers Π with the property that the consecutive averages $\{\bar{x}_n\}$ of every sequence $\{x_n\}_{n\geq 1}$ with terms in $\{a_1, ..., a_r\}$ cannot increase from a value $\bar{x}_k < \Pi$ to a value $\bar{x}_l > \Pi$ without taking the value $\bar{x}_s = \Pi$ for some s with $k < s < l$.

1. Introduction

Let $r \geq 1$ be an integer and let $a_1, a_2, ..., a_r$ be real numbers with

$$a_1 < a_2 < \cdots < a_r.$$

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Define

$$SEQ(a_1, \ldots, a_r)$$

to be the set of all sequences $\{x_n\}_{n\geq 1}$ such that $x_n \in \{a_1, a_2, \ldots, a_r\}$ for all $n \geq 1$. For example, $SEQ(0, 1)$ is the set of all binary sequences.

To each sequence $\{x_n\}_{n\geq 1}$ we associate the sequence of consecutive averages $\{\overline{x}_n\}_{n\geq 1}$ defined by

$$\overline{x}_n = \frac{x_1 + \cdots + x_n}{n}.$$

Clearly, if $\{x_n\}_{n\geq 1} \in SEQ(a_1, \ldots, a_r)$, then

$$a_1 \leq \overline{x}_n \leq a_r$$

for all $n \geq 1$.

We are now in a position to define the sets which will be studied in the current article.

Definition. For $a_1 < a_2 < \cdots < a_r$ let us define

$$IV(a_1, \ldots, a_r)$$

to be the set of all numbers $\prod \in (a_1, a_r)$ with the following "intermediate value property": *if* $\{x_n\}_{n\geq 1} \in SEQ(a_1, \ldots, a_r)$ *and if* $\overline{x}_k < \prod < \overline{x}_l$ *for some integers* $k < l$, *then there exists an integer* s *with* $k < s < l$ *such that* $\overline{x}_s = \prod$.

A Putnam Exam problem [1] asks whether $\frac{4}{5}$ is in $IV(0, 1)$. Indeed the answer turns out to be affirmative. More generally, we can make the following statement:

Theorem 1. $IV(0, 1) = \left\{\frac{k}{k+1} : k \geq 1\right\} \subset (0, 1)$.

In the present paper we will fully generalize the above Theorem 1, providing a complete description of all "sets of intermediate values" $IV(a_1, \ldots, a_r)$. In particular we will determine necessary and sufficient conditions under which $IV(a_1, \ldots, a_r) \neq \emptyset$.

Note. By definition, the numbers $\prod \in IV(a_1, ..., a_r)$ are precisely those which cannot be "skipped" or "jumped over" by *increasing* averages. In the last section we will discuss the case of intermediate values which cannot be skipped by *decreasing* averages. This being said, in the next three sections, the term "skipped" will signify "skipped" by averages going up (e.g., $\prod = 0.7$ being skipped at the step between the third and the fourth averages of the sequence $0, 1, 1, 1, ...$).

2. Case of Binary Sequences

To prove Theorem 1, we first show that

$$IV(0, 1) \subseteq \left\{\frac{k}{k+1} : k \geq 1\right\}.$$

Indeed, if

$$\prod \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right) \cup \cdots,$$

then the consecutive averages of the sequence

$$0, 1, 1, 1, 1, ...$$

will skip $\prod$.

We now prove the reverse inclusion, namely,

$$\left\{\frac{k}{k+1} : k \geq 1\right\} \subseteq IV(0, 1).$$

That is, we prove that if $\prod = \frac{k}{k+1}$, $k \geq 1$, then $\prod$ cannot be skipped by a sequence of consecutive averages $\{\bar{x}_n\}_{n \geq 1}$. We will proceed by contradiction. Assuming $\{\bar{x}_n\}_{n \geq 1}$ skips $\prod = \frac{k}{k+1}$ it follows that

$$\bar{x}_n < \frac{k}{k+1} < \bar{x}_{n+1} \tag{1}$$

for some $n \geq 1$. First note that x_{n+1} must be 1, because if $x_{n+1} = 0$, then the average $\bar{x}_{n+1}$ cannot be larger than $\bar{x}_n$. Thus, if we denote $S = x_1 + \cdots + x_n$, then (1) can be rewritten as follows:

$$\frac{S}{n} < \frac{k}{k+1} < \frac{S+1}{n+1}. \tag{2}$$

By cross-multiplication (2) is equivalent with the system of the following two inequalities:

$$(k+1)S < nk, \tag{3}$$

and

$$(n+1)k < (k+1)(S+1). \tag{4}$$

From (3) and (4) it follows that

$$nk - 1 < (k+1)S < nk,$$

which is impossible, as all three terms are integers, and as there can be no integer falling between consecutive integers. This concludes the proof of Theorem 1.

From Theorem 1, a simple linearity argument leads us to the following result.

Theorem 2. *If* $a < b$, *then*

$$IV(a, b) = \left\{\frac{1}{k+1}a + \frac{k}{k+1}b : k \geq 1\right\}.$$

In the next two sections we will consider sequences with terms in a set with three elements.

3. Case of Ternary Sequences

Let $0 < \mu < 1$. In order to find $IV(0, \mu, 1)$ we distinguish between the case of an irrational μ and the case of a rational μ. The easier case is the case of an irrational μ. Then there are no intermediate values for the consecutive averages of sequences with terms in the set $\{0, \mu, 1\}$. In other words:

Theorem 3. *If* $0 < \mu < 1$ *is irrational*, *then* $IV(0, \mu, 1) = \varnothing$.

Proof. We already know that every $\prod \in (0, 1)$ that is *not* of the form $\frac{k}{k+1}$ can be skipped by the averages of some sequence in $SEQ(0, 1) \subset SEQ(0, \mu, 1)$. It will be enough to show that if μ is irrational, then every

$\prod \in (0, 1)$ that is of the form $\frac{k}{k+1}$ can be skipped by the averages of some sequence in $SEQ(0, \mu, 1)$. Indeed, every $\prod = \frac{k}{k+1} < \mu$ will be skipped by the consecutive averages of the sequence

$$0, \mu, \mu, \mu, \ldots$$

(the averages form an increasing sequence of irrationals with limit μ), while every $\prod = \frac{k}{k+1} > \mu$ will be skipped by the consecutive averages of the sequence

$$\mu, 1, 1, 1, \ldots$$

(here, the averages form an increasing sequence of irrationals with limit 1). This concludes the proof of Theorem 3.

Next, we consider the case $0 < \mu = \frac{p}{q} < 1$ with p, q relatively prime positive integers. Again, every $\prod \in (0, 1)$ that is *not* of the form $\frac{k}{k+1}$ can be skipped, so the question we have to answer is which numbers of the form $\prod = \frac{k}{k+1}$ can be skipped by the consecutive averages of some sequence in $SEQ\left(0, \frac{p}{q}, 1\right)$.

First note that the sequence

$$x_1 = \frac{p}{q},\ x_2 = x_3 = \cdots = 1$$

has consecutive averages of the form

$$\frac{\frac{p}{q} + l}{l+1} = \frac{p+ql}{q+ql}, \tag{5}$$

where $l = 0, 1, 2, \ldots$. Then every $\prod = \frac{k}{k+1} > \frac{p}{q}$ that is not of the form $\frac{p+ql}{q+ql}$ can be skipped by the sequence of increasing averages (5), so it cannot be in $IV\left(0, \frac{p}{q}, 1\right)$. We will have to determine when a fraction $\frac{p+ql}{q+ql}$ is of the form $\frac{k}{k+1}$. The equality

$$\frac{p+ql}{q+ql} = \frac{k}{k+1}$$

can be rewritten in the following equivalent form:

$$p(k+1) = q(k-l). \tag{6}$$

From (6), keeping in mind that p, q are relatively prime, we get

$$k+1 = qt,$$

and

$$k - l = pt$$

for some integer t. In this case, $\frac{k}{k+1}$ is of the form $\frac{qt-1}{qt} = 1 - \frac{1}{qt}$. As a consequence, if a $\prod = \frac{k}{k+1} > \frac{p}{q}$ is in the set of intermediate values $IV\left(0, \frac{p}{q}, 1\right)$, then q divides $k+1$.

We will now prove that there is no $\prod$ in $IV\left(0, \frac{p}{q}, 1\right)$ that is of the form $\frac{k}{k+1}$ and is less than $\frac{p}{q}$. First, notice that if $\frac{p}{q} < \frac{1}{2}$, then there is nothing to prove $\left(\text{in } (0, 1) \text{ there is no number of the form } \frac{k}{k+1} \text{ that is less than } \frac{1}{2}\right)$, so we may assume $\frac{p}{q} > \frac{1}{2}$.

Consider the sequence

$$x_1 = 0,\ x_2 = x_3 = \cdots = \frac{p}{q}. \tag{7}$$

The averages of the sequence (7) are increasing, they approach $\frac{p}{q}$ and are of the form

$$\frac{pl}{q(l+1)}, \quad l \geq 0.$$

If $\prod = \frac{k}{k+1} < \frac{p}{q}$ is not of the form $\frac{pl}{q(l+1)}$, then $\prod$ will be skipped by the increasing averages of the sequence (7). Thus we only need to consider the case in which

$$\frac{k}{k+1} = \frac{pl}{q(l+1)},$$

that is,

$$\frac{p}{q} = \frac{k(l+1)}{l(k+1)}. \tag{8}$$

Note that in (8) we have $k < l$, since $\frac{p}{q} < 1$. We will prove that under the above conditions $\frac{k}{k+1}$ can be skipped by the successive averages of a sequence $\{x_i\}_{i \geq 1}$ in $SEQ\left(0, \frac{p}{q}, 1\right)$. Assume that the n-th average is the average of u zeros, $v\,\frac{p}{q}$'s and w ones (in particular, $u + v + w = n$). Without loss of generality we assume that $x_{n+1} = 1$. Keeping in mind (8) we need to show that there exist positive integers u, v, w such that

$$\frac{v\frac{k(l+1)}{l(k+1)} + w}{n} < \frac{k}{k+1} < \frac{v\frac{k(l+1)}{l(k+1)} + w + 1}{n+1}. \tag{9}$$

Note that (9) can be rewritten as follows:

$$\frac{vk(l+1) + wl(k+1)}{n} < kl < \frac{vk(l+1) + wl(k+1) + l(k+1)}{n+1}. \tag{10}$$

Using $n = u + v + w$ it turns out that (10) is equivalent to

$$0 < ukl - vk - wl < l. \tag{11}$$

Since the sufficiently large elements in the additive semigroup generated by k and l consist of all multiples of the $\gcd(k, l) \leq k < l$ larger than a certain value [2, p. 219, Problem 16], it turns out that it is possible to pick up positive integers u, v, w such that (11) holds true.

Up to this point we know that every $\prod \in (0, 1)$ that is *not* of the form $1 - \dfrac{1}{qt}$ can be skipped by the consecutive averages of some sequence in $SEQ\left(0, \dfrac{p}{q}, 1\right)$, in other words,

$$IV\left(0, \frac{p}{q}, 1\right) \subseteq \left\{1 - \frac{1}{qt} : t \geq 1\right\}. \tag{12}$$

The reverse inclusion

$$\left\{1 - \frac{1}{qt} : t \geq 1\right\} \subseteq IV\left(0, \frac{p}{q}, 1\right) \tag{13}$$

will be proved by contradiction. Assume that $\dfrac{qt - 1}{qt}$ can be skipped by the consecutive averages of some sequence in $SEQ\left(0, \dfrac{p}{q}, 1\right)$. Without loss of generality we may assume that $\dfrac{qt - 1}{qt}$ is in between the average

$$\overline{x}_n = \frac{x_1 + \cdots + x_n}{n}$$

with u of the $x_1, \ldots, x_n$ being zeros, v being $\dfrac{p}{q}$ and w being ones $(u + v + w = n)$ and the average

$$\overline{x}_{n+1} = \frac{x_1 + \cdots + x_n + 1}{n + 1}$$

with u of the $x_1, \ldots, x_n$ being zeros, v being $\dfrac{p}{q}$ and $w + 1$ being ones $(x_{n+1} = 1)$:

$$\frac{v\dfrac{p}{q} + w}{n} < \frac{qt - 1}{qt} < \frac{v\dfrac{p}{q} + w + 1}{n + 1}. \tag{14}$$

Equivalently, (14) can be rewritten as follows:

$$\frac{pv + qw}{n} < \frac{qt - 1}{t} < \frac{pv + qw}{n + 1}, \tag{15}$$

which is equivalent with the system consisting of the following two inequalities:

$$pvt < qut + qvt - n, \tag{16}$$

and

$$qut + qvt - n - 1 < pvt. \tag{17}$$

From (16) and (17) it follows

$$pvt < qut + qvt - n < pvt + 1,$$

which is again a contradiction (an integer in between two consecutive integers). This concludes the proof of the reverse inclusion (13). From (12) and (13) we get

Theorem 4. *If* $0 < \frac{p}{q} < 1$ *with* p, q *relatively prime positive integers, then*

$$IV\left(0, \frac{p}{q}, 1\right) = \left\{1 - \frac{1}{qt} : t = 1, 2, 3, \ldots\right\}.$$

A straightforward linearity argument based on the previous two theorems leads us to the following intermediate value theorem characterizing all sets $IV(a, b, c)$.

Theorem 5. *Let* $a < b < c$ *and let* $\mu := \frac{b-a}{c-a}$. *If* μ *is irrational, then*

$$IV(a, b, c) = \varnothing.$$

If $\mu = \frac{p}{q}$ *with* p, q *relatively prime positive integers, then*

$$IV(a, b, c) = \left\{\left(1 - \frac{1}{qt}\right)c + \frac{1}{qt}a : t = 1, 2, 3, \ldots\right\}.$$

4. The General Intermediate Value Theorem

We will now completely characterize the intermediate value sets of the form $IV(0, \mu_1, \ldots, \mu_r, 1)$, where $0 < \mu_1 < \cdots < \mu_r < 1$. First, notice that we can formulate right away the following result.

Theorem 6. *If* μ_i *is irrational for some* $i = 1, 2, \ldots, r,$ *then*

$$IV(0, \mu_1, \ldots, \mu_r, 1) = \varnothing.$$

Proof. Follows from Theorem 3, if μ_i is rational, then every $\prod \in (0, 1)$ can be skipped by the averages of some sequence in

$$SEQ(0, \mu_i, 1) \subset SEQ(0, \mu_1, \ldots, \mu_r, 1).$$

Now let us assume that all μ_i's are rational:

$$\mu_i = \frac{p_i}{q_i}, \; i = 1, \ldots, r,$$

with $\gcd(p_i, q_i) = 1$ for all $i = 1, \ldots, r.$

Let M be the least common multiple of the denominators of the reduced fractions $\frac{p_i}{q_i}$, $i = 1, \ldots, r.$ Then we will prove that the following result holds true.

Theorem 7. *With the above notations, we have*

$$IV(0, \mu_1, \ldots, \mu_r, 1) = \left\{1 - \frac{1}{Mt} : t = 1, 2, 3, \ldots\right\}. \tag{18}$$

Proof. Let $i \in \{1, 2, \ldots, r\}$. Then from Theorem 4 it follows that for $A \geq 2$, the element

$$\prod = 1 - \frac{1}{A}$$

will be skipped by the averages of some sequence in $SEQ\left(0, \frac{p_i}{q_i}, 1\right) \subset$ $SEQ(0, \mu_1, \ldots, \mu_r, 1)$ as long as q_i does not divide A. Thus, if $\prod = 1 - \frac{1}{A}$ cannot be skipped by the averages of the sequences in $SEQ(0, \mu_1, \ldots, \mu_r, 1)$, then $q_1 \mid A, q_2 \mid A, \ldots, q_r \mid A$, that is, $M \mid A$, or

$$\prod = 1 - \frac{1}{Mt}$$

for some $t \geq 1$. Thus we have proved that

$$IV(0, \mu_1, \ldots, \mu_r, 1) \subseteq \left\{1 - \frac{1}{Mt} : t = 1, 2, 3, \ldots\right\}. \tag{19}$$

To complete the proof we will prove the reverse inclusion. In other words, we will show that if $\prod \in (0, 1)$ is of the form $\frac{Mt-1}{Mt}$, then $\prod$ cannot be skipped by the consecutive averages of any sequence in $SEQ(0, \mu_1, \ldots, \mu_r, 1)$. We proceed by contradiction.

Assume that $\frac{Mt-1}{Mt}$ can be skipped by the consecutive averages of some sequence in $SEQ(0, \mu_1, \ldots, \mu_r, 1)$. Without loss of generality we may assume that $\frac{Mt-1}{Mt}$ is in between the average

$$\bar{x}_n = \frac{x_1 + \cdots + x_n}{n}$$

with u of the $x_1, \ldots, x_n$ being zeros, v_1 being $\frac{p_1}{q_1}$, v_2 being $\frac{p_2}{q_2}, \ldots, v_r$ being $\frac{p_r}{q_r}$ and w being ones $(u + v_1 + \cdots + v_r + w = n)$, and the average

$$\bar{x}_{n+1} = \frac{x_1 + \cdots + x_n + 1}{n+1}$$

with u of the $x_1, \ldots, x_{n+1}$ being zeros, v_1 being $\frac{p_1}{q_1}$, v_2 being $\frac{p_2}{q_2}, \ldots, v_r$ being $\frac{p_r}{q_r}$ and $w+1$ being ones (we took $x_{n+1} = 1$ which leads to the greatest possible increase in the average):

$$\frac{v_1 \frac{p_1}{q_1} + \cdots + v_r \frac{p_r}{q_r} + w}{n} < \frac{Mt-1}{Mt} < \frac{v_1 \frac{p_1}{q_1} + \cdots + v_r \frac{p_r}{q_r} + w + 1}{n+1}. \tag{20}$$

For every $i = 1, \ldots, r$ let us define

$$Q_i := \frac{\operatorname{lcm}(q_1, \ldots, q_r)}{q_i} = \frac{M}{q_i}.$$

With this notation, a multiplication of all terms in (20) by M gives

$$\frac{p_1Q_1v_1 + \cdots + p_rQ_rv_r + Mw}{n} < \frac{Mt-1}{t}$$
$$< \frac{p_1Q_1v_1 + \cdots + p_rQ_rv_r + Mw + M}{n+1},$$

which, by cross-multiplication turns out to be equivalent to the following system of inequalities:

$$p_1Q_1v_1t + \cdots + p_rQ_rv_rt + Mwt < Mnt - n, \tag{21}$$

and

$$Mnt + Mt - n - 1 < p_1Q_1v_1t + \cdots + p_rQ_rv_rt + Mwt + Mt. \tag{22}$$

Finally, from (21) and (22) it follows that

$$p_1Q_1v_1t + \cdots + p_rQ_rv_rt + Mwt < Mnt - n$$
$$< p_1Q_1v_1t + \cdots + p_rQ_rv_rt + Mwt + 1$$

which is, again, a contradiction (there is no integer in between two consecutive integers). This shows that (19) is true. From (18) and (19), Theorem 7 follows.

From the above result, a linearity argument leads us to the following arithmetic intermediate value theorem.

Theorem 8. *Let* $a_1 < a_2 < \cdots < a_r$ $(r \geq 3)$. *For* $i = 2, \ldots, r-1$, *let*

$$\mu_i := \frac{a_i - a_1}{a_r - a_1}.$$

Then the following hold true:

(a) *If for some* $i = 2, \ldots, r-1$ *the number* μ_i *is irrational, then* $IV(a_1, \ldots, a_r) = \varnothing$.

(b) *If* $\mu_2, \ldots, \mu_{r-1}$ *are all rational numbers,* $\mu_i = \frac{p_i}{q_i}$, *with* $\gcd(p_i, q_i) = 1$ *for* $i = 2, \ldots, r-1$ *and* $M = \operatorname{lcm}(q_2, \ldots, q_{r-1})$, *then*

$$IV(a_1, \ldots, a_r) = \left\{\frac{1}{Mt}a_1 + \left(1 - \frac{1}{Mt}\right)a_r : t = 1, 2, 3, \ldots\right\}.$$

5. Further Comments

Note that for $a_1 < a_2 < \cdots < a_r$ the sets $IV(a_1, \ldots, a_r)$ represent the values $\prod$ with the property that the consecutive averages $\{\bar{x}_n\}$ of every sequence $\{x_n\}_{n\geq 1} \in SEQ(a_1, \ldots, a_r)$ cannot *increase* from a value $\bar{x}_k < \prod$ to a value $\bar{x}_l > \prod$ without taking the value $\bar{x}_s = \prod$ for some s with $k < s < l$. Similarly we can define the sets

$$DV(a_1, \ldots, a_r)$$

representing the values $\prod$ with the property that the consecutive averages $\{\bar{x}_n\}$ of every sequence $\{x_n\}_{n\geq 1} \in SEQ(a_1, \ldots, a_r)$ cannot *decrease* from a value $\bar{x}_k > \prod$ to a value $\bar{x}_l < \prod$ without taking the value $\bar{x}_s = \prod$ for some s with $k < s < l$.

The connection between the sets $IV(a_1, \ldots, a_r)$ and $DV(a_1, \ldots, a_r)$ can be expressed in a simple way as follows:

$$DV(a_1, \ldots, a_r) = -IV(-a_r, \ldots, -a_1). \tag{23}$$

The proof of (23) is straightforward if we use the transformation

$$\{x_n\}_{n\geq 1} \mapsto \{-x_n\}_{n\geq 1}. \tag{24}$$

Clearly, (24) is a one-to-one correspondence between $SEQ(a_1, \ldots, a_r)$ and $SEQ(-a_r, \ldots, -a_1)$. Under this correspondence, the sequence of averages of $\{x_n\}_{n\geq 1}$ skips (going up) $\prod \in (a_1, a_r)$, if and only if the sequence of averages of $\{-x_n\}_{n\geq 1}$ skips (going down) $-\prod \in (-a_r, -a_1)$.

We can use (24) to translate Theorems 2 and 8 for decreasing trends. Thus, we obtain the following two results.

Theorem 9. *If* $a < b$, *then*

$$DV(a, b) = \left\{\frac{k}{k+1}a + \frac{1}{k+1}b : k \geq 1\right\}.$$

Theorem 10. *Let* $a_1 < a_2 < \cdots < a_r$ $(r \geq 3)$. *For* $i = 2, \ldots, r-1$, *let*

$$\mu_i := \frac{a_i - a_1}{a_r - a_1}.$$

Then the following hold true:

(a) *If for some* $i = 2, \ldots, r-1$ *the number* μ_i *is irrational*, *then* $DV(a_1, \ldots, a_r) = \varnothing$.

(b) *If* $\mu_2, \ldots, \mu_{r-1}$ *are all rational numbers*, $\mu_i = \dfrac{p_i}{q_i}$, *with* $\gcd(p_i, q_i) = 1$ *for* $i = 2, \ldots, r-1$ *and* $M = \operatorname{lcm}(q_2, \ldots, q_{r-1})$, *then*

$$DV(a_1, \ldots, a_r) = \left\{\left(1 - \frac{1}{Mt}\right)a_1 + \frac{1}{Mt}a_r : t = 1, 2, 3, \ldots\right\}.$$

References

[1] 2004 Putnam Exam, Problem A1.

[2] I. Niven, H. S. Zuckerman and H. L. Montgomery, An Introduction to the Theory of Numbers, 5th ed., Wiley, 1991.

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