Far East Journal of Theoretical Statistics
Volume 28, Number 1, 2009, Pages 107-116
Published Online: June 4, 2009
This paper is available online at http://www.pphmj.com

SOME SIMULATION RESULTS OF WEIBULL AND MIXED WEIBULL DISTRIBUTIONS

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Abstract

The Weibull variate is commonly used as a lifetime distribution in reliability applications. The two-parameter (shape and scale) Weibull distribution can represent a decreasing, constant or increasing failure rate. Further, flexibility can be introduced into the Weibull distribution by adding the third parameter which is a location parameter. In this paper, we discuss the idea of Weibull distribution which is one of the continuous distributions with two and three parameters. We also try to explain the idea of a mixed Weibull distribution which can be generated from two or more Weibull distributions. This idea has been suggested and used recently in statistical distributions as it gives a number of parameters more than Weibull distribution. We also mention the idea of how to combine two or more Weibull distributions to obtain a mixed Weibull distribution using a mixing parameter. This distribution is more flexible in reliability studies. We have several versions of mixed Weibull distributions that are used in the field of statistics and engineering.

2000 Mathematics Subject Classification: 60E05.

Keywords and phrases: two-parameter Weibull distribution, three-parameter Weibull distribution, mixed Weibull distribution.

Received January 13, 2009

Introduction

The Weibull distribution is one of the widely used distributions in reliability. This distribution has two or three parameters; scale (characteristic life), shape (power) and location parameters. Scale and shape parameters must be greater than zero, while location parameter can be equal to zero. From Weibull distribution, we can find mixed Weibull distributions. These distributions are the result of combining two or more Weibull distributions. To get a mixed Weibull distribution, a mixing parameter is used. This parameter must be greater than zero and less than one.

The standard Weibull distribution is when the shape parameter equals one. Other distributions, like exponential and Rayleigh, can be found from Weibull distribution. It is also related to the standard extreme value variate. Evan et al. [3].

Weibull Distribution

The two-parameter Weibull distribution is

$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \quad x \geq 0 \tag{1}$$

with $\alpha > 0$, $\beta > 0$. The parameters α and β are the scale and shape parameters of the distributions. This distribution is named after Waloddi Weibull who was the first to promote the usefulness of this distribution to model data sets of widely differing character.

The initial study by Weibull [8] appeared in a Scandinavian journal and dealt with the strength of materials. A similar model was proposed earlier by Rosen and Rammler [7] in the context of modeling, the variability in the diameter of powder particles being greater than a specific size.

Gumbel [4] refers to the Weibull distribution as the third asymptotic distribution of the smallest extremes. A report by Weibull [9] lists over 1000 references to the applications of the basic Weibull distribution.

The two-parameter Weibull distribution can be written in alternate parametric form as indicated below, Murthy et al. [5];

$$F(x) = 1 - e^{-(\lambda x)^{\beta}} \quad \text{with } \lambda = 1/\alpha \tag{2}$$

or

$$F(x) = 1 - e^{-\frac{x^{\beta}}{\alpha^{\beta}}} \tag{3}$$

or

$$F(x) = 1 - e^{-\lambda' x^{\beta}} \quad \text{with} \quad \lambda' = \left(\frac{1}{\alpha}\right)^{\beta}. \tag{4}$$

Further flexibility can be introduced by adding a third parameter which is a location parameter and is usually denoted by the symbol γ. The probability density function is zero for $x < \gamma$ and then follows a Weibull distribution with origin at γ. In reliability applications, gamma is often referred to as the minimum life, but this does not guarantee that no failures will occur below this value in the future.

The three-parameter Weibull distribution is given by the distribution function;

$$F(x) = 1 - e^{-\left[\frac{(x-\gamma)}{\alpha}\right]^{\beta}}, \quad x \geq \gamma. \tag{5}$$

The three parameters $\alpha > 0$, $\beta > 0$ and $\gamma \geq 0$ are the scale, shape and location parameters of the distribution, respectively.

The probability density functions of the two- and three-parameter Weibull distributions are as follows, Al-Fawzan [1] and Razali and Al-Wakil [6];

$$f(x) = \left(\beta \frac{x^{\beta-1}}{\alpha^{\beta}}\right) e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \quad x \geq 0 \tag{6}$$

and

$$f(x) = \left[\frac{\beta(x-\gamma)^{\beta-1}}{\alpha^{\beta}}\right] e^{-\left[\frac{(x-\gamma)}{\alpha}\right]^{\beta}}, \quad x \geq 0,\ \gamma \geq 0. \tag{7}$$

The negative exponential, two-parameter Weibull, three-parameter Weibull and bi-Weibull distributions form a family of distributions of gradually increasing complexity. The two-parameter Weibull distribution extends the range of models to include (one of) decreasing, constant or increasing hazard function. The bi-Weibull distribution allows (two of) decreasing, constant and increasing hazard function.

To estimate the parameter of the Weibull distribution, there are a number of methods that can be classified in two categories; graphical and analytical methods. The graphical methods can be used because of their simplicity and speed, however, they involve a great probability of error. These methods include Weibull probability plotting and hazard plotting technique. Analytical methods are motivated by the availability of high speed computers. They include maximum likelihood estimator (MLE), moments, least squares, Bayesian and others.

Mixed Weibull Distribution

Mixed Weibull distribution or mixture Weibull distribution is one of the new studies which have been used a lot recently in statistical research and its applications are very common in reliability studies.

A mixed Weibull distribution has five or more parameters. These are the shape, scale and location parameters. This type of distribution is even more useful because multiple causes of failure can be simultaneously modeled. The Weibull mixture model has been referred to by many other names such as additive-mixed Weibull distribution, bimodal-mixed Weibull, mixed-mode Weibull distribution, Weibull distribution of the mixed type, multi-modal Weibull distribution, and so forth.

In general, a mixture distribution is a distribution made of combining two or more component distributions. The probability density function of this mixture distribution can be shown to be

$$f(x) = w_1 f_1(x) + \cdots + w_n f_n(x),$$

where

$$\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad w_i > 0. \tag{8}$$

In equation (8), w_i is the mixing parameter which represents the proportion of mixing of the component distributions. While $f_i(x)$ are the probability density functions of the component distribution *i*. *n* is the number of component distributions being mixed.

The probability density function of the mixture Weibull distribution of the two distributions in (6) and (7) is as follows:

$$f(x) = w(\beta_1 x^{\beta_1 - 1}/\alpha^{\beta_1})e^{-(x/\alpha_1)^{\beta_1}}$$

$$+ (1-w)\beta_2(x-\gamma)^{\beta_2 - 1}/\alpha_2^{\beta_2} e^{-[(x-\gamma)/\alpha_2]^{\beta_2}}, \quad (9)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $\gamma \geq 0$.

In equation (9), α_1 and α_2 are the scale parameters, β_1 and β_2 are the shape parameters, and γ is the location parameter while w is the mixing parameter. According to Boes [2] who assumed that only the mixing proportions are unknown for the two-component case, then we have

$$f(x) = wf_1(x) + (1-w) f_2(x).$$

By integrating a family of equations of the form;

$$F(x) = wF_1(x) + (1-w)F_2(x),$$

then this leads to the following relationship;

$$w = \frac{F(x) - F_2(x)}{F_1(x) - F_2(x)}. \quad (10)$$

This gives necessary and sufficient conditions on F_1 and F_2 for the uniform attainment of the Cramer-Rao bound on the variance of w.

There are many methods for estimating the parameters of the mixture distribution using both graphical and analytical approaches. The principles and procedures of the analytical methods are the same except for some details.

Simulation Experiments

We generated data for two-parameter and three-parameter Weibull distributions, giving a number of values to the parameters with different sample sizes and we estimated the parameters using maximum likelihood estimation (MLE). We obtained some statistical measurements such as total deviation (TD), coefficient of variation (CV) and the standard error (SE). The results are as follows:

Table 1. Two-parameter Weibull distribution

No.	β	α	Sample size, *n*	$\hat{\beta}$	$\hat{\alpha}$	TD	CV	SE
1	2	10	20	1.870	11.292	0.194	0.517	1.194
			50	2.250	10.550	0.180	0.467	0.630
			100	1.988	9.422	0.064	0.516	0.434
2	3	50	20	1.720	50.147	0.430	0.406	3.956
			50	2.739	46.903	0.149	0.392	2.352
			100	2.852	52.650	0.102	0.358	1.686
3	3.5	100	20	3.328	96.057	0.089	0.236	5.275
			50	3.827	95.479	0.139	0.304	3.784
			100	3.273	100.400	0.069	0.306	2.755

From Table 1, we can see that for the two-parameter Weibull distribution, the minimum TD is 0.064 when the sample size is 100 and the shape parameter is 2 and the scale parameter is 10. Also the CV and SE are relatively low. This indicates that the overall level of accuracy is acceptable.

Table 2. Three-parameter Weibull distribution

No.	β	α	γ	Sample size, *n*	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	TD	CV	SE
1	2	10	1	20	1.510	8.190	1.989	1.415	0.385	0.804
				50	1.868	9.519	1.855	0.969	0.471	0.653
				100	2.279	10.149	0.079	1.076	0.448	0.433
2	3	50	1	20	2.400	40.131	8.360	7.757	0.326	3.066
				50	2.939	41.960	10.621	9.802	0.372	2.379
				100	2.362	46.378	2.298	1.583	0.311	1.454
3	3.5	100	1	20	3.214	89.941	9.227	8.410	0.284	5.388
				50	3.089	96.056	1.063	0.220	0.372	2.379
				100	4.341	100.840	4.895	4.144	0.298	2.694

For the three-parameter Weibull distribution, Table 2 shows that the minimum TD is 0.220 when the sample size is 50, the shape parameter is 3.5, scale parameter is 100 and location parameter is 1.

Following are the graphs of the probability density functions of the two-parameter and three-parameter Weibull distributions;

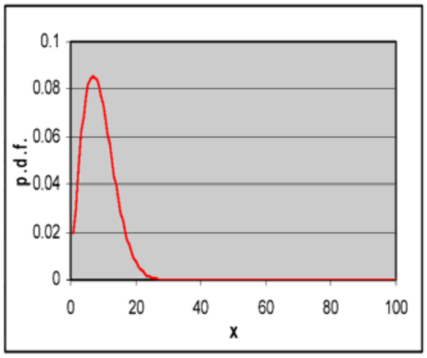

Figure 1. Probability density function of Weibull distribution with $\beta = 2$, $\alpha = 10$, $n = 100$.

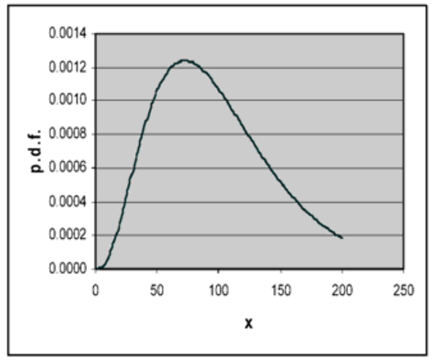

Figure 2. Probability density function of Weibull distribution with $\beta = 3.5$, $\alpha = 100$, $\gamma = 1$, $n = 50$.

Now we will refer to the mixed Weibull distribution of 6-parameters; the first component of Weibull distribution has two parameters; scale, $\alpha_1 = 1$, and shape, $\beta_1 = 3$, and the second component of Weibull distribution has three parameters; scale, $\alpha_2 = 2$, shape, $\beta_2 = 2$, and location, $\gamma = 1$. In addition, we used different values for the mixing parameter (w); 0.10, 0.40 and 0.80. The probability density function was obtained using equation (9). The results are as follows:

Table 3. Probability density function of mixed Weibull distribution with the parameters $\alpha_1 = 1$, $\beta_1 = 3$, $\alpha_2 = 2$, $\beta_2 = 2$, $\gamma = 1$, and mixing parameter; $w = 0.10,\ 0.40,\ 0.80$

Obs. No.	x_i	$f(x)$		
		$w = 0.10$	$w = 0.40$	$w = 0.80$
1	0.03	0.0002700	0.001080	0.002160
2	0.12	0.0043125	0.017250	0.034500
3	0.22	0.0143662	0.057465	0.114930
4	0.35	0.0352076	0.140831	0.281661
5	0.73	0.1083475	0.433390	0.866780
6	0.79	0.1143540	0.457416	0.914832
7	1.25	0.1772387	0.339769	0.556476
8	1.41	0.2130594	0.262547	0.328531
9	1.52	0.2393891	0.228542	0.214079
10	1.79	0.3072483	0.215181	0.092424
11	1.80	0.3096219	0.215915	0.090972
12	1.94	0.3399218	0.229153	0.081462
13	2.38	0.3857684	0.257187	0.085745
14	2.40	0.3859563	0.257310	0.085781
15	2.87	0.3510614	0.234041	0.078014
16	2.99	0.3327386	0.221826	0.073942
17	3.14	0.3064803	0.204320	0.068107
18	3.17	0.3008915	0.200594	0.066865
19	4.72	0.0526345	0.035090	0.011697
20	5.09	0.0280999	0.018733	0.006244

We also plotted the values of the distribution in one graph, as shown in Figure 3.

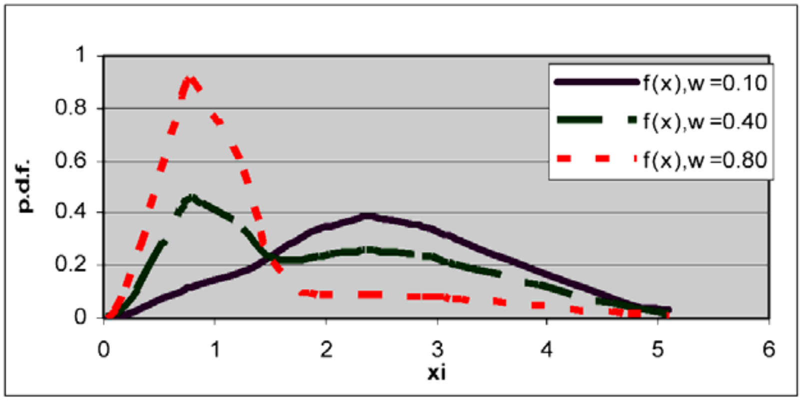

Figure 3. Probability density function of mixed Weibull distribution with $\alpha_1 = 1$, $\beta_1 = 3$, $\alpha_2 = 2$, $\beta_2 = 2$, $\gamma = 1$ and 3 values of mixing parameter *w*.

Conclusion

Total deviation, TD, gives an indication of the accuracy of parameter estimation. From Table 1, we can see that, for the two-parameter Weibull distribution, the lowest TD and SE are found when $\alpha = 10$, $\beta = 2$ and $n = 100$, which gives the best estimation for these parameters. For the three-parameter Weibull distribution, Table 2 shows that the best estimation for the parameters is when $\alpha = 100$, $\beta = 3.5$, $\gamma = 1$ and $n = 50$, which gives the lowest values for TD and SE. Figures 1 and 2 are the graphical representation of these cases.

Table 3 gives the results of combining two Weibull distributions, the first with two parameters; $\alpha_1 = 1$, $\beta_1 = 3$ and the second with three parameters $\alpha_2 = 2$, $\beta_2 = 2$, $\gamma = 1$, and taking three different values for the mixing parameter, *w*; 0.10, 0.40 and 0.80. These results were plotted in Figure 3.

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