JP Journal of Fixed Point Theory and Applications
Volume 4, Number 1, 2009, Pages 33-47
Published Online: June 04, 2009
This paper is available online at http://www.pphmj.com

FIXED POINT THEOREMS FOR ORDERED CONTRACTIVE MAPPINGS ON NONCOMMUTATIVE BANACH SPACES

NI FEI and JIANG LINING

Department of Mathematics
Beijing Institute of Technology
Beijing, 100081, P. R. China
e-mail: jianglining@bit.edu.cn

Abstract

Introducing a concept of a noncommutative Banach space, we obtain several fixed point theorems for continuous or discontinuous ordered contractive maps in an ordered noncommutative Banach space. A sufficient condition for the fixed point to be unique is given.

1. Preliminaries

Amann [1] introduced the concepts of ordered topological linear space and ordered Banach space, and gave a number of solutions of nonlinear equations in ordered Banach spaces. Based on his work, many authors studied the properties of fixed points of nonlinear equations in ordered Banach spaces [2-5, 7-8]. Furthermore, Zhang [9] introduced some types of ordered contractive maps and obtained some fixed point theorems in ordered Banach spaces. Since these fixed point results are based on the existence of the ordered structures which are compatible with the related topological structures, illumined by [1] and [9], the paper defines the ordered contractive maps and obtains the corresponding fixed point

2000 Mathematics Subject Classification: 47H10.

Keywords and phrases: fixed point, order contractive map, noncommutative Banach space.

Supported by Program for New Century Excellent Talents in University of China.

Received November 19, 2008

theorems for ordered contractive maps in noncommutative Banach spaces. We now give the definition of a noncommutative Banach space.

Definition 1.1. Let E be a group. Then E is called a *noncommutative Banach space* if the following conditions are satisfied:

1. There exists a metric d on E so that (E, d) is a complete metric space;

2. The metric d is invariant under the translation operation. That is, $\forall x, y, z \in E$, $d(xz, yz) = d(x, y)$;

3. There exists a binary continuous operation

$$F : \mathbb{R} \times E \to E, \quad (\alpha, g) \mapsto g^{\alpha},$$

which extends the group multiplications in E;

4. The metric d is sub-homogeneous, that is, for $x \in E$, there exists a constant $C_x > 0$ such that for $\alpha \in \mathbb{R}$,

$$d(x^{\alpha}, e) \leq C_x |\alpha| d(x, e).$$

It is clear that a Banach space is a noncommutative Banach space. The following is a nontrivial example on noncommutative Banach space.

Example 1.1. Suppose that H is a Hilbert space and $U(H)$ is the unitary group of H. As a subset of $L(H)$, $U(H)$ is a complete metric space if one defines

$$d(S, T) = \| T^{-1}S - I \| = \| S - T \| \quad (S, T \in U(H)).$$

Furthermore, for $T \in U(H)$ and $\alpha \in \mathbb{R}$, set

$$T^{\alpha} = \int_0^{2\pi} e^{i\alpha\theta} dE_{\theta},$$

where E_{θ} stands for the spectral measure associated with the operator T [6], then $U(H)$ is a noncommutative Banach space.

Proof. It suffices to prove that $U(H)$ possesses Properties 3 and 4 of Definition 1.1.

First, suppose that $d(T_n, T) \to 0$, where $T_n \in U(H)$. Since for $k \in \mathbb{N}$,

$$d(T_n^k, T^k) = \| T_n T_n^{k-1} - TT_n^{k-1} + TT_n^{k-1} - TT^{k-1} \|$$

$$\leq \| T_n - T \| \| T_n^{k-1} \| + \| T \| \| T_n^{k-1} - T^{k-1} \|,$$

by using induction, we obtain for an arbitrary polynomial $P(x)$, that $d(P(T_n), P(T)) \to 0$. Since for $\varepsilon > 0$, there exists a polynomial $P_0(x)$ such that $\sup_{x \in [0, 2\pi]} | P_0(x) - x^{\alpha} | < \frac{\varepsilon}{3}$,

$$d(P_0(T), T^{\alpha}) \leq \frac{\varepsilon}{3}; \quad d(P_0(T_n), T_n^{\alpha}) \leq \frac{\varepsilon}{3}.$$

Also, for $\frac{\varepsilon}{3} > 0$, $\exists N \in \mathbb{N}$ so that if $n > N$, then $\| P_0(T_n) - P_0(T) \| < \frac{\varepsilon}{3}$. Thus, when $n > N$,

$$d(T_n^{\alpha}, T^{\alpha}) \leq \| T_n^{\alpha} - P(T_n) + P(T_n) + P(T) - P(T) - T^{\alpha} \|$$

$$\leq \| T_n^{\alpha} - P(T_n) \| + \| P(T_n) - P(T) \| + \| P(T) - T^{\alpha} \|$$

$$\leq \varepsilon.$$

Therefore, for $\alpha \in \mathbb{R}$, $d(T_n, T) \to 0$ implies that $d(T_n^{\alpha}, T^{\alpha}) \to 0$.

Next, since the metric d is pseudo-homogeneous, we assume $T \neq I$. For $\alpha \in \mathbb{R}$,

$$d(T^{\alpha}, I) = \left\| \int_0^{2\pi} (e^{i\theta\alpha} - 1) dE_{\theta} \right\|$$

$$\leq \sup_{\theta \in [0, 2\pi]} | e^{i\theta\alpha} - 1 |.$$

Since the exponential function is a periodic function, we consider only the case when $| \alpha | \in [0, 1]$

$$\sup_{\theta \in [0, 2\pi]} | e^{i\theta\pi} - 1 | = \begin{cases} 2 < 2\pi | \alpha |, & \text{if } | \alpha | > \frac{1}{2}, \\ | e^{i2\pi\alpha} - 1 | \leq 2\pi | \alpha |, & \text{if } 0 \leq | \alpha | \leq \frac{1}{2}. \end{cases}$$

In all, set $C_T = \frac{2\pi}{d(T, I)}$, then for any $\alpha \in \mathbb{R}$, $d(T^{\alpha}, I) \leq C_T | \alpha | d(T, I)$.

Similarly, we prove that for $T \in U(H)$, $\lim_{\alpha \to \alpha_0} T^{\alpha} = T^{\alpha_0}$ and $U(H)$ is a noncommutative Banach space.

2. Ordered Contractive Maps on Noncommutative Banach Spaces

In this section, we introduce an order structure in a noncommutative Banach space and obtain basic properties of ordered contractive maps.

Definition 2.1. Suppose that E is a noncommutative Banach space. A set $P \subseteq E$ is called *convex* if $\forall x, y \in P$, $x^p y^q \in P$, where $p, q \in \mathbb{R}^+$ and $p + q = 1$. Furthermore, $P \subseteq E$ is called a *cone* if P is closed, convex and invariant under exponential operation by element of $[0, \infty)$, and if $P \cap P^{-1} = \{e\}$, where $P^{-1} = \{x^{-1} \mid x \in P\}$.

It is easy to see that a cone is a semigroup. Each cone can induce a partial ordering in E through the rule $x \lesssim y$ if and only if $y^{\beta} x^{-\beta} \in P$ for $\beta \in [0, 1]$. This ordering is antisymmetry, reflexive and transitive.

Definition 2.2. If there exists a constant $N > 0$ such that for any $e \lesssim x \lesssim y$, $d(x, e) \leq N d(y, e)$, P is called *positive*, and the constant N is called the *positive constant* of P.

Let "$\lesssim$" be the partial ordering determined by a cone P. For $u, v \in E$, if one of $u \lesssim v$ and $v \lesssim u$ holds, then we say that u and v are comparable and write:

$$\vee(u, v) = \begin{cases} u, & \text{when } v \lesssim u, \\ v, & \text{when } u \lesssim v. \end{cases}$$

Lemma 2.1. *If u and v are comparable, then uv^{-1} and vu^{-1} are comparable, and*

$$e \lesssim \vee(uv^{-1}, vu^{-1}).$$

Proof. Suppose that $v \lesssim u$. Then $\forall \alpha \in [0, 1]$,

$$(uv^{-1})^{\alpha}(vu^{-1})^{-\alpha} = (uv^{-1})^{\alpha}((uv^{-1})^{-1})^{-\alpha} = (uv^{-1})^{2\alpha}.$$

Since $2\alpha > 0$ and $uv^{-1} \in P$, $(uv^{-1})^{2\alpha} \in P$. Thus, the elements uv^{-1} and vu^{-1} are comparable, and $vu^{-1} \lesssim uv^{-1}$. Also, $\forall \alpha \in [0, 1]$, $(uv^{-1})^{\alpha} e^{-\alpha} = (uv^{-1})^{\alpha} \in P$, so $e \lesssim uv^{-1}$, and $e \lesssim \vee (uv^{-1}, vu^{-1})$. □

Definition 2.3. Let E be a noncommutative Banach space and P be a positive cone of E with the positive constant N. A map $A : E \to E$ is called a β-*ordered contractive map* if there exists a constant $0 < \beta < 1$ such that for $u, v \in E$, if u and v are comparable, then Au and Av are also comparable, and moreover

$$\vee (Av(Au)^{-1}, Au(Av)^{-1}) \lesssim \vee (vu^{-1}, uv^{-1})^{\beta}.$$

Here, the β is called the *constant* of the ordered contractive map.

Remark 2.1. The ordered contractive map need not be continuous.

Lemma 2.2. *Suppose that for all* $n \in \mathbb{N}$, u_n *and* v_n *are comparable. If* $v_n \to v_0$, $u_n \to u_0$, *then* u_0 *and* v_0 *are comparable. That is to say, the ordering structure is compatible with the metric given in E.*

Proof. Since $\forall n \in \mathbb{N}$, one of $u_n \lesssim v_n$ and $v_n \lesssim u_n$ holds, there exist subsequences $\{v_{n_k}\}$ and $\{u_{n_k}\}$ such that for $\forall 0 \le \beta \le 1$, either $u_{n_k}^{\beta} v_{n_k}^{-\beta} \in P$ or $v_{n_k}^{\beta} u_{n_k}^{-\beta} \in P$ holds. Without loss of generality, suppose that $u_{n_k}^{\beta} v_{n_k}^{-\beta} \in P$. Then

$$d(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_0^{\beta} v_0^{-\beta}) \le d(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_{n_k}^{\beta} v_0^{-\beta}) + d(u_{n_k}^{\beta} v_0^{-\beta}, u_0^{\beta} v_0^{-\beta})$$

$$= d(u_{n_k}^{\beta}, u_0^{\beta}) + d(v_{n_k}^{-\beta}, v_0^{-\beta}).$$

The last equation holds because the metric is invariant under the translation operation.

Because $v_{n_k} \to v_0$ and $u_{n_k} \to u_0$, we have $v_{n_k}^{-\beta} \to v_0^{-\beta}$ and $u_{n_k}^{-\beta} \to u_0^{-\beta}$. Since the multiplication operation on E is continuous,

$$\lim_{k \to \infty} d(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_0^{\beta} v_0^{-\beta}) = 0.$$

Notice, the fact that the cone P is closed, $u_0^{\beta} v_0^{-\beta} \in P$. This implies that u_0 and v_0 are comparable. □

Lemma 2.3. *If* $x, y \in P$ *and* $x \lesssim y$, *then* $\forall 0 < \beta < 1$, $x^{\beta} \lesssim y^{\beta}$.

Proof. Since $x \lesssim y$, $\forall \alpha \in [0, 1]$, $y^{\alpha} x^{-\alpha} \in P$. For $0 < \beta < 1$, $\alpha\beta \in [0, 1]$, so $y^{\alpha\beta} x^{-\alpha\beta} \in P$, namely $x^{\beta} \lesssim y^{\beta}$. □

Lemma 2.4. *If* x *and* y *are comparable, then* $d(\vee(xy^{-1}, yx^{-1}), e) = d(x, y)$.

Proof. Suppose $\vee(x, y) = x$. Then

$$yx^{-1} \lesssim xy^{-1}$$

and

$$\vee(xy^{-1}, yx^{-1}) = xy^{-1}.$$

Since the metric d is invariant under the translation operation,

$$d(x, y) = d(xy^{-1}, yy^{-1}) = d(\vee(xy^{-1}, yx^{-1}), e).$$

This completes the proof. □

3. Theorems about the Fixed Points

Throughout this section, we suppose that E is a noncommutative Banach space which is partially ordered by a positive cone P with the positive constant N, and give several theorems on the fixed points of the ordered contractive maps on E.

Theorem 3.1. *Suppose that the* β*-ordered contractive map* $A : E \to E$ *is continuous. If there exists an element* $x_0 \in E$ *such that* x_0 *and* Ax_0 *are comparable, then the sequence* $A^n x_0$ *converges to some fixed point* x^* *of A. Moreover, there is a number* C_{x_0} *depending on the choice of* x_0, *so that*

$$d(x_0, x^*) \leq \left(\frac{C_{x_0} \cdot N \cdot \beta}{1 - \beta} + 1\right) d(x_0, Ax_0).$$

Proof. Consider the sequence

$$x_1 = Ax_0,\ x_2 = Ax_1,\ \dots,\ x_{n+1} = Ax_n,\ \dots.$$

Since x_0 and $x_1 = Ax_0$ are comparable and the map A is a β-ordered contractive map, x_1 and $x_2 = Ax_1$ are comparable, and hence, x_n and $x_{n+1} = Ax_n$ are comparable. Since

$$\begin{aligned}\vee(x_n x_{n+1}^{-1},\ x_{n+1}x_n^{-1}) &= \vee(Ax_{n-1}(Ax_n)^{-1},\ Ax_n(Ax_{n-1})^{-1})\\ &\lesssim \vee(x_{n-1}x_n^{-1},\ x_n x_{n-1}^{-1})^{\beta},\end{aligned}$$

using Lemma 2.3

$$\begin{aligned}\vee(x_n x_{n+1}^{-1},\ x_{n+1}x_n^{-1}) &\lesssim \vee(x_{n-1}x_n^{-1},\ x_n x_{n-1}^{-1})^{\beta}\\ &\lesssim \vee(x_{n-2}x_{n-1}^{-1},\ x_{n-1}x_{n-2}^{-1})^{\beta^2}\\ &\lesssim \cdots\\ &\lesssim \vee(x_0x_1^{-1},\ x_1x_0^{-1})^{\beta^n}.\end{aligned}$$

Thus

$$d(\vee(x_n x_{n+1}^{-1},\ x_{n+1}x_n^{-1}),\ e) \le Nd(\vee(x_0x_1^{-1},\ x_1x_0^{-1})^{\beta^n},\ e). \tag{1}$$

Because the metric d is sub-homogeneous, there exists a constant C_{x_0}, which depends on the choice of x_0, so that

$$d(\vee(x_0x_1^{-1},\ x_1x_0^{-1})^{\beta^n},\ e) \le C_{x_0} \cdot \beta^n d(\vee(x_0x_1^{-1},\ x_1x_0^{-1}),\ e). \tag{2}$$

Notice that the metric d is invariant under the translation operation

$$d(\vee(x_n x_{n+1}^{-1},\ x_{n+1}x_n^{-1}),\ e) = d(x_n,\ x_{n+1}). \tag{3}$$

In the same way

$$d(\vee(x_0x_1^{-1},\ x_1x_0^{-1}),\ e) = d(x_0,\ x_1). \tag{4}$$

So, the inequality (2) turns into

$$\begin{aligned}d(x_n,\ x_{n+1}) &= d(\vee(x_n x_{n+1}^{-1},\ x_{n+1}x_n^{-1}),\ e)\\ &\le N \cdot d(\vee(x_0x_1^{-1},\ x_1x_0^{-1})^{\beta^n},\ e)\end{aligned}$$

$$\leq N \cdot C_{x_0} \cdot \beta^n d(\vee (x_0 x_1^{-1}, x_1 x_0^{-1}), e)$$

$$= N \cdot C_{x_0} \cdot \beta^n d(x_0, x_1).$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence since $\beta \in (0, 1)$. Suppose that $x_n \to x^*$, then

$$Ax^* = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

which implies that x^* is a fixed point of A. Moreover

$$d(x^*, x_0) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_n, x_{n+1}) + \cdots$$

$$= \sum_{n=1}^{\infty} d(x_n, x_{n+1}) + d(x_0, x_1)$$

$$\leq \sum_{n=1}^{\infty} C_{x_0} \cdot N \cdot \beta^n d(x_0, x_1) + d(x_0, x_1)$$

$$= \left(\frac{C_{x_0} \cdot N \cdot \beta}{1 - \beta} + 1\right) d(x_0, Ax_0).$$ □

Corollary 3.2. *Conditions and assumptions are the same as in Theorem* 3.1. *Let* $\tilde{x}$ *be another fixed point of* A. *If* $\tilde{x}$ *and* x^* *are comparable, then* $\tilde{x} = x^*$.

Proof. Since $\tilde{x}$ and x^* are comparable, one can suppose that $\tilde{x} \lesssim x^*$. Using the definition of contractive map

$$\vee (A\tilde{x}(Ax^*)^{-1}, Ax^*(A\tilde{x})^{-1}) \lesssim \vee (x^*\tilde{x}^{-1}, \tilde{x}x^{*-1})^{\beta},$$

namely

$$\vee (x^*\tilde{x}^{-1}, \tilde{x}x^{*-1}) \lesssim \vee (x^*\tilde{x}^{-1}, \tilde{x}x^{*-1})^{\beta}.$$

Since $\tilde{x} \lesssim x^*$, $\vee(\tilde{x}x^{*-1}, x^*\tilde{x}^{-1}) = x^*\tilde{x}^{-1}$, then $x^*\tilde{x}^{-1} \lesssim (x^*\tilde{x}^{-1})^{\beta}$, $(x^*\tilde{x}^{-1})^{\beta-1} \in P$. Notice that $1 - \beta \in [0, 1]$, $(x^*\tilde{x}^{-1})^{1-\beta} \in P$. By $(x^*\tilde{x}^{-1})^{\beta-1} \in P$ and $(x^*\tilde{x}^{-1})^{1-\beta} \in P$, $x^*\tilde{x}^{-1} = e$, and $x^* = \tilde{x}$. □

Theorem 3.3. *Suppose that* $A : E \to E$ *is a* β*-ordered contractive map. If there exists an element* $x_0 \in E$ *so that* $\forall n$, x_0 *and* $A^n x_0$ *are comparable, then* A *has some fixed point, and the sequence* $\{A^n x_0\}$ *converges to one fixed point* x^* *of* A. *Moreover, there exists a constant* C_{x_0} *such that*

$$d(x^*, x_0) \leq \left(\frac{C_{x_0} \cdot N \cdot \beta}{1-\beta} + 1\right) d(x_0, Ax_0).$$

Proof. Similar to the proof of Theorem 3.1, the sequence $\{x_n = A^n x_0\}$ is a Cauchy sequence. By the completeness of E, let $x_n \to x^* \in E$. Now, we prove that x^* is a fixed point of A.

For all m, n, suppose that $m > n$, using the given condition, x_0 and x_{m-n} are comparable. Then Ax_n and Ax_{m-n} are comparable, and so are $x_n = A^n x_0$ and $x_m = A^n x_{m-n}$. Let $m \to \infty$, using Lemma 2.2, $\forall n$, x_n and x^* are comparable, therefore Ax_n and Ax^* are comparable, and so

$$e \precsim \vee (Ax_n (Ax^*)^{-1}, Ax^* (Ax_n)^{-1}) \precsim \vee (x_n x^{*-1}, x^* x_n^{-1})^{\beta}.$$

Since P is a positive cone,

$$d(\vee (Ax_n (Ax^*)^{-1}, Ax^* (Ax_n)^{-1}), e) \leq C_{x_0} \cdot N \cdot \beta d(\vee (x_n x^{*-1}, x^* x_n^{-1}), e),$$

that is,

$$d(x_{n+1}, Ax^*) = d(Ax_n, Ax^*) \leq C_{x_0} \cdot N \cdot \beta d(x_n, x^*) \to 0.$$

Therefore, $x^* = Ax^*$ and x^* is a fixed point of A. At last, similar to the proof of Theorem 3.1, we can get the estimation of $d(x^*, x_0)$ and we omit it here. □

Theorem 3.4. *Let* $A : E \to E$ *be a continuous map and satisfy the following condition*:

(C1) *If* u *and* v *are comparable, then* Au *and* Av *are comparable. Also, if* u *and* Au *are comparable, and* v *and* Av *are comparable, then there* *exists a* $\lambda \in \left(0, \frac{1}{2}\right)$ *so that for* $\forall \beta \in [0, 1]$,

$$\vee (Av(Au)^{-1}, Au(Av)^{-1})^{\beta} \lesssim \vee (Au \circ u^{-1}, u \circ (Au)^{-1})^{\lambda\beta}$$
$$\circ \vee (Av \circ v^{-1}, v(Av)^{-1})^{\lambda\beta}.$$

If there exists an element $x_0 \in E$, *such that* x_0 *and* Ax_0 *are comparable, then the sequence* $\{A^n x_0\}$ *converges to a fixed point* x^* *of A. Moreover, there exists a constant* C_{x_0} *such that*

$$d(x_0, x^*) \leq \left(1 + \frac{C_{x_0} \cdot N \cdot \lambda}{1 - 2\lambda}\right) d(x_0, Ax_0).$$

Proof. Set $\beta = \frac{\lambda}{1-\lambda}$, then

$$1 + \frac{N\beta}{1-\beta} = 1 + \frac{\lambda \cdot N}{1 - 2\lambda}.$$

Consider the sequence:

$$x_1 = Ax_0, x_2 = Ax_1 = A^2 x_0, \dots, x_{n+1} = Ax_n, \dots.$$

Since x_0 and Ax_0 are comparable, for $n \in \mathbb{N}$, x_n and Ax_n are comparable, and

$$\begin{aligned} e &\lesssim \vee (x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}) \\ &= \vee (Ax_{n-1}(Ax_n)^{-1}, Ax_n(Ax_{n-1}^{-1})) \\ &\lesssim \vee (Ax_{n-1} x_{n-1}^{-1}, x_{n-1}(Ax_{n-1})^{-1})^{\lambda} \vee (Ax_n x_n^{-1}, x_n(Ax_n)^{-1})^{\lambda} \\ &= \vee (x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{\lambda} \vee (x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{\lambda}. \end{aligned}$$

Therefore

$$e \lesssim \vee (x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}) \lesssim \vee (x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{\lambda} \vee (x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{\lambda},$$

and $\forall \beta \in [0, 1]$,

$$\vee (x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1})^{\beta} \lesssim \vee (x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{\beta\lambda} \vee (x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{\beta\lambda}.$$

That is,

$$\vee (x_n x_{n-1}^{-1},\, x_{n-1}x_n^{-1})^{\beta\lambda} \vee (x_{n+1}x_n^{-1},\, x_n x_{n+1}^{-1})^{-(1-\lambda)\beta} \in P,$$

and so

$$\vee (x_{n+1}x_n^{-1},\, x_n x_{n+1}^{-1})^{(1-\lambda)} \lesssim \vee (x_n x_{n-1}^{-1},\, x_{n-1}x_n^{-1})^{\lambda}.$$

Since $0 < \lambda < \frac{1}{2}$, $0 < \frac{\lambda}{1-\lambda} < 1$, using Lemma 2.3

$$\begin{aligned} e &\lesssim (x_{n+1}x_n^{-1},\, x_n x_{n+1}^{-1}) \\ &\lesssim \vee (x_n x_{n-1}^{-1},\, x_{n-1}x_n^{-1})^{\lambda/1-\lambda} \\ &\lesssim \cdots \\ &\lesssim \vee (x_0 x_1^{-1},\, x_1 x_0^{-1})^{(\lambda/1-\lambda)^n}. \end{aligned}$$

Using inequality (2) in the proof of Theorem 3.1, there exists a constant C_{x_0} such that

$$d(\vee (x_{n+1}x_n^{-1},\, x_n x_{n+1}^{-1}),\, e) \leq C_{x_0} \cdot N \cdot \left(\frac{\lambda}{1-\lambda}\right)^n d(x_0 x_1^{-1},\, x_1 x_0^{-1}).$$

Since the metric d is invariant under the translation operation,

$$d(x_n,\, x_{n+1}) \leq C_{x_0} \cdot N \cdot \left(\frac{\lambda}{1-\lambda}\right)^n d(x_0,\, x_1).$$

This implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of E, let $x_n \to x^* \in E$. Then

$$Ax^* = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus, A has a fixed point in E and the sequence $\{A^n x\}$ converges to a fixed point of A. □

Theorem 3.5. *Let* $A : E \to E$ *be a map satisfying Condition* (C1) *of Theorem* 3.4. *If there exists an element* $x_0 \in E$ *so that* $\forall n \in \mathbb{N}$, x_0 *and* $A^n x_0$ *are comparable, then* A *has a fixed point in* E *and the sequence*

$\{A^n x_0\}$ converges to a fixed point x^* of A. Moreover, there exists a constant C_{x_0} such that

$$d(x_0, x^*) \leq \left(1 + \frac{N \cdot C_{x_0} \cdot \lambda}{1 - 2\lambda}\right) d(x_0, Ax_0).$$

Proof. Similar to the proof of Theorem 3.3, $\{x_n = A^n x_0\}$ is a Cauchy sequence. Let $x_n \to x^* \in E$. Now, we prove that x^* is a fixed point of A. As found before, for all n, x_n and x^* are comparable. Using Condition (C1), for all n, x_{n-1} and $x_n = Ax_{n-1}$ are comparable. Let $n \to \infty$. Then using Lemma 2.2, x^* and Ax^* are comparable. Hence

$$\begin{aligned} e &\precsim \vee (Ax_n(Ax^*)^{-1}, Ax^*(Ax_n)^{-1}) \\ &\precsim \vee (Ax_n x_n^{-1}, x_n(Ax_n)^{-1})^{\lambda} \circ \vee (Ax^* x^{*-1}, x^*(Ax^*)^{-1})^{\lambda}. \end{aligned}$$

Let $n \to \infty$. Then we obtain

$$\begin{aligned} e &\precsim \vee (x^*(Ax^*)^{-1}, Ax^*(x^*)^{-1}) \\ &\precsim \vee (Ax^* x^{*-1}, x^*(Ax^*)^{-1})^{\lambda} \circ \vee (Ax^* x^{*-1}, x^*(Ax^*)^{-1})^{\lambda}, \end{aligned}$$

that is,

$$e \precsim \vee (x^*(Ax^*)^{-1}, Ax^*(x^*)^{-1}) \precsim \vee (Ax^* x^{*-1}, x^*(Ax^*)^{-1})^{2\lambda}.$$

Thus, $\vee (Ax^* x^{*-1}, x^*(Ax^*)^{-1})^{2\lambda - 1} \in P$. Since $2\lambda - 1 < 0$ and $\vee ((Ax)^* x^{*-1}, x^*(Ax^*)^{-1}) \in P$, $x^*(Ax^*)^{-1} = Ax^*(x^*)^{-1} = e$, namely $Ax^* = x^*$. Therefore, x^* is a fixed point of A. □

Remark 3.1. In Theorems 3.3, 3.4 and 3.5, the estimations of $d(x_0, x^*)$ are the same as that in Theorem 3.1. This is because $\{x_n\}$ is the Cauchy sequence which makes $d(x_n, x^*) \leq C_{x_0} \cdot N \cdot \beta^n d(x_1, x_0)$ holds. In Theorems 3.4 and 3.5, $\beta = \frac{\lambda}{1 - \lambda}$.

Theorem 3.6. *Suppose that* $u_0, v_0 \in E$ *with* $u_0 \precsim v_0$, *and* $[u_0, v_0] = \{u \in E \mid u_0 \precsim u \precsim v_0\}$ *is an ordered interval in E. If* $A : [u_0, v_0] \to [u_0, v_0]$ *is a* β*-ordered contractive map, then A has a unique fixed point. Moreover, for all* $x \in [u_0, v_0]$, *the sequence* $\{A^n x\}$ *converges to the only fixed point of A.*

Proof. Define sequences:

$$u_1 = Au_0,\ u_2 = Au_1,\ \ldots,\ u_{n+1} = Au_n,\ \ldots,$$

$$v_1 = Av_0,\ v_2 = Av_1,\ \ldots,\ v_{n+1} = Av_n,\ \ldots,$$

then $\{u_n\}, \{v_n\} \subset [u_0, v_0]$. Since $u_0 \precsim v_0$ and A is a β-ordered contractive map, for all n, u_n and v_n are comparable, and

$$\begin{aligned} e &\precsim \vee (u_n v_n^{-1}, v_n u_n^{-1}) \\ &= \vee (Au_{n-1}(Av_{n-1})^{-1}, Av_{n-1}(Au_{n-1})^{-1}) \\ &\precsim \vee (u_{n-1}v_{n-1}^{-1}, v_{n-1}u_{n-1}^{-1})^{\beta} \\ &\precsim \cdots \\ &\precsim \vee (u_0 v_0^{-1}, v_0 u_0^{-1})^{\beta^n}. \end{aligned}$$

Because P is positive, there exists a constant $C_{u_0 v_0^{-1}}$ and a positive integer N such that

$$d(u_n, v_n) = d(u_n v_n^{-1}, e) = d(v_n u_n^{-1}, e) \le C_{u_0 v_0^{-1}} \cdot N \cdot \beta^n d(u_0, v_0).$$

Since again $u_0 \precsim v_1$ for all n, u_n and u_{n+1} are comparable, and

$$\begin{aligned} e &\precsim \vee (u_n u_{n+1}^{-1}, u_{n+1}u_n^{-1}) \\ &= \vee(Au_{n-1}(Au_n)^{-1}, Au_n(Au_{n-1}^{-1})) \\ &\precsim \vee (u_{n-1}u_n^{-1}, u_n u_{n-1}^{-1})^{\beta} \\ &\precsim \cdots \\ &\precsim \vee (u_0 u_1^{-1}, u_1 u_0^{-1})^{\beta^n}. \end{aligned}$$

Thus

$$d(u_n, u_{n+1}) \leq \beta^n N d(u_0, u_1).$$

Notice that $\beta < 1$, $\{u_n\}$ is a Cauchy sequence with a limit point $u^* \in [u_0, v_0]$. Similarly, $\{v_n\}$ is a Cauchy sequence with a limit point $v^* \in [u_0, v_0]$. Then

$$d(u^*, v^*) = \lim_{n\to\infty} d(u_n, v_n) \leq \lim_{n\to\infty} C_{u_0 v_0^{-1}} \cdot N \cdot \beta^n d(u_0, v_0) = 0.$$

This implies that $u^* = v^*$.

Now, we prove that u^* is a fixed point of A. For all $m > n$, since u_0 and u_{m-n} are comparable, $A^n u_0 = u_n$ and $A^n u_{m-n} = u_m$ are comparable. Let $m \to \infty$. Then u_n and u^* are comparable, and Au_n and Au^* are also comparable

$$e \lesssim \vee\, (Au_n(Au^*)^{-1}, Au^*(Au_n)^{-1}) \lesssim \vee\, (u_n u^{*-1}, u^* u_n^{-1})^{\beta}.$$

So

$$\lim_{n\to\infty} d(u_{n+1}, Au^*) = \lim_{n\to\infty} d(Au_n, Au^*) \leq \lim_{n\to\infty} C_{u_n u^{*-1}} \cdot N \cdot \beta d(u_n, u^*) = 0.$$

Thus, $Au^* = u^*$, and u^* is a fixed point of A.

For all $x \in [u_0, v_0]$, since x and u_0 are comparable, $A^n x$ and $A^n u_0$ are comparable, and

$$e \lesssim \vee\, (A^n x(A^n u_0)^{-1}, A^n u_0(A^n x)^{-1}) \lesssim [\vee\, (u_0 x^{-1}, x u_0^{-1})]^{\beta^n} \to e.$$

Therefore, $A^n x \to u^*$.

Now, we prove that the fixed point of A is unique. Suppose that v is another fixed point of A in $[u_0, v_0]$, then

$$d(u^*, v) \leq d(u^*, A^n u_0) + d(A^n u_0, A^n v).$$

Notice that $u_0 \lesssim v$, and therefore we have

$$\begin{aligned} e &\lesssim \vee\, (A^n u_0(A^n v)^{-1}, A^n v(A^n u_0)^{-1}) \\ &= \vee((AA^{n-1}u_0)(AA^{n-1}v)^{-1}, (AA^{n-1}v)(AA^{n-1}u_0)^{-1}) \end{aligned}$$

$$\lesssim \vee (A^{n-1}u_0(A^{n-1}v)^{-1},\ A^{n-1}v(A^{n-1}u_0)^{-1})^{\beta}$$

$$\lesssim \dots$$

$$\lesssim \vee (u_0v^{-1},\ vu_0^{-1})^{\beta^n}.$$

Thus

$$\begin{aligned} d(A^n u_0,\ A^n v) &= d(\vee\ A^n u_0(A^n v)^{-1},\ A^n v(A^n u_0)^{-1}) \\ &\le Nd(\vee\ (u_0v^{-1},\ vu_0^{-1})^{\beta^n},\ e) \\ &\le NC_{u_0v^{-1}}\beta^n d(u^*,\ v). \end{aligned}$$

Since u^* and v are fixed points, $d(A^n u_0,\ A^n v) \to d(u^*,\ v)$, we have $d(u^*,\ v) = 0$. The uniqueness is proved. □

References

[1] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Funct. Anal. 11 (1972), 346-384.

[2] Y. H. Du, Fixed points of increasing operators in ordered Banach spaces and applications, Appl. Anal. 38 (1990), 1-20.

[3] D. J. Guo, Fixed points of mixed monotone operators with applications, Appl. Anal. 31 (1988), 215-244.

[4] D. J. Guo and V. Lakshmikantham, Couple fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (1987), 623-632.

[5] Juan J. Nieto and Rosana Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223-239.

[6] Walter Rudin, Functional Analysis, 2nd ed., McGraw-Hill, Inc., New York, 1991.

[7] Y. R. Syau, Some fixed point theorems of T-monotone operators, J. Math. Anal. Appl. 205 (1997), 325-329.

[8] J. G. Wang, Fixed points of increasing operators in ordered Banach spaces, Acta Math. Sinica (Chin. Ser.) 43 (2000), 43-48 (in Chinese).

[9] X. Zhang, Fixed points of ordered contractive maps, Acta Math. Sinica (Chin. Ser.) 48 (2005), 973-978 (in Chinese).