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T-LATTICES AND FIXED POINT THEORY

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Abstract

In this paper, we introduce the notion of a T-lattice as a generalization of that of a complete lattice and obtain some of its properties. A generalization of Knaster-Tarski's fixed point theorem is also obtained.

1. Introduction

The Knaster-Tarski theorem [6] states that any order-preserving function on a complete lattice has a fixed point and also a smallest fixed point. In [1], a constructive proof of Tarski's fixed point theorem is given showing the existence. The extension to set-valued functions got developed by Smithson [4] and Zhou [8]. Earlier, Vives [7] proved a stronger version of the extension which is applied to games with strict strategic complementarities. Smithson had used a weaker monotonicity requirement than that of Zhou, but he does not obtain a lattice structure of the set of fixed points. Echenique [2] gave a shorter and constructive proof of Tarski's fixed point theorem and Zhou's extension of Tarski's fixed point theorem. Our results generalize the known results of Tarski and Knaster in the context of the so-called complete T-lattice (see Theorem 1 in Section 3). We will also discuss some consequences of such theorem.

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The paper is organized as follows. In Section 2, we fix notations, introduce the T-lattice notion that generalizes the notion of complete lattices and give some preliminary results. In Section 3, we prove a fixed point theorem for a T-lattice which generalizes Knaster-Tarski's fixed point theorem for complete lattices [6] along with a concluding remark. An application of Theorem 1 is given in Section 4.

2. Notation and Preliminaries

Let $(E, \leq)$ be an ordered set with a least element 0 and a greatest element 1, and A and B be two subsets of E. Then we say that B is larger than A and we write $A \prec B$ if for every $a \in A$, $b \in B$, we have $a \leq b$. Note that the above relationship is not necessarily an order because it is not reflexive, in general.

Next, let T be a given operator on $(E, \leq)$ reversing the order and consider the following subsets K_r and K_l of E:

$$K_r = \{x \in E;\, Tx \leq x\} \quad \text{and} \quad K_l = \{x \in E;\, x \leq Tx\}.$$

Since $T1 \leq 1$ and $T1 \leq T^2 1$, K_r and K_l are nonempty and we have

$$TK_r \subset K_l \quad \text{and} \quad TK_l \subset K_r,$$

where TA denotes the image of A under the map T for $A \subset E$.

Definition 2.1. We say that $(E, \leq)$ is a *T-lattice* if for every $A \subset E$ such that $TA \prec A$ (resp. $A \prec TA$), there exists an element $c \in K_r$ (resp. $c \in K_l$) such that $TA \prec \{c\} \prec A$ (resp. $A \prec \{c\} \prec TA$).

Example 2.1. Let $(E, \leq)$:

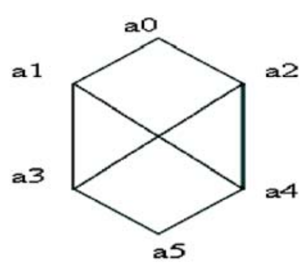

Figure 1

and consider the map T_0 defined on E by: $T_0(x) = a_5$ for $x \in E \setminus \{a_5, a_4\}$ and $T_0(x) = a_0$, otherwise. Then it is clear that T_0 reverses the order and the corresponding set K_r is reduced to $E \setminus \{a_5, a_4\}$ and K_l to $\{a_4, a_5\}$. Hence, $(E, \leq)$ is a T_0-lattice.

Now, we mention that any subset A of E satisfying $TA \prec A$ (and so in particular $Tx \leq x$ for all $x \in A$) is a subset of K_r. Furthermore, for any chain C of K_r, we have $TC \prec C$ since for any element x and y, $x \leq y$ implies that $Ty \leq Tx \leq x \leq y$. Similarly, we have $C \prec TC$ for all chains C of K_l. Thus, we introduce below the notion of a complete T-lattice.

Definition 2.2. A given T-lattice $(E, \leq)$ is said to be *complete* if each subset $A \subset E$ satisfying $TA \prec A$ (resp. $A \prec TA$) admits an infimum element in K_r (a supremum element in K_l).

Example 2.2. The T_0-lattice defined in the previous example is a complete T_0-lattice.

Hence, if $(E, \leq)$ is a given complete T-lattice and C is any given chain of K_r, then C admits an infimum element in K_r. We also show in a similar way that the supremum element for any chain of K_l is in K_l. Furthermore, we have

Proposition 2.1. *Let $(E, \leq)$ be a complete T-lattice. Then any subset A of K_r (resp. A of K_l) admits a supremum element in K_r (resp. an infimum in K_l).*

Proof. First, note that the set $M = M(A)$ of upper bounds of a subset A of K_r is nonempty, since $1 \in M(A)$. Now, let $m_1, m_2 \in M$, hence we have $x \leq m_1$ and $x \leq m_2$ for every $x \in A$. It follows from the fact that T reverses the order that $Tm_1 \leq Tx \leq x \leq m_2$. Therefore, $TM \prec M$ and so M has an infimum element. □

Lemma 1. *If $(E, \leq)$ is a complete lattice with greatest element 1 and least element 0, then $(E, \leq)$ is a complete T_l-lattice with T_l an operator on E such that $T_l x = 0$.*

Proof. Under the assumptions of the above lemma, we have $K_r = E$ and $K_l = \{0\}$. Thus, for every subset $A \subset E$, we have $TA \prec A$ which shows that A admits an infimum element and a supremum element in E. This completes the proof. □

3. Main Result

Let $(E, \leq)$ be an ordered set. Denote by $\mathcal{P}_{\text{res}}(E)$ the set of all mappings $F : E \to E$ preserving the order on E, more exactly, we have

$$\mathcal{P}_{\text{res}}(E) = \{F : E \to E;\ Fx \leq Fy \text{ for } x, y \in E \text{ such that } x \leq y\}.$$

We say that $(E, \leq)$ possesses the *fixed point property* if for every $F \in \mathcal{P}_{\text{res}}(E)$, there exists $x = x_F \in E$ such that $Fx = x$, that is, if the set of fixed points of F is nonempty. Below, we give sufficient conditions for an ordered set to possess this property.

Theorem 1. *Let $(E, \leq)$ be a complete T-lattice with the least element* 0 *and greatest element* 1. *If $E = K_r \cup K_l$, then $(E, \leq)$ has the fixed point property*.

The proof of the above theorem relies essentially on the following one.

Theorem 2. *Let $(E, \leq)$ be an ordered set such that every chain of E admits a supremum element and $F \in \mathcal{P}_{\text{res}}(E)$. Assume further that there is $x \in E$ such that $x \leq Fx$. Then F has a fixed point*.

Indeed, let $A = \{x \in E : x \leq Fx\}$ and C be a maximal chain of A. Hence, the chain C admits a supremum element s in E. For all $x \in C$, we have $x \leq s$, applying the mapping F, we get $x \leq Fx \leq Fs$ that implies $s \leq Fs$, thus we obtain $s \in A$. Since C is maximal, $s = Fs$.

Proof of Theorem 1. For all mappings F on E preserving the order, we have $0 \leq F(0)$. We set $A = \{x \in E \setminus x \leq F(x)\}$ which is a nonempty set, and the least element 0 is in A. We also have for any $x \in A$, $Fx \in A$. So, if there exists $x_0 \in K_r$ such that $x_0 \in A$, we obtain $TFx_0 \leq Tx_0 \leq x_0 \leq Fx_0$ that implies $Fx_0 \in K_r$, hence the restriction of $F \setminus A \cap K_r$

is a mapping preserving the order. According to Proposition 2.1, every chain of K_r admits a supremum element in K_r. Let C be chain in $A \cap K_r$ and s be a supremum element of C. We have $x \leq s$ for every $x \in C$. We apply the mapping F, $x \leq Fx \leq Fs$ which gives $s \leq Fs$, then $s \in A$. By applying the previous theorem on the set $A \cap K_r$, F has a fixed point in $A \cap K_r$. We now assume that $A \cap K_r = \varnothing$ (i.e., $A \subset K_l$). All chains C of A admit a supremum element s in K_l since $C \prec TC$ and we have $x \leq s$ for all $x \in C$. We then apply the mapping F, $x \leq Fx \leq Fs$ that gives $s \leq Fs$ and hence $s \in A$. By applying the previous theorem on the set A, it follows that F has a fixed point in A. □

Example 3.1. Let $(E, \leq)$ be the complete T_0-lattice that was defined in the Example 2.1 (see Figure 1) with greatest element a_0 and least element a_5, we have $E = K_r \cup K_l$. Then $(E, \leq)$ has the fixed point property.

This result generalizes the following theorem (Knaster-Tarski's theorem [6]:

Theorem 3. *All complete lattices possess the fixed point property.*

Proof. Let $(L, \leq)$ be a complete lattice with a least element 0 and a greatest element 1. Then we have $(L, \leq)$ a complete T_l-lattice with $T_l(x) = 0$ for all $x \in L$. Then $L = K_r$, and therefore, $(L, \leq)$ has the fixed point property. This ends the proof. □

Concluding remark

Let $(E = (L_1 \cup L_2), \leq)$ be an ordered set with least element 0 and greatest element 1. Also, let L_1 and L_2 be two complete lattices such that: if x, y are comparable with $x \in L_1$ and $y \in L_2$, then we have necessarily $y \leq x$ or $x \leq y$. Hence, if $y \leq x$, it follows that $1 \in L_1$ and $0 \in L_2$.

Now, if we assume that $(E = (L_1 \cup L_2), \leq)$ is not a complete lattice and T is an operator on E such that $Tx = 0$ for $x \in L_1 \setminus L_2$, $Tx = 1$ for

$x \in L_2 \setminus L_1$ and $Tx = x$ for $x \in L_1 \cap L_2$, then it is clear that T_0 reverses the order and the corresponding set $K_r = \{x \in E;\, T_0 x \leq x\}$ is reduced to L_1 and $K_l = \{x \in E;\, x \leq T_0 x\}$ to L_2. Since, every subset of L_1 has an infimum element and every subset of L_2 has a supremum element. Thus, $(E = (L_1 \cup L_2), \leq)$ is a complete T-lattice. Therefore, by applying Theorem 1, $(E = (L_1 \cup L_2), \leq)$ has the fixed point property.

4. An Application

Let (X, d) be a bounded hyperconvex space. We set $X = B(x_0, r)$ with $x_0 \in X$ and let $f : X \to X$ be a nonexpansive mapping and strictly nonexpansive on orbits (i.e., for every $x \in X$ such that $f(x) \neq x$, we have $d(f^2(x), f(x)) < d(f(x), x)$). We put $k_0 = \sup \dfrac{d(f^{n+1}(x_0), f^n(x_0))}{d(f^n(x_0), f^{n-1}(x_0))}$, then $0 < k_0 < 1$ and

$$\begin{aligned} & d(f^m(x_0), f^n(x_0)) \\ & \leq \sum_{i=n}^{m} d(f^i(x_0), f^{i+1}(x_0)) \\ & \leq \frac{1}{1-k_0} d(f^n(x_0), f^{n-1}(x_0)) - \frac{1}{1-k_0} d(f^{m-1}(x_0), f^m(x_0)) < \infty \end{aligned}$$

for all $m > n$ which shows that $(f^n(x_0))$ is a Cauchy sequence. Note that a hyperconvex metric space is complete and the sequence $(f^n(x_0))$ has a limit $\alpha \in X$. We put $l_0 = \dfrac{1}{1-k} d(x_0, f(x_0))$, $l_n = \dfrac{1}{1-k} d(f^n(x_0), f^{n+1}(x_0))$ for $n \in \mathbb{N}^*$, then the sequence $(l_n, f^n(x_0))$ has a limit $(0, \alpha)$ on $\{0\} \times X$.

Let $L = \{(l_n, f^n(x_0)) : \forall n \geq n_0\}$ for n_0 large, $M = \{0\} \times X$ and $N = \{(-l_n, f^n(x_0)) : \forall n \geq n_0\}$. Then we take $E = L \cup M \cup N$. Consider the order relation $<_1$, defined as follows $(t, x) <_1 (s, y) \Leftrightarrow d(x, y) \leq (s - t)$ and $t \leq s$. Let T be the operator defined on E by $T(l_n, f^n(x_0))$

$= (-l_n, f^n(x_0))$, $T(-l_n, f^n(x_0)) = (l_n, f^n(x_0))$ and $T(0, x) = (0, x)$, $\forall x \in X$. Hence, $(E, <_1)$ is a complete T-lattice, and the set $K_r = L \cup M$ and $K_l = M \cup N$.

We consider the mapping F on E defined by $F(l_n, f^n(x_0)) = (l_{n+1}, f^{n+1}(x_0))$, $F(-l_n, f^n(x_0)) = (-l_{n+1}, f^{n+1}(x_0))$ for all $n \geq n_0$ and $F(0, x) = (0, f(x))$ for all $x \in X$. Since F preserves the order, using Theorem 1, we see that F admits a fixed point and consequently, so does f. □

References

[1] P. Cousot and R. Cousot, Constructive versions of Tarski's fixed point theorems, Pacific J. Math. 82 (1979), 43-57.

[2] F. Echenique, A short and constructive proof of Tarski's fixed-point theorem, Internat. J. Game Theory 33 (2005), 215-218.

[3] R. Sine, Hyperconvexity and approximate fixed points, Nonlinear Anal. 13 (1989), 863-869.

[4] R. E. Smithson, Fixed points of order preserving multifunctions, Proc. Amer. Math. Soc. 28 (1971), 304-310.

[5] P. M. Soardi, Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73 (1979), 25-29.

[6] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific J. Math. 5 (1955), 285-309.

[7] X. Vives, Nash equilibrium with strategic complementarities, J. Math. Econom. 19 (1990), 305-321.

[8] L. Zhou, The set of Nash equilibria of a supermodular game is a complete lattice, Games Econom. Behav. 7 (1994), 295-300.