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COARSELY EMBEDDABILITY AND PROPERTY (T) OF TOPOLOGICAL GROUPS

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Abstract

In this paper, we study the relationship between the coarsely embeddability and Property (T) of a topological group G. We show that a topological group G with a metric d has not Property (T) if G is coarsely embedded into a Hilbert space and $\sup_{g, h \in G}(d(g, h)) = \infty$. We also prove that a topological group G with Property (T) is a bounded metric group for every metric if G is coarsely embedded into a Hilbert space.

1. Introduction

In the mid 60's, Kazhdan defined the following Property (T) for locally compact groups and used this to prove that a large class of lattices is finitely generated.

Definition 1.1 [3]**.** A topological group G has Property (T) if there exist a compact subset Q and a real number $\varepsilon > 0$ such that, whenever π is a continuous

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unitary representation of G on a Hilbert space H for which there exists a vector $\xi \in H$ of norm 1 with $\sup_{q \in Q} \| \pi(q)\xi - \xi \| < \varepsilon$, then there exists an invariant vector, namely, a vector $\eta \neq 0$ in H such that $\pi(g)\eta = \eta$ for all $g \in G$.

In [4], for a locally compact group G, if G has Property (T), then G is compactly generated. In particular, a discrete group G with Property (T) is finitely generated. We also know that a locally compact group G with Property (T) is amenable if and only if G is compact.

In [7], Yu defined "Property A" which can be regarded as a generalization of amenability for discrete spaces with bounded geometry.

Definition 1.2 [7]**.** A discrete metric space Γ is said to have *Property A* if for any $r > 0$, $\varepsilon > 0$, there exists a family of finite subsets $\{A_\gamma\}_{\gamma \in \Gamma}$ of $\Gamma \times N$ (N is the set of all natural numbers) such that

(1) $\gamma \times 1 \in A_\gamma$ for all $\gamma \in \Gamma$;

(2) $\dfrac{\sharp(A_\gamma - A_{\gamma'}) + \sharp(A_{\gamma'} - A_\gamma)}{\sharp(A_\gamma \cap A_{\gamma'})} < \varepsilon$ for all γ and $\gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq r$,

where for each finite set A, $\sharp A$ is the number of elements in A;

(3) $\exists R > 0$ such that if $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$ for some $\gamma \in \Gamma$, then $d(x, y) \leq R$.

Therefore, for a locally compact discrete group with a word length metric, amenability implies Property (A) and Property (A) can imply coarsely embedding into a Hilbert space.

We will naturally ask: what is the equivalent condition of a group G with Property (T) being coarsely embedding into a Hilbert space?

In order to solve above questions, we introduce theory of coarse embeddings.

Coarse embeddings were defined by Gromov [2] to express the idea of inclusion in the large scale geometry of groups.

Definition 1.3 [2]**.** Let X, Y be metric spaces. A (not necessarily continuous function) $f : X \to Y$ is a coarse embedding if there exist non-decreasing functions

$\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying

(1) $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$,

(2) $\lim_{t\to\infty} \rho_1(t) = \infty$.

Yu showed later that the case when a finitely generated group with a word length metric is being embedded into the Hilbert space is of great importance in solving the Novikov Conjecture [7]. Nowak generalized characterizations of coarse embeddability into l_2 in terms of negative definite functions from locally finite to general metric space [5].

Due to relationship among coarse embeddability, negative definite functions and Property (T), in this paper, we study Property (T) of topological groups by using the connection between negative definite functions and coarse embeddings of topological groups, and then solve above questions partially.

2. Preliminaries

We first introduce some of basic notations and terminologies, the details can be found in [4].

Definition 2.1. Let π be an orthogonal representation of the topological group G on a real Hilbert space H^0.

(1) A continuous mapping $b : G \to H^0$ such that

$$b(gh) = b(g) + \pi(g)b(h), \quad \text{for all } g, h \in G$$

is called a 1-*cocycle* with respect to π.

(2) A 1-cocycle $b : G \to H^0$ for which there exists $\xi \in H^0$ such that $b(g) = \pi(g)\xi - \xi$, for all $g \in G$ is called a 1-*coboundary* with respect to π.

(3) The space $Z^1(G, \pi)$ of all 1-cocycles with respect to π is a real vector space under the pointwise operations, and the set $B^1(G, \pi)$ of all 1-coboundaries is a subspace of $Z^1(G, \pi)$. The quotient vector space $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$ is called the *first cohomology group* with coefficients in π.

(4) Let $b \in Z^1(G, \pi)$. The affine isometric action associated to a cocycle $b \in Z^1(G, \pi)$ is the affine isometric action α of G on H defined by

$$\alpha(g)x = \pi(g)x + b(g), \quad g \in G, x \in H,$$

where H is the canonical affine Hilbert space associated to H^0.

Definition 2.2. A continuous real valued kernel Ψ on a topological space X is conditionally of negative type if $\Psi(x, x) = 0$, $\Psi(x, y) = \Psi(y, x)$, for all $x, y \in X$, and $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Psi(x_i, x_j) \leq 0$, for any elements $x_1, x_2, ..., x_n$ in X, and any real numbers $c_1, c_2, ..., c_n$ with $\sum_{i=1}^{n} c_i = 0$. A continuous real value function ψ on a topological group G is conditionally of negative type if the kernel on G defined by $(g, h) \mapsto \psi(h^{-1}g)$ is conditionally of negative type.

Example 2.3. Let G be a topological group and α be an affine isometric action of G on a real Hilbert space H, according to [4, Example C.2.2 ii], for any $\xi \in H$, the function

$$\psi : G \to R, \; g \to \| \alpha(g)\xi - \xi \|^2$$

is conditionally of negative type.

In particular, for any orthogonal representation π on H and for any $b \in Z^1(G, \pi)$, the function $g \mapsto \| b(g) \|^2$ is conditionally of negative type.

Definition 2.4. A topological group G has Property (FH) if every affine isometric action of G on a real Hilbert space has a fixed point.

The following theorem describes connection among bounded functions conditionally of negative type, Property (FH) and cohomology groups.

Theorem 2.5 [4]**.** *Let G be a topological group. The following statements are equivalent*:

(1) $H^1(G, \pi) = 0$ *for every orthogonal representation* π *of G*,

(2) *G has Property* (*FH*),

(3) *every function conditionally of negative type on G is bounded*.

Corollary 2.6 [4]**.** *Let G be a topological group. If G has Property* (T)*, then every function conditionally of negative type on G is bounded.*

Proof. By virtue of [4, Theorem 2.12.4]: if G has Property (T), then G has Property (FH) and Theorem 2.5, it is obvious. □

3. Coarse Embeddings of Topological Group and Property (T)

The following theorem will show the connection between coarse embeddability of topological groups and Property (T).

Theorem 3.1. *A topological group G with a metric d admits a coarse embedding into a Hilbert space, and* $\sup_{g, h \in G}(d(g, h)) = \infty$, *then G has not Property* (T).

We will apply the following lemma to prove above theorem:

Lemma 3.2 [5]**.** *A metric space X admits a coarse embedding into a Hilbert space if and only if there exist a conditionally of negative type Ψ on X and non-decreasing functions* $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2,$ *satisfying*

(1) $\rho_1(d_X(x, y)) \leq \Psi(x, y) \leq \rho_2(d_X(x, y))$ *for all* $x, y \in X$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty.$

Corollary 3.3. *A topological group G admits a coarse embedding into a Hilbert space if and only if there exist a function conditionally of negative type* $\psi : G \to H$ *and non-decreasing functions* $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2,$ *satisfying*

(1) $\rho_1(d_G(h, g)) \leq \psi(h^{-1}g) \leq \rho_2(d_X(h, g))$ *for all* $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty.$

Proof. We know that a continuous real value function ψ on a topological group G is conditionally of negative type if the kernel on G defined by $(g, h) \mapsto \psi(h^{-1}g)$ is conditionally of negative type, we denote $\psi(h^{-1}g) = \Psi(h, g)$, according to Lemma 3.2, the result is obvious. □

Proof of Theorem 3.1. Suppose G has Property (T), according to Corollary 2.6, every function conditionally of negative type on G is bounded. According to condition, G admits a coarse embedding into a Hilbert space, with respect to Corollary 3.3, there exist a function conditionally of negative type $\psi : G \to H$ and non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, satisfying

(1) $\rho_1(d_G(h, g)) \leq \psi(h^{-1}g) \leq \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty$,

with respect to $\sup_{g, h \in G}(d(g, h)) = \infty$ and Corollary 3.3 (2), function conditionally of negative type ψ is an unbounded function on G. Contradiction. □

Theorem 3.4. *A topological group G admits a coarse embedding into a Hilbert space and has Property* (*T*), *then G is a bounded metric group for every metric*.

Proof. With respect to Corollary 2.6, every function conditionally of negative type on G is bounded. According to condition, G admits a coarse embedding into a Hilbert space, with respect to Corollary 3.3, there exist a function conditionally of negative type $\psi : G \to H$ and non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, satisfying

(1) $\rho_1(d_G(h, g)) \leq \psi(h^{-1}g) \leq \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty$. Therefore, $\sup_{g, h \in G} d(g, h) < \infty$ for every metric d on G. □

Example 3.5. The groups R^n, Z^n do not have Property (T).

Example 3.6. The integer Heisenberg group $H = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ does not have Property (T).

In fact, according to [1, Page 32, Example and Theorem 43], H admits a coarse embedding into a Hilbert space. It is obvious that H with word length metric and $\sup_{g \in G}(l(g)) = \infty$. Hence, H does not have Property (T). □

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