JP Journal of Algebra, Number Theory and Applications
Volume 13, Number 2, 2009, Pages 209-219
Published Online: May 22, 2009
This paper is available online at http://www.pphmj.com

REMARKS ON THE BORDIGA SCROLLS OF DEGREE TEN

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Abstract

Let $\mathcal{U}$ be a very ample vector bundle of rank 2 on $\mathbb{P}^2$ with $c_1(\mathcal{U}) = 4$ and $c_2(\mathcal{U}) = 6$. The associated scroll (X, H), a Bordiga scroll of degree 10, turns out to be isomorphic to the blowing-up of $\mathbb{P}^3$ along a twisted cubic C. Then the jumping lines and conics of $\mathcal{U}$ can be described in a very easy and explicit way through the geometry of C.

0. Introduction

Rational surfaces polarized by a very ample line bundle whose adjunction theoretic reduction is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ are known as Bordiga surfaces. They have degree d, $6 \leq d \leq 16$, and sectional genus 3. For $7 \leq d \leq 10$ such surfaces occur as hyperplane sections of threefolds which are scrolls over $\mathbb{P}^2$ [8, Proposition 1.3]. We refer to them as Bordiga scrolls. Let (X, H) be such a scroll. Then $X = \mathbb{P}(\mathcal{U})$, where $\mathcal{U}$ is a very ample vector bundle of rank 2 on $\mathbb{P}^2$ and H is the tautological line bundle on X. Moreover, $c_1(\mathcal{U}) = 4$ and $c_2(\mathcal{U}) = 16 - d$. Recently, in connection

2000 Mathematics Subject Classification: Primary 14C20; Secondary 14J40, 14F05.

Keywords and phrases: vector bundles on the projective plane, jumping line, jumping conic, scroll, Fano threefold.

Received January 12, 2009

with results on ample vector bundles with a section vanishing on a projective manifold of sectional genus 3 [10], case $d = 10$ attracted the attention of Maeda [11]. He proved [11, Theorem 3] that if $c_2(\mathcal{U}) = 6$, then X is isomorphic to $\mathbb{P}^3$ blown-up along a twisted cubic C. This fact suggests an unexpected way to translate geometric properties of C in terms of the rank-2 vector bundle $\mathcal{U}$. In particular, it allows us to describe the jumping lines as well as the jumping conics of $\mathcal{U}$ in a very precise way (see Theorems (2.3) and (2.6)). The argument simply involves quadric surfaces through C and quartic surfaces singular along C, respectively. From this point of view, this paper can be regarded as a supplement to [11].

Varieties are always assumed to be defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. Tensor products of line bundles are denoted additively. The pullback of a vector bundle $\mathcal{F}$ on a projective variety Y by an embedding $Z \hookrightarrow Y$ is denoted by $\mathcal{F}_Z$. We denote by K_Y the canonical bundle of a smooth variety Y.

Let L be a very ample line bundle on a smooth projective variety Y and let $V \subseteq H^0(Y, L)$ be a subspace of sections providing an embedding $\varphi_V : Y \to \mathbb{P}^N = \mathbb{P}(V)$ (meant as the set of codimension 1 vector subspaces of V). Let $J_1(L)$ be the first jet bundle of L and

$$j_1 : V \otimes \mathcal{O}_Y \to J_1(L) \tag{0.1}$$

the sheaf homomorphism associating to every section $s \in V$ its 1-jet $j_1(s)(x)$ evaluated at x for every $x \in Y$.

Consider (0.1) in the special case $Y = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $V = H^0(Y, L)$. Note that j_1 is surjective, due to the very ampleness of L; moreover, $\mathrm{Ker}(j_{1,x}) = 0$ for every $x \in \mathbb{P}^n$ because $|L - x|$ does not contain singular elements. It follows that j_1 is an isomorphism of vector bundles. So we have

$$J_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}. \tag{0.2}$$

1. Reconstructing the Scroll from the Twisted Cubic

Let $\mathcal{U}$ be a very ample vector bundle of rank 2 on $\mathbb{P}^2$ with Chern classes $c_1(\mathcal{U}) = 4$, $c_2(\mathcal{U}) = 6$, let (X, H) be as in the introduction and let $\pi : X \to \mathbb{P}^2$ be the scroll projection. According to [11, Theorem 3], X is the blow-up of $\mathbb{P}^3$ along a twisted cubic C. Let $\sigma : X \to \mathbb{P}^3$ be the blowing-up. For every $x \in \mathbb{P}^3 \setminus C$ there exists a unique secant line λ_x to C passing through x. Let p, q be the (distinct or coinciding) points, where λ_x meets C. The unordered pair $\{p, q\}$ defines a point $\psi(x) \in C^{(2)}$, the symmetric product of C with itself. Note that $C^{(2)} \cong \mathbb{P}^2$, since $C \cong \mathbb{P}^1$. Therefore $x \mapsto \psi(x)$ defines a rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, which is a morphism on $\mathbb{P}^3 \setminus C$. Of course ψ is not defined on C because for every $x \in C$ there are infinitely many secant lines to C passing through x. Note also that for $x \in \mathbb{P}^3 \setminus C$, we have $\psi(y) = \psi(x)$ for all $y \in \lambda_x \setminus C$. What we said proves the following fact.

Remark (1.1). The scroll projection $\pi : X \to \mathbb{P}^2$ is the resolution of the indeterminacies of the rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Moreover, every fiber of π is the proper transform of a secant or a tangent line to C via σ.

(1.2) Let $E = \sigma^{-1}(C)$ be the exceptional divisor of the blowing-up $\sigma : X \to \mathbb{P}^3$. We have $E = \mathbb{P}(N_{C/\mathbb{P}^3})$, where $N_{C/\mathbb{P}^3}$ is the normal bundle of $C \subset \mathbb{P}^3$. Recall that

$$N_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}, \qquad (1.2.1)$$

since C lies on a quadric cone (e.g., see [6, Corollary 2.2]). Hence $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Remark (1.1) implies that $\pi|_E : E \to \mathbb{P}^2$ is a morphism of degree 2, ramified along a section of the $\mathbb{P}^1$-bundle $E \to C$. The branch divisor of $\pi|_E$ is the curve $G \subset \mathbb{P}^2$, which is the image of C in its symmetric product $C^{(2)} = \mathbb{P}^2$ via the diagonal map $C \to C \times C$ and the projection to $C^{(2)}$. Hence G is a smooth conic. Describing $C \subset \mathbb{P}^3$ as the locus of points $(t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3)$, where $(t_0 : t_1) \in \mathbb{P}^1$ and recalling that the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{1(2)} = \mathbb{P}^2$ is given by

$$((t_0 : t_1), (s_0 : s_1)) \mapsto (t_0 s_0 : t_0 s_1 + t_1 s_0 : t_1 s_1),$$

we see that G consists of points $(t_0^2 : 2t_0 t_1 : t_1^2)$. In other words, we can choose homogeneous coordinates $(x_0 : x_1 : x_2)$ on $\mathbb{P}^2$ such that G has equation

$$x_1^2 - 4x_0 x_2 = 0. \tag{1.2.2}$$

Sometimes we will refer to G as the *fundamental conic* of (X, H).

(1.3) Now set $L = \sigma^* \mathcal{O}_{\mathbb{P}^3}(1)$, $M = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. Clearly, both L and M are nef. Moreover, M^2 is the class of a fiber of π, hence

$$M^2 E = 2 \tag{1.3.1}$$

by what we said in (1.2). Recalling [9, Lemma 2.2.14, (b)] and (1.2.1) we have also

$$L^3 = 1, \quad L^2 E = 0, \quad LE^2 = -3, \quad E^3 = -10. \tag{1.3.2}$$

Furthermore,

$$L \mathfrak{f} = 0 \quad \text{and} \quad E \mathfrak{f} = -1 \tag{1.3.3}$$

for every fiber $\mathfrak{f}$ of $E \to C$.

Taking into account the two structures of X deriving from π and σ, we have that $\mathrm{Pic}(X) \cong \mathbb{Z} \times \mathbb{Z}$, two bases being given by $\{L, E\}$ and $\{H, M\}$, respectively.

Lemma (1.4). *The two bases above are related as follows*:

$$H = 3L - E, \qquad M = 2L - E.$$

Proof. We can write $H = aL - bE$ and $M = xL - yE$ for some integers a, b, x, y. Consider the canonical bundle K_X. Due to the scroll structure of $(X = \mathbb{P}(\mathcal{U}), H)$, we have $K_X + 2H = \pi^*(K_{\mathbb{P}^2} + \det \mathcal{U}) = M$. Thus

$$K_X = (x - 2a)L + (2b - y)E.$$

On the other hand, due to the blowing-up σ, we have

$$K_X = \sigma^* K_{\mathbb{P}^3} + E = -4L + E.$$

Comparing the two expressions above gives $x = 2(a - 2)$, $y = 2b - 1$; hence $M = 2(a - 2)L - (2b - 1)E$. To compute a and b we use (1.3.1) combined with (1.3.2) to get

$$2 = M^2 E = (2(a - 2)L - (2b - 1)E)^2 E = 2(2b - 1)(6a - 10b - 7). \quad (1.4.1)$$

As we said M is nef, hence $M\mathfrak{f} \geq 0$ for every fiber $\mathfrak{f}$ of $E \to C$. Taking into account (1.3.3), this shows that $2b - 1 \geq 0$. Thus (1.4.1) implies $2b - 1 = 6a - 10b - 7 = 1$, since we are interested only in integral solutions. This gives $a = 3$, $b = 1$ and therefore $x = 2$, $y = 1$. □

Remarks (1.5). (i) By subtracting the two relations provided by Lemma (1.4) we get $L = H - M$. It follows that L can be seen as the tautological line bundle of $\mathcal{U}(-1)$, hence $\mathcal{U}(-1)$ is nef, so being L.

(ii) Let $\mathcal{J}_C$ be the ideal sheaf of $C \subset \mathbb{P}^3$. From the exact sequence

$$0 \to \mathcal{J}_C(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0$$

we get $h^0(\mathbb{P}^3, \mathcal{J}_C(2)) = 3$. Thus by Lemma (1.4), taking the proper transform via σ induces a bijection

$$|\mathcal{J}_C(2)| \leftrightarrow |M| = \pi^*|\mathcal{O}_{\mathbb{P}^2}(1)|. \quad (1.5.1)$$

For any line $\ell \subset \mathbb{P}^2$ set $M_\ell := \pi^{-1}(\ell)$. Then M_ℓ is the proper transform via σ of a quadric surface of $\mathbb{P}^3$ containing C.

2. The Role of the Tangent Developable Detecting Jumping Lines and Conics

(2.1) Look at the twisted cubic C as the image of the embedding $\varphi : \mathbb{P}^1 \to \mathbb{P}^3 = \mathbb{P}(V)$, where $V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and let $\mathrm{Tan}(C) \subset \mathbb{P}^3$ be the tangent ruled developable to C. As is known, $\mathrm{Tan}(C)$ is a surface of degree 4, singular exactly along C, and for which C is a locus of cusps. Consider $T := \sigma^{-1}(\mathrm{Tan}(C))$ and note that $\pi(T) = G$ is the fundamental conic defined in (1.2). Moreover, $T = \mathbb{P}(\mathcal{U}_G)$ and the scroll projection π of (X, H) induces a scroll structure on (T, H_T). Note also that T is the (smooth) normalization of $\mathrm{Tan}(C)$ via $\sigma|_T : T \to \mathrm{Tan}(C)$. Now, look at (0.1) with $X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(3)$ and $V = H^0(X, L)$. Recall that the projective tangent line to C at the point $p = \varphi(x)$ $(x \in \mathbb{P}^1)$ is just $\mathbb{P}(\mathrm{Im}(j_{1,x}))$, where j_1 is the sheaf homomorphism (0.1) with the present data. By taking projectivizations we thus see that (0.1) induces a birational morphism $\mathbb{P}(J_1(\mathcal{O}_{\mathbb{P}^1}(3))) \to \mathrm{Tan}(C)$ and by the universal property of the normalization we conclude that $T = \mathbb{P}(J_1(\mathcal{O}_{\mathbb{P}^1}(3)))$.

Lemma (2.2). $J_1(\mathcal{O}_{\mathbb{P}^1}(3)) = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$; *in particular*, T *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The assertion follows from (0.2) and [4, Lemma 1.2]. □

Now, let us come to the jumping lines of $\mathcal{U}$. For any line $\ell \subset \mathbb{P}^2$, we have $\mathcal{U}_\ell = \mathcal{O}_\ell(2)^{\oplus 2}$ or $\mathcal{O}_\ell(3) \oplus \mathcal{O}_\ell(1)$ according to whether ℓ is general or a jumping line. As in Remark (1.5)(ii), let M_ℓ be the proper transform of a quadric surface $Q \in |\mathcal{J}_C(2)|$ via σ.

Theorem (2.3). *Let* $\ell \subset \mathbb{P}^2$ *be any line. Then the following facts are equivalent*:

(1) ℓ *is a jumping line of* $\mathcal{U}$;

(2) *the quadric* $Q \in |\mathcal{J}_C(2)|$ *corresponding to* M_ℓ *is a quadric cone*;

(3) ℓ *is tangent to the fundamental conic* G.

Proof. Set $M_\ell = \pi^{-1}(\ell)$ as in Remark (1.5). Then $M_\ell = \mathbb{P}(\mathcal{U}_\ell)$. Hence $M_\ell \cong \mathbb{F}_e$, the Segre-Hirzebruch surface of invariant e, where $e = 2$ or 0 according to whether ℓ is jumping line or not. On the other hand, by Remark (1.5) these circumstances depend on the fact that the quadric $Q \in |\mathcal{J}_C(2)|$ corresponding to M_ℓ is a quadric cone or a smooth quadric. This proves the equivalence of (1) and (2). Next, note that if $Q \in |\mathcal{J}_C(2)|$ is a quadric cone, then its vertex p has to lie on C. Hence there is a single line common to Q and $\mathrm{Tan}(C)$: namely, the projective tangent line to C at p. Taking the proper transforms, this means that the two ruled surfaces M_ℓ and T have a single fiber in common. Since $\pi(T) = G$, looking at their bases, this is equivalent to saying that ℓ is tangent to G. On the other hand, if Q is a smooth quadric, then $Q = A \times B$, with $A, B \cong \mathbb{P}^1$ and $C \in |A + 2B|$, up to exchanging the rulings A and B. Thus the morphism $C \to B$ induced by the second projection of Q has two distinct branch points. This means that there are two distinct fibers of the ruling $|A|$ which are tangent to C. Looking at the proper transforms M_ℓ and T and projecting down to $\mathbb{P}^2$ via π this means that the line ℓ intersects G in two distinct points. This proves the equivalence of (2) and (3). □

It turns out from Theorem (2.3) that the jumping lines of $\mathcal{U}$ are parameterized in $\mathbb{P}^{2\vee}$ by the dual conic of G. Note that $h^0(\mathbb{P}^2, \mathcal{U}(-2)) = 0$ as shown in [11, Theorem 1]. Moreover, $\mathcal{U}(-2)$ is normalized, because $c_1(\mathcal{U}(-2)) = 0$. Hence $\mathcal{U}(-2)$ is stable. Then the locus of $\mathbb{P}^{2\vee}$ parameterizing the jumping lines is a conic (a curve of degree $c_2(\mathcal{U}(-2)) = 2$) by a result of Barth [1, Theorem 2]. Theorem (2.3) simply provides a precise description of it.

There are other interesting surfaces contained in X: namely, the elements of $|L|$. Let $P \subset \mathbb{P}^3$ be a plane not tangent to C, let p_1, p_2, p_3 be the distinct points constituting $P \cap C$, and let $\widetilde{P} = \sigma^{-1}(P)$. Note that p_1, p_2, p_3 are not collinear. The composition

$$\theta_P := \pi|_{\widetilde{P}} \circ \sigma^{-1}|_P : P \to \mathbb{P}^2$$

is the quadratic transformation blowing-up p_1, p_2, p_3 and contracting the proper transforms of the secants $\langle p_i, p_j\rangle$. Set $x_k = \pi(\sigma^{-1}(\langle p_i, p_j\rangle))$, where $k \in \{1, 2, 3\}$ is distinct from i and j. Looking, e.g., at p_1 we note that the secants $\langle p_1, p_2\rangle$ and $\langle p_1, p_3\rangle$ are two generators of the quadric cone $Q_1 \in |\mathcal{J}_C(2)|$ of vertex p_1. Then the line $\ell_1 = \langle x_2, x_3\rangle$ is the base curve of the proper transform M_{ℓ_1} of Q_1. It thus follows from Theorem (2.3) that ℓ_1 is tangent to the fundamental conic G. Of course the same is true for the lines ℓ_2 and ℓ_3. Hence, we have

Remark (2.4). For a general plane $P \subset \mathbb{P}^3$, the triangle of $\mathbb{P}^2$ generated by the quadratic transformation θ_P is circumscribed to G.

(2.5) Triangles circumscribed to G enter also in the description of the jumping conics of $\mathcal{U}$. First of all, looking at $\mathcal{U}(-2)$ instead of $\mathcal{U}$, we know from [12, Theorem I.8] that the locus J of jumping conics is a quadric hypersurface in $\mathbb{P}^5 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$. In particular, J contains the 3-fold parameterizing reducible conics with one component at least tangent to G, and cuts the Veronese surface of conics of rank 1 exactly along the curve parameterizing those supported on a tangent line to G, counted twice [12, Lemma I.3 and Corollary I.6]. Moreover, J contains irreducible conics [12, Corollary I.9]. Note also that G is not a jumping conic because $\mathcal{U}_G = \mathcal{O}_G(2)^{\oplus 2}$ by Lemma (2.2). To understand which condition irreducible jumping conics have to satisfy we focus on the linear system $|2M|$.

By Lemma (1.4) we know that $2M = 4L - 2E$, so we have to look at the linear system $|\mathcal{J}_C^2(4)|$ of quartic surfaces $F \subset \mathbb{P}^3$ having C as a double curve. Clearly, taking the proper transform via σ defines an injection $|\mathcal{J}_C^2(4)| \hookrightarrow |2M|$. Note that $h^0(2M) = h^0(\pi^*\mathcal{O}_{\mathbb{P}^2}(2)) = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, by the projection formula. Moreover, there is an isomorphism

$$H^0(\mathbb{P}^3, \mathcal{J}_C^2(4)) \cong \mathrm{Sym}^2(H^0(\mathbb{P}^3, \mathcal{J}_C(2)))$$

([3, Lemma 6.4]). Taking into account (1.5.1), this fact says that the above injection is a bijection and

$$|2M| = |\pi^*\mathcal{O}_{\mathbb{P}^2}(2)| = \pi^*|\mathcal{O}_{\mathbb{P}^2}(2)|.$$

Hence every $N \in |2M|$ is of the form $\pi^{-1}(\gamma)$ for some conic $\gamma \subset \mathbb{P}^2$. So we can write $N = N_\gamma$. Note that $N_\gamma = \mathbb{P}(\mathcal{U}_\gamma)$, hence if γ (i.e., F) is irreducible, then $N_\gamma \cong \mathbb{F}_e$, for some $e \geq 0$.

Irreducible jumping conics are characterized as follows.

Theorem (2.6). *Let $\gamma \subset \mathbb{P}^2$ be any irreducible conic. Then the following facts are equivalent*:

(1) γ *is a jumping conic of* $\mathcal{U}$;

(2) *the quartic surface* $F \in |\mathcal{J}_C^2(4)|$ *corresponding to* N_γ *is contained in a special linear complex*;

(3) γ *and* G *are Poncelet related*, *with* G *as the inner conic* (*i.e.*, *there exist infinitely many triangles inscribed in* γ *and circumscribed to* G).

Proof. Quartic ruled surfaces $F \subset \mathbb{P}^3$ are known since longtime (e.g., see [5, pp. 302-303]). If the double locus of F is a twisted cubic, there are only two possibilities according to whether F admits a line as directrix or not. Let $N_\gamma = \sigma^{-1}(F)$. Looking at F as the projection of a quartic scroll of $\mathbb{P}^5$, we have $N_\gamma \cong \mathbb{F}_e$ with $e = 2$ or 0 accordingly. So γ is a jumping conic if and only if we are in the former case. Note that in both cases the curve representing F in the Grassmannian $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ of lines of $\mathbb{P}^3$ is contained in a hyperplane, being a rational curve of degree 4. This exactly means that F is contained in a linear complex. But this complex is special only in the former case [5, pp. 48-50]. This gives the equivalence of (1) and (2). The equivalence of (2) and (3) is proven in [2, Section 6]. □

Remark (2.7). Fix homogeneous coordinates in $\mathbb{P}^2$ such that G is represented by (1.2.2), and suppose that γ has equation

$$\sum a_{ij}x_ix_j = 0 \ (a_{ij} = a_{ji}).$$

Then the condition that γ is Poncelet related with G with the latter as the inner conic is expressed by the following relation [2, Section 6], deriving also from a classical result of Cayley [7]:

$$a_{11}(a_{11} + 2a_{02}) - 4a_{01}a_{12} + a_{00}a_{22} = 0.$$

Therefore this is the equation describing the quadric hypersurface $J \subset \mathbb{P}^5$ introduced in (2.5) with homogeneous coordinates

$$(a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22}).$$

3. A Final Remark

According to Remark (1.1), σ restricted to every fiber of π is an isomorphism. Thus it is easy to see that the morphism $(\pi, \sigma) : X \to \mathbb{P}^2 \times \mathbb{P}^3$ is an embedding. On the other hand, let $\mathcal{X}$ be any projective manifold endowed with an ample vector bundle $\mathcal{E}$ of rank ≥ 2 and an ample line bundle $\mathcal{H}$ such that $\mathcal{E}$ has a global section vanishing scheme-theoretically on X with $\mathcal{H}_X = H$. Then

$$(\mathcal{X}, \mathcal{E}, \mathcal{H}) = (\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(1, 1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(1, 1))$$

by [10, Theorem 1]. In this setting, X can be regarded as a general threefold section of $\mathbb{P}^2 \times \mathbb{P}^3$ Segre embedded in $\mathbb{P}^{11}$, while π and σ, the two Mori contractions of X, are just the morphisms induced by the two projections. Moreover, $\mathcal{U}$ fits into an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \to \mathcal{U} \to 0.$$

In terms of extendability of our original pair (X, H), [10, Theorem 1 and Theorem 2, case (7)] imply the following fact.

Proposition (3.1). *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety whose general threefold section with a linear space is X, with $H = (\mathcal{O}_{\mathbb{P}^N}(1))_X$. Then Y is either $\mathbb{P}^2 \times \mathbb{P}^3$ Segre embedded in $\mathbb{P}^{11}$ or a general hyperplane section of it. Moreover, in the latter case, $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2})$, where $T_{\mathbb{P}^2}$ is the tangent bundle.*

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