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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A KIND OF FOURTH-ORDER BOUNDARY VALUE PROBLEMS

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Abstract

This paper considers the existence and uniqueness of solutions for the fourth-order two point boundary value problem

$$\begin{cases} \dfrac{d^4u}{dx^4} = f(x, u, u', u'') + e(x), \ 0 < x < 1, & (1) \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, & (2) \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is the function satisfying Carathéodory condition and $e(x) \in L^1[0, 1]$. With the condition of the nonlinear increasing function and Leray-Schauder principle, the existence and uniqueness of solutions of a kind of fourth-order boundary value problems are discussed.

1. Introduction

The static elastic beam can be described by a fourth-order boundary value problem. Since the both sides of the support beam have different conditions, there exist various boundary value problems. We consider the

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problem about the static elastic beam with one simple support and one sliding support.

Some results of the existence and uniqueness for the boundary value problem (1)-(2) can be found in [1, 3-4]. In this paper, we discuss the existence and uniqueness for the boundary value problem (1)-(2).

2. The Main Result

In this paper, the following normed spaces are considered: $C[0,1]$, $C^1[0, 1]$, $C^2[0, 1]$, $L^1[0, 1]$ and $L^\infty[0, 1]$. $\|\cdot\|_1$ and $\|\cdot\|_2$ denote usual $L^1[0, 1]$ and $L^2[0, 1]$ norms, respectively. Let $\omega^{4,1}(0, 1)$ be the Sobolev space which consists of such kinds of functions $y : [0, 1] \to \mathbb{R}$, where y', y'' and y''' are all absolutely continuous on $[0, 1]$ and $\frac{d^4 y}{dx^4} \in L^1[0, 1]$. We use the following lemmas:

Lemma 2.1 [2]**.** *If* $y \in L^1[0, 1]$ *and* $y(0) = 0$ *or* $y(1) = 0$, *then*

$$\| y \|_2^2 \le \frac{4}{\pi^2} \| y' \|_2^2. \tag{3}$$

Lemma 2.2. *Suppose* $e(x) \in L^1[0, 1]$. *Then there exists only one solution for the linear boundary value problem*

$$\frac{d^4 y}{dx^4} = e(x), \quad x \in [0, 1] \tag{4}$$

with

$$y(0) = y'(1) = y''(0) = y'''(1). \tag{5}$$

It is easy to show that $u = u(x)$ is the unique solution of boundary value problem (4)-(5), where

$$u(x) = \int_0^x (x-t)^3 e(t)dt + \frac{1}{6} Ax^3 + Bx,$$

$$A = -\int_0^1 e(t)dt,$$

$$B = \frac{1}{2} \int_0^1 e(t)dt - \int_0^1 (1-t)^2 e(t)dt.$$

The main result of the paper is as follows:

Theorem 2.3. *Suppose functions* $f, h, g : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ *satisfy the Carathéodory condition and* $e \in L^1[0, 1]$. *If* f *has the following decomposition*:

$$f(x, y, z, u) = h(x, y, z, u) + g(x, y, z, u)$$

with

(i) *for any* $(x, y, z, u) \in [0, 1] \times \mathbb{R}^3$ *and* $\alpha, \beta, \gamma \in (0, 1)$, $m \in L^\infty[0, 1]$,

$$| h(x, y, z, u) | \le m(| y |^\alpha + | z |^\beta + | u |^\gamma), \tag{6}$$

(ii) *for* $a, b, c, d \in L^\infty[0, 1]$, $\gamma \in [1, 2)$,

$$yg(x, y, z, u) \le ay^2 + b| yz | + c| yu | + d| y |^\gamma, \tag{7}$$

(iii) *there exist* L^2*-Carathéodory condition function* $w : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *and function* $\gamma(x) \in L^1[0, 1]$ *such that*

$$| f(x, y, z, u) | \le w(x, y, z)| u | + \gamma(x), \tag{8}$$

for any $y, z, u \in \mathbb{R}$ *and almost all* $x \in [0, 1]$, *and*

$$4\| a \|_\infty + 2\pi\| b \|_\infty + \pi^2\| c \|_\infty < \frac{\pi^2 - 8}{4}, \tag{9}$$

then there exists a unique solution for boundary value problem (1)-(2) *in* $C^2[0, 1]$.

3. The Proof of Theorem 2.3

Let $X = C^2[0, 1]$, $Y = L^1[0, 1]$ and $D(L) = \{u \mid u \in \omega^{4,1}(0, 1), u(0) = u'(1) = u''(0) = u'''(1) = 0\}$. Then $D(L) \subset X$. We define linear operator $L : D(L) \to Y$ by

$$L(u) = \frac{d^4u}{dx^4}, \quad u \in D(L)$$

and define nonlinear operator $N : X \to Y$ by

$$N(u)(x) = f(x, u, u', u''), \quad x \in (0, 1).$$

Let $K = L^{-1}$. Since L is one-to-one linear operator, by Lemma 2.2 and the Arzéla-Ascoli's theorem, it follows that KN is a continuous operator. By the theorem of Leray-Schauder, we can prove that the boundary value problem (1)-(2) has a solution in $C^2[0, 1]$. For this, we need only to obtain all the possible solutions of

$$\frac{d^4u}{dx^4} = \lambda f(x, u, u', u'') + \lambda e(x), \quad x \in [0, 1] \tag{10}$$

$$u(0) = u'(1) = u''(0) = u'''(1) = 0$$

in $C^2[0, 1]$ which do not depend on any priori estimate of $\lambda \in [0, 1]$.

So, suppose $u(x)$ is the solution with respect to some $\lambda \in [0, 1]$ of equations (1)-(2), we prove that there exists $M > 0$ such that

$$\max\{\| u \|_\infty, \| u' \|_\infty, \| u'' \|_\infty\} \le M.$$

Multiplying (10) by u and integrating them on $[0, 1]$, we have

$$\begin{aligned}\int_0^1 u \frac{d^4u}{dx^4} dx &= \lambda \int_0^1 uf(x, u', u'', u''')dx + \lambda \int_0^1 u(x)e(x)dx \\ &\le \int_0^1 | u || h(x, u', u'', u''') | dx \\ &\quad + \int_0^1 | u || g(x, u', u'', u''') | dx + \int_0^1 | u || e(x) | dx \\ &\le \frac{1}{2}\int_0^1 u^2 dx + \int_0^1 | h(x, u', u'', u''') |^2 dx \\ &\quad + \int_0^1 | u || g(x, u', u'', u''') | dx + \| u \|_\infty \| e(x) \|_1.\end{aligned}$$

In terms of conditions of the theorem, we get

$$\int_0^1 u \frac{d^4u}{dx^4} dx = uu''' |_0^1 - \int_0^1 u'u''' dx = -u'u'' |_0^1 + \int_0^1 u''u'' dx \le \| u'' \|_2^2,$$

$$| h(x, u', u'', u''') | \le m^2(| u |^\alpha, | u' |^\beta, | u'' |^\gamma)^2 \le 4m^2(| u |^{2\alpha}, | u' |^{2\beta}, | u'' |^{2\gamma}).$$

From $\int_0^1 |u(x)|^{2\alpha} dx \le \| u \|_2^{2\alpha}$ and Lemma 2.1, we have

$$\int_0^1 |u||h(x, u, u', u'')| dx$$

$$\le \frac{1}{2}\| u \|_2^2 + 2\| m \|_\infty^2 (\| u \|_2^{2\alpha} + \| u' \|_2^{2\beta} + \| u'' \|_2^{2\gamma})$$

$$\le \frac{1}{2}\left(\frac{4}{\pi^2}\right)^2 \| u'' \|_2^2 + 2\| m \|_\infty^2 \left(\frac{4}{\pi^2}\right)^{2\alpha} \| u'' \|_2^{2\alpha}$$

$$+ \left(\frac{4}{\pi^2}\right)^{\beta} \| u'' \|_2^{2\beta} + \| u'' \|_2^{2\gamma}$$

$$\int_0^1 ug(x, u, u', u'') dx$$

$$\le \| a \|_\infty \int_0^1 u^2 dx + \| b \|_\infty \int_0^1 | uu' | dx$$

$$+ \| c \|_\infty \int_0^1 | uu'' | dx + \| d \|_\infty \int_0^1 | u |^\gamma dx$$

$$\le \| a \|_\infty \| u \|_2^2 + \| b \|_\infty \| u \|_2 \| u' \|_2 + \| c \|_\infty \| u \|_2 \| u'' \|_2 + \| d \|_\infty \| u \|_2^\gamma$$

$$\le \left(\frac{4}{\pi^2}\right)^2 \| a \|_\infty \| u'' \|_2^2 + \left(\frac{2}{\pi}\right)^3 \| b \|_\infty \| u'' \|_2^2$$

$$+ \left(\frac{2}{\pi}\right)^2 \| c \|_\infty \| u'' \|_2^2 + \left(\frac{2}{\pi}\right)^{2\gamma} \| d \|_\infty \| u'' \|_2^\gamma.$$

Then

$$\| u'' \|_2^2 \le \left(\frac{8}{\pi^4} + \frac{16}{\pi^4}\| a \|_\infty + \frac{8}{\pi^3}\| b \|_\infty + \frac{4}{\pi^2}\| c \|_\infty\right) \| u'' \|_2^2 + \left(\frac{2}{\pi}\right)^{2\gamma} \| b \|_\infty \| u'' \|_2^\gamma$$

$$+ 2\| m \|_\infty^2 \left(\left(\frac{4}{\pi^2}\right)^{2\alpha} \| u'' \|_2^{2\alpha} + \left(\frac{4}{\pi^2}\right)^{\beta} \| u'' \|_2^{2\beta} + \| u'' \|_2^{2\gamma}\right) + \frac{4}{\pi^2}\| u'' \|_2 \| e \|_1.$$

For $\alpha, \beta \in (0, 1)$ and $\gamma \in (1, 2)$, by the condition

$$4\| a \|_\infty + 2\pi\| b \|_\infty + \pi^2\| c \|_\infty < \frac{\pi^2 - 8}{4},$$

we know that there exists $M_1 > 0$ such that

$$\| u'' \|_2 < M_1. \tag{11}$$

From (8), (10) and (11), we know that there exists $M > 0$ such that $\left\| \frac{d^4 u}{dx^4} ds \right\| \leq M.$

Then from

$$u''(x) = \int_0^x u'''(t)dt = \int_0^x \int_0^t \frac{d^4 u}{ds^4} dsdt,$$

$$u'(x) = \int_x^1 u''(s)ds, \quad u(x) = \int_0^x u'(s)ds,$$

we get that $\sup\{| u(x) |, | u'(x) |, | u''(x) |\} \leq M.$

Next, if the nonlinear term satisfies conditions in Theorem 2.3 for almost all $x \in [0, 1]$ and for each fixed (δ, u) the function y strictly decreases, then there exists, for the boundary value problem (1)-(2), only one solution in $C^2[0, 1]$.

In fact, if $u_1(x)$ and $u_2(x)$ are all solutions of the boundary value problem (1)-(2), then

$$\frac{d^4 u_1}{dx^4} = f(x, u_1, u_1', u_1'') + e(x), \quad 0 < x < 1,$$

$$u_1(0) = u_1'(1) = u_1''(0) = u_1'''(1) = 0$$

and

$$\frac{d^4 u_2}{dx^4} = f(x, u_2, u_2', u_2'') + e(x), \quad 0 < x < 1,$$

$$u_2(0) = u_2'(1) = u_2''(0) = u_2'''(1) = 0.$$

Since the function f strictly decreases, we have

$$\|(u_1 - u_2)''\|_2 = \int_0^1 \left(\frac{d^4 u_1}{dx^4} - \frac{d^4 u_2}{dx^4}\right)(u_1 - u_2)dx$$

$$= \int_0^1 [f(x, u_1, u_1', u_1'') - f(x, u_2, u_2', u_2'')](u_1 - u_2)dx \leq 0,$$

and so $\|(u_1 - u_2)''\|_2 = 0.$

Note that

$$\sup_{x \in [0, 1]} \{|u_1(x) - u_2(x)|\} \leq \sup_{x \in [0, 1]} \{|u_1'(x) - u_2'(x)|\} \leq \|(u_1 - u_2)''\|_2,$$

and therefore for almost all $x \in [0, 1]$, $u_1(x) = u_2(x)$. Further, since $u_1(x)$ and $u_2(x)$ are all continuous functions, we have $u_1(x) = u_2(x)$ for every $x \in [0, 1]$.

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