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A REMARK ON THE ZERO ORDER HANKEL TRANSFORM

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Abstract

We prove that for $n = 2m$ when m is a positive integer, the zero order Hankel transform of r^n is a Dirac distribution. As an application, we discuss the Green function of the Helmholtz equation in spherical geometry. We also present the Hankel transform of some Bessel and Gaussian functions obtained from Weber integrals of Bessel functions.

1. Introduction

The *zero order Hankel transform* of the function $f(r)$ is defined by the relation [2, 3]

$$F(k) = \int_0^\infty f(r) J_0(kr) r dr \tag{1}$$

with the inverse

$$f(r) = \int_0^\infty F(k) J_0(kr) k dk \tag{1a}$$

and

$$\int_0^\infty J_0(kr) J_0(k'r) r dr = \delta(k - k')/k \tag{2}$$

in which J_0 is the Bessel function of the first kind of order zero and δ is the Dirac distribution.

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The Plancherel theorem states

$$\int_0^\infty f(r)g(r)rdr = \int_0^\infty F(k)G(k)kdk. \tag{3}$$

The relation (1) is valid at every point at which $f(r)$ is continuous provided that the *function* defined on $(0, \infty)$ is piecewise continuous and of bounded variation in every finite subinterval on $(0, \infty)$ and that the integral $\int_0^\infty |f(r)|r^{1/2}dr$ is finite.

An extensive table of zero order Hankel transforms exists [3].

We use here the following definitions of the *Hankel transform*:

$$\underline{H}\{f(r)\} = \lim_{a\Rightarrow 0}\int_0^\infty \exp(-ar)f(r)J_0(kr)rdr, \quad a \geq 0, \tag{4}$$

$$\underline{H}\{F(k)\} = \lim_{q\Rightarrow 0}\int_0^\infty \exp(-qk)F(k)J_0(kr)kdk, \quad q \geq 0. \tag{4a}$$

The expressions (4) and (4a) reduce to (1) and (1a) when limit and integration commute which requires the uniform convergence of these processes.

Let us compare (1) and (4) for $f(r) = r^{\mu-2}$ with μ real. Then we get from (1)

$$F(k) = k^{-\mu}\int_0^\infty J(kt)t^{\mu-1}dt = 2^{\mu-1}k^{-\mu}\Gamma(\mu/2)/\Gamma(1-\mu/2) \quad [6, \text{p. } 351] \tag{5}$$

in which Γ is the function gamma and $0 < \mu < 1/2$.

Now, according to (4) with $a \geq 0$,

$$\underline{H}(r^{\mu-2}) = \lim_{a\Rightarrow 0}\int_0^\infty \exp(-ar)J_0(kr)r^{\mu-1}dr \tag{6}$$

but [6, p. 385]

$$\int_0^\infty \exp(-ar)J_0(kr)r^{\mu-1}dr$$

$$= \Gamma(\mu)(a^2+k^2)^{-\mu/2}\Gamma(\mu)\,{}_2F_1[\mu/2, (1-\mu)/2, 1; k^2(a^2+k^2)^{-1}], \tag{6a}$$

${}_2F_1$ is the hypergeometric function reducing for $a = 0$ to

$$ {}_2F_1(\mu/2,\ (1-\mu)/2,\ 1;\ 1) $$

and [5, p. 161]:

$$ {}_2F_1(\mu/2,\ (1-\mu)/2,\ 1;\ 1) = \Gamma(1/2)/\Gamma(1-\mu/2)\Gamma[(1+\mu)/2]. \tag{6b} $$

Substituting (6a) into (6) and taking into account (6b) gives

$$ \underline{H}(r^{\mu-2}) = k^{-\mu}\Gamma(\mu)\Gamma(1/2)/\Gamma(1-\mu/2)\Gamma[(1+\mu)/2] \tag{7} $$

but [5, p. 35]

$$ \Gamma(\mu) = 2^{\mu-1}\pi^{-1/2}\Gamma(\mu/2)\Gamma[(1+\mu)/2]. \tag{7a} $$

Taking into account (7a), the expression (7) reduces to (5) with no more constraint on μ.

Another interesting example is given by the function

$$ f(r) = J_\nu(r)/r^{\lambda+1} \quad \nu \geq 0 \quad \lambda \text{ real} \tag{8} $$

for which (4) becomes

$$ \underline{H}\{J_\nu(kr)/r^{\lambda+1}\} = \lim_{a\Rightarrow 0}\int_0^\infty \exp(-ar)J_0(kr)J_\nu(kr)r^{-\lambda}dr. \tag{9} $$

It is proved [6, p. 402] that for $0 < \lambda < \nu + 1$, limit and integration commute $\lim\int = \int\lim$ so that (4) reduces to (1) and [6, p. 403]

$$ \underline{H}\{J_\nu(kr)/r^{\lambda+1}\} = A(\lambda,\ \nu)k^{\lambda-1} \tag{10} $$

with

$$ A(\lambda,\ \nu) = \Gamma(\lambda)\Gamma[(\nu-\lambda+1)/2]/2^\lambda\Gamma^2[(\nu+\lambda+1)/2]\Gamma[(\lambda-\nu+1)/2]. \tag{10a} $$

We prove in this note that the Hankel transform of r^{2m} is a Dirac distribution when m is a positive integer.

2. Hankel Transform of the Monomials r^n, n Positive Integer

We start with the Hankel transform of $r^{\mu-2}$ when μ is an arbitrary real positive number which has according to (6), the expression (5) that we may write

$$\underline{H}(r^{\mu-2}) = -2^{\mu} k^{-\mu} \Gamma(\mu/2)/\mu\Gamma(-\mu/2) \tag{11}$$

since

$$\Gamma(1-\mu/2) = -\mu\Gamma(-\mu/2)/2. \tag{11a}$$

For $\mu > 0$, we get from (11), $\lim_{k\Rightarrow 0} \underline{H}(r^{\mu-2}) \Rightarrow \infty$ but $k = 0$ in (6) also gives $[\underline{H}(r^{\mu-2})]_0 \Rightarrow \infty$, so we have

$$\lim_{k\Rightarrow 0} \int = \int \lim_{k\Rightarrow 0}. \tag{12}$$

2.1. Odd integer μ

With $\mu = 2m+1$, $m > 1$, the relation (11) becomes

$$\underline{H}\{r^{2m-1}\} = -2^{2m+1} k^{2m+1} \Gamma(m+1/2)/(2m+1)\Gamma(-m-1/2). \tag{13}$$

Using the relation [1, p. 255]

$$\Gamma(m+1/2) = 1.3.5\ldots(2m-1)\pi^{1/2}/2m \tag{14a}$$

and [5, p. 35] for z non integer

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z), \tag{14b}$$

we get

$$1/\Gamma(-m-1/2) = \sin[(m+3/2)\pi]\Gamma(m+3/2)/\pi \tag{15}$$

with

$$\Gamma(m+3/2) = 1.3.5\ldots(2m-3)\pi^{1/2}/2^{m-1}. \tag{15a}$$

Substituting (14a) and (15) into (13) gives the well known result [2, 3]

$$\underline{H}\{r^{2m-1}\} = A_m k^{-2m-1}, \tag{16}$$

$$A_m = -2^{2m+1}\Gamma(m+1/2)\Gamma(m+3/2)\sin[(m+3/2)\pi]/(2m+1)\pi, \tag{16a}$$

for instance, for $m = 1$ and $m = 2$: $\underline{H}\{r\} = -1/k^3$, $\underline{H}\{r^3\} = 9/k^5$.

2.2. Even integer μ

With $\mu = 2m + 2,\ m \geq 0,$ the relation (11) becomes

$$\underline{H}\{r^{2m}\} = -2^{2m+2}\Gamma(m+1)/2(m+1)k^{2m+2}\Gamma(-m-1) = 0 \qquad (17)$$

since $1/\Gamma(-m-1) = 0.$

Now, according to (12) for $k \Rightarrow 0,\ (17) \Rightarrow \infty$ which suggests that $\underline{H}\{r^{2m}\}$ is a Dirac distribution and in particular for $m = 0,$

$$\underline{H}\{r^0\} = \delta\,(k)/k. \qquad (18)$$

To check this result, we just have to use the dual transform (4a) with $F(k) = \delta(k)/k$ which gives

$$\underline{H}\{\delta(k)/k\} = \lim_{q\Rightarrow 0}\int_0^\infty \exp(-qk)\delta(k)J_0(kr)dk = r^0 \qquad (19)$$

in agreement with (18).

To get $\underline{H}\{r^{2m}\}$ for $m > 0,$ we introduce the differential operator

$$\Delta_k = \partial_k^2 + k^{-1}\partial_k \qquad (20)$$

with the generalization of (12)

$$\Delta_k \int = \int \Delta_k. \qquad (20a)$$

Now, taking into account (6), the relation (18) is

$$\lim_{a\Rightarrow 0}\int_0^\infty \exp(-ar)J_0(kr)r^{\mu-1}dr = \delta(k)/k. \qquad (21)$$

Let us apply Δ_k to (21) : according to the Bessel equation satisfied by $J_0(kr)$ and using the derivatives $\delta'(k) = -\delta(k)/k,\ \delta''(k) = 2\delta(k)/k^2,$ we get

$$\Delta_k J_0(kr) = -r^2 J_0(kr), \quad \Delta_k[\delta(k)/k] = -2\delta''(k)/k. \qquad (22)$$

So, taking into account (22), applying Δ_k to (21) gives

$$\underline{H}\{r^2\} = -2\delta''(k)/k. \qquad (23)$$

To check (23), we still use (4a)

$$\underline{H}\{\delta''(k)/k\} = -2\lim_{q\Rightarrow 0}\int_0^\infty \exp(-qk)\delta''(k)J_0(kr)dk$$

$$= -2\lim_{q\Rightarrow 0}\partial_k^2[\exp(-qk)J_0(kr)]_{k=0}$$

$$= -2[\partial_k^2 J_0(kr)]_{k=0} = r^2 \tag{24}$$

since $\partial_z^2 J_0(z) = -J_0(z) + 1/z\, J_1(z)$ so that $[\partial_z^2 J_0(z)]_{z=0} = -1/2$.

The generalization of this result to r^{2m} is obtained with the iterated operator $\Delta_k^{(2m)}$ such as

$$\Delta_k^{(2m)} J_0(kr) = (-1)^m r^{2m} J_0(kr), \quad \Delta_k^{(2m)}[\delta(k)/k] = N(m)\delta^{(2m)}(k)/k, \tag{25}$$

the first relation (25) is trivial while the second one uses the derivatives of the Dirac distribution $\delta^{(n)}(z) = (-1)^n n!\,\delta(z)/z^n$ for instance,

$$\Delta_k^{(2)}[\delta(k)/k] = 8/3\,\delta^{(4)}(k)/k, \tag{26}$$

the calculation of $N(m)$ for $m > 2$ has still to be made.

Then taking into account (25) with $\Delta_k^{(m)}\int = \int \Delta_k^{(m)}$ gives

$$\underline{H}\{r^{2m}\} = (-1)^m N(m)\delta^{(2m)}(k)/k \tag{27}$$

with the dual

$$\underline{H}\{(-1)^m N(m)\delta^{(2m)}(k)/k\} = r^{2m} \tag{27a}$$

implying according to (4a)

$$(-1)^m N(m)[\partial_k^{(2m)} J_0(kr)]_{k=0} = r^{2m} \tag{28}$$

whose (24) is a particular case.

We get, for instance, from (27) for $m = 2$ taking into account (26)

$$\underline{H}\{r^4\} = 8/3\,\delta^{(4)}(k)/k \tag{29}$$

and from (28)

$$8/3\,[\partial_k^{(4)} J_0(kr)]_{k=0} = r^4 \tag{29a}$$

since

$$\partial_z^{(4)} J_0(z) = J_0(z) - 2J_1(z)/z - 3J_0(z)/z^2 + 6J_1(z)/z^3 \tag{30}$$

with

$$J_0(z) = 1 - z^2/4\,\ldots, \qquad J_1(z) = z/2 - z^3/16\,\ldots \tag{30a}$$

so that $[\partial_z^{(4)} J_0(z)]_{z=0} = 3/8$.

3. Application

Let us consider the Green function of the Helmholtz equation in spherical geometry. Then using polar coordinates, this equation is

$$[\partial_r^2 + r^{-1}\partial_r + k^2]\,g(r, r_0) = -4\pi\delta(r - r_0)/r \tag{31}$$

with the Hankel transform

$$A(k, r_0) + B(k, r_0) + k^2 G(k, r_0) = -4\pi J_0(kr_0) \tag{32}$$

in which

$$A(k, r_0) = \lim_{a \Rightarrow 0} \int_0^\infty r dr \exp(-ar) J_0(kr)\partial_r^2 g(r, r_0),$$

$$B(k, r_0) = \lim_{a \Rightarrow 0} \int_0^\infty dr \exp(-ar) J_0(kr)\partial_r g(r, r_0),$$

$$G(k, r_0) = \lim_{a \Rightarrow 0} \int_0^\infty r dr \exp(-ar) J_0(kr) g(r, r_0). \tag{33}$$

Using integration by parts, a simple calculation gives

$$B(k, r_0) = b(k, r_0) + k\int_0^\infty drg(r, r_0)J_1(kr), \tag{34}$$

$$b(k, r_0) = [g(r, r_0)J_0(kr)]_0^\infty \tag{34a}$$

and

$$A(k,\ r_0) = a_1(k,\ r_0) - a_2(k,\ r) - k\int_0^\infty drg(r,\ r_0)J_1(kr) - k^2G(k,\ r_0) \quad (35)$$

with

$$a_1(k,\ r_0) = [r\partial_r g(r,\ r_0)J_0(kr)]_0^\infty,$$

$$a_2(k,\ r_0) = [g(r,\ r_0)\{J_0(kr) - krJ_1(kr)\}]_0^\infty. \quad (35a)$$

Substituting (34) and (35) into (32) gives

$$a_1(k,\ r_0) - a_2(k,\ r_0) + b(k,\ r_0) = -4\pi J_0(kr_0), \quad (36)$$

and taking into account (34a) and (35a), this equation becomes

$$[r\partial_r g(r,\ r_0)J_0(kr) + krg(r,\ r_0)\,J_1(kr)]_0^\infty = -4\pi J_0(kr_0). \quad (37)$$

Now, according to the right hand side of (31), the solutions of Helmholtz equation in the half spaces $r > r_0$ and $r < r_0$ are obtained by exchanging r and r_0 which suggests

$$g(r,\ r_0) = f(r)J_0(kr_0), \quad r \geq r_0 \quad (38a)$$

$$= f(r_0)J_0(kr), \quad r \leq r_0. \quad (38b)$$

But (38b) implies that $r\partial_r g(r,\ r_0)J_0(kr) + krg(r,\ r_0)J_1(kr) = 0$ since

$$\partial_r J_0(kr)J_0(kr) + kJ_0(kr)J_1(kr) = 0 \quad (39)$$

so taking into account (38a), the relation (37) reduces to

$$\lim_{r\Rightarrow\infty}[r\partial_r g(r,\ r_0)J_0(kr) + krg(r,\ r_0)J_1(kr)] = -4\pi \quad (40)$$

with the solution in which A is an amplitude and Y_0 is the Bessel function of the second kind of order zero

$$f(r) = AY_0(kr). \quad (41)$$

Substituting (41) into (40) and using the Wronskian relation [1]

$$Y_0(kr)J_1(kr) - Y_1(kr)J_0(kr) = 2/\pi kr \quad (42)$$

gives $A = -2\pi^2$ so that finally

$$g(r, r_0) = -2\pi^2 Y_0(kr)J_0(kr_0), \quad r \geq r_0$$

$$= -2\pi^2 Y_0(kr_0)J_0(kr), \quad r \leq r_0. \tag{43}$$

Taking into account (39), a second set of solutions [4] is obtained with the Hankel function $H_0(kr) = J_0(kr) + iY_0(kr)$,

$$g(r, r_0) = 2i\pi^2 H_0(kr)J_0(kr_0), \quad r \geq r_0$$

$$= 2i\pi^2 H_0(kr_0)J_0(kr), \quad r \leq r_0. \tag{44}$$

Hankel transform is a powerful tool to handle problems in cylindrical geometry.

Remark. The first Weber integral [6, p. 394] for $\mathrm{Re.}\,\mu > 0$ is

$$\int_0^\infty J_0(kr)\exp(-p^2r^2)r^{\mu-1}dr = \Phi_\mu(p, k) \tag{45}$$

in terms of the confluent hypergeometric function ${}_1F_1$,

$$\Phi_\mu(p, k) = [\Gamma(\mu/2)/2p^\mu]\exp(-k^2/4p^2)\,{}_1F_1(1-\mu/2, 1; k^2/4p^2) \tag{45a}$$

which the Hankel transform of $\exp(-p^2r^2)r^{\mu-1}$ giving for $\mu = 2$,

$$F\{\exp(-p^2r^2)\} = 1/2\,p^2\exp(-k^2/4p^2). \tag{46}$$

The second Weber integral [6, p. 399] gives for $\mathrm{Re.}\,\mu$,

$$\int_0^\infty J_0(kr)J_0(cr)\exp(-p^2r^2)r^{\mu-1}dr = \Omega_\mu(p, k) \tag{47}$$

in which with $\varpi^2 = k^2 + c^2 - 2kc\cos\phi$,

$$\Omega_\mu(p, k) = [\Gamma(\mu/2)/2\pi p^\mu]\int_0^\pi d\phi\,\exp(-\varpi^2/4p^2)\,{}_1F_1(1-\mu/2, 1; \varpi^2/4p^2). \tag{47a}$$

Hankel transform of the function $J_0(cr)\exp(-p^2r^2)r^{\mu-2}$ taking for $\mu = 2$ the simple form

$$F\{J_0(cr)\exp(-p^2r^2)\} = 1/2p^2\exp[-(k^2 + c^2)/4p^2]I_0(kc/2p^2) \qquad (48)$$

in which I_0 is the modified Bessel function of the first kind of order zero.

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