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A UNIFIED REPRESENTATION THEOREM ON NEW ALGEBRAIC BASES, FOR (CO)INTEGRATED PROCESSES UP TO THE SECOND ORDER

MARIA GRAZIA ZOIA

Dipartimento di Matematica ed Econometria
Università Cattolica
Milano, Italy
e-mail: maria.zoia@unicatt.it

Abstract

The paper establishes a unified representation theorem for (co)integrated processes up to the second order which provides a compact and informative insight into the solution of VAR models with unit roots, and sheds light on the cointegration features of the engendered processes. The theorem is primarily stated by taking a one-lag specification as a reference frame, and it is afterwards extended to cover the case of an arbitrary number of lags via a companion-form based approach. All proofs are obtained by resorting to an innovative and powerful algebraic apparatus tailored to the derivation of the intended results.

1. Introduction

In the wake of Granger's original representation theorem, published in the Eighties (Engle and Granger [6]), the analysis of vector autoregressive – VAR – models with unit roots has risen to a major branch of modern econometrics, whose track bears the mark of Johansen's contributions (Johansen [16, 17, 18]).

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Representation theorems offer a time-series mirror image of the final form of structural models, insofar as they provide closed-forms solution to VAR systems, link the integration order of the engendered solution to the parameter space of the parent model, and bring to the foreground the cointegration relationships inherent in the system.

The development of representation theorems from Granger's seminal work has followed two major directions. The former, aimed at extending the original approach beyond first-order integrated – $I(1)$ – processes, has eventually led to Johansen's well-known results (ibid.) and more recently to Faliva and Zoia's $I(2)$ and unified representation theorems [11, 12]. The latter has addressed the issue of solving VAR systems with unit roots by resorting to *ad hoc* and tailor-made algebraic tooling, such as the Smith-McMillan form (Engle and Yoo [7], Haldrup and Salmon [14], Hansen [15]), Jordan and companion forms (Archontakis [1], Gregoir [13]), partitioned inversion and Laurent expansion about a pole of a matrix-polynomial inverse (Faliva and Zoia [9, 10]).

This paper fits in with the aforementioned framework inasmuch as an overall insight into VAR-model solutions and their (co)integration features is obtained from an innovative formulation of a general representation theorem, via a tailor-made analytical apparatus centred on orthogonal-complement algebra, a noteworthy matrix decomposition, and ad hoc matrix-polynomial inversion formulas about a pole.

The aim of the paper is to provide a unified representation theorem for $I(\upsilon)$ processes with $\upsilon = 1, 2$ capable of shedding light on the integration and cointegration characteristics of the solutions of VAR systems via the closed-form expressions of the parameter matrices involved. A simple-lag VAR model – which can be neatly solved resorting to the algebraic toolkit of the Appendix – is first investigated; reached conclusions are then extended to the case of an arbitrary number of lags by a companion-form based approach.

The article develops as follows: an overall glance at the outcomes of the paper is cast in Section 2; Section 3 establishes a unified representation theorem of new conception for cointegrated processes; proofs rest on an effective algebraic apparatus as devised in the Appendix.

2. Reference Model and Basic Results on VAR Solutions

Let us consider an n-dimensional vector autoregressive (VAR) model specified as follows:

$$\underset{(n,\,n)}{\boldsymbol{A}(L)}\ \underset{(n,\,1)}{\boldsymbol{y}_t} = \underset{(n,\,1),}{\boldsymbol{\varepsilon}_t} \tag{1}$$

where $\boldsymbol{\varepsilon}_t$ is a white noise process,

$$\boldsymbol{A}(L) = \boldsymbol{I} + \boldsymbol{A}_1 L, \quad \boldsymbol{A}_1 \neq \boldsymbol{0} \tag{2}$$

is a linear polynomial in the lag operator L, whose total effect matrix

$$\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{A}_1 \tag{3}$$

has index $\upsilon \leq 2$ and whose characteristic polynomial $\det \boldsymbol{A}(z)$ has a possibly multiple unit-root with all other roots outside the unit circle.

Solving (1) yields

$$\boldsymbol{y}_t = \boldsymbol{N}_\upsilon \boldsymbol{\omega}_1 t^{\upsilon-1} + \sum_{j=1}^{\upsilon} \boldsymbol{N}_j(\boldsymbol{\eta}_{jt} + \boldsymbol{\omega}_j) + \sum_{j=0}^{\infty} \boldsymbol{M}_j L^j, \tag{4}$$

where the $\boldsymbol{M}_j$'s are coefficient matrices with exponentially decreasing entries, the $\boldsymbol{\omega}_j$'s denote arbitrary vectors,

$$\boldsymbol{\eta}_{1t} = \sum_{\tau \leq t} \boldsymbol{\varepsilon}_\tau, \quad \boldsymbol{\eta}_{2t} = \sum_{\tau \leq t} \boldsymbol{\eta}_{1\tau} \tag{5}$$

are first and second order random walks, respectively,

$$\boldsymbol{N}_\upsilon = \boldsymbol{N} - \boldsymbol{N}_{\upsilon-1}, \tag{6}$$

$$\boldsymbol{N}_{\upsilon-1} = \begin{cases} \boldsymbol{N}(\boldsymbol{I} + \boldsymbol{A}) & \text{if } \upsilon = 2, \\ \boldsymbol{0} & \text{if } \upsilon = 1, \end{cases} \tag{7}$$

$$\boldsymbol{N} = \boldsymbol{C}_{\upsilon\perp}(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp})^{-1}\boldsymbol{B}'_{\upsilon\perp}, \tag{8}$$

$\boldsymbol{B}_{\upsilon\perp}$ and $\boldsymbol{C}_{\upsilon\perp}$ denote orthogonal complements of full column-rank matrices $\boldsymbol{B}_\upsilon$ and $\boldsymbol{C}_\upsilon$ obtained by a rank factorization of $\boldsymbol{A}^\upsilon$, that is,

$$\boldsymbol{A}^\upsilon = \boldsymbol{B}_\upsilon \boldsymbol{C}'_\upsilon, \quad r(\boldsymbol{A}^\upsilon) = r(\boldsymbol{B}_\upsilon) = r(\boldsymbol{C}_\upsilon). \tag{9}$$

The solution $\boldsymbol{y}_t$ is an integrated (I) process, namely

$$\boldsymbol{y}_t \sim I(\upsilon). \tag{10}$$

Indeed the said solution turns out to exhibit a multi-fold integration and cointegration (CI) structure, whose core features are

$$\boldsymbol{C}'_{\upsilon}\boldsymbol{y}_t \sim I(0) \to \boldsymbol{y}_t \sim CI(\upsilon, \upsilon), \quad cr = r(\boldsymbol{A}^{\upsilon}), \tag{11}$$

$$\boldsymbol{B}'_{\perp}\boldsymbol{y}_t \sim I(1) \to \boldsymbol{y}_t \sim CI(2, 1), \quad cr = n - r(\boldsymbol{A}), \text{ under } \upsilon = 2, \tag{12}$$

where cr stands for cointegration rank and $\boldsymbol{B}$ is a full column-rank matrix obtained by a rank factorization of $\boldsymbol{A}$, that is,

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}', \quad r(\boldsymbol{A}) = r(\boldsymbol{B}) = r(\boldsymbol{C}). \tag{13}$$

We should look at (1) as a companion-form reparametrization of an isomorphic q-lag m-dimensional VAR model

$$\underset{(m,1)}{\tilde{\boldsymbol{y}}_t} + \sum_{k=1}^{q} \boldsymbol{P}_k \tilde{\boldsymbol{y}}_{t-k} = \underset{(m,1)}{\tilde{\boldsymbol{\varepsilon}}_t} \tag{14}$$

and solve, we would obtain

$$\tilde{\boldsymbol{y}}_t = \tilde{\boldsymbol{N}}_{\upsilon}\tilde{\boldsymbol{\omega}}_1 t^{\upsilon-1} + \sum_{j=1}^{\upsilon} \tilde{\boldsymbol{N}}_j(\tilde{\boldsymbol{\eta}}_{jt} + \tilde{\boldsymbol{\omega}}_j) + \sum_{j=0}^{\infty} \tilde{\boldsymbol{M}}_j L^j. \tag{15}$$

Here, the $\tilde{\boldsymbol{M}}_j$'s are coefficient matrices with exponentially decreasing entries, the $\tilde{\boldsymbol{\omega}}_j$'s denote arbitrary vectors, $\tilde{\boldsymbol{\eta}}_{1t}$ and $\tilde{\boldsymbol{\eta}}_{2t}$ are first and second order random walks, respectively, and

$$\tilde{\boldsymbol{N}}_{\upsilon} = \tilde{\boldsymbol{N}} - \tilde{\boldsymbol{N}}_{\upsilon-1}, \tag{16}$$

$$\tilde{\boldsymbol{N}}_{\upsilon-1} = \begin{cases} \boldsymbol{J}\boldsymbol{N}(\boldsymbol{I} + \boldsymbol{A})\boldsymbol{J}' & \text{if } \upsilon = 2, \\ \boldsymbol{0} & \text{if } \upsilon = 1, \end{cases} \tag{17}$$

$$\tilde{\boldsymbol{N}} = \boldsymbol{J}\boldsymbol{N}\boldsymbol{J}', \tag{18}$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0}_{m, m(q-1)} \end{bmatrix}. \tag{19}$$

Likewise $\boldsymbol{y}_t$, the process $\tilde{\boldsymbol{y}}_t$ is characterized by integration and cointegration properties, namely

$$\tilde{\boldsymbol{y}}_t \sim I(\upsilon), \tag{20}$$

$$(\boldsymbol{J}\boldsymbol{C}_\upsilon)'\tilde{\boldsymbol{y}}_t \sim I(0) \Rightarrow \tilde{\boldsymbol{y}}_t \sim CI(\upsilon, \upsilon), \tag{21}$$

$$\tilde{\boldsymbol{B}}'_\perp \dot{\boldsymbol{P}}\tilde{\boldsymbol{y}}_t \sim I(\upsilon - 1) \Rightarrow \tilde{\boldsymbol{y}}_t \sim CI(\upsilon, \upsilon - 1), \text{ under } \upsilon = 2, \tag{22}$$

where $\dot{\boldsymbol{P}} = \sum_{k=1}^{q} k\boldsymbol{P}_k$ and $\tilde{\boldsymbol{B}}$ is a full column-rank matrix obtained by a rank factorization of $\boldsymbol{P} = \boldsymbol{I} + \sum_{j=1}^{q} \boldsymbol{P}_j$, that is,

$$\boldsymbol{P} = \tilde{\boldsymbol{B}}\tilde{\boldsymbol{C}}', \quad r(\boldsymbol{P}) = r(\tilde{\boldsymbol{B}}) = r(\tilde{\boldsymbol{C}}). \tag{23}$$

In the next section, the results claimed above will be proved on a sound basis, whose algebraic core is set forth in the Appendix.

3. A Unified Representation Theorem

In this section, we establish the main result, namely a unified representation theorem for (co)integrated processes up to the second order whose outcomes have been anticipated in Section 2. The basic theorem takes a one-lag VAR model with unit roots as a reference frame, and the extension to a multi-lag specification is developed as a corollary.

Theorem 3.1. *Consider an n-dimensional VAR model specified as follows*:

$$\underset{(n,\, n)}{\boldsymbol{A}(L)} \; \underset{(n,\, 1)}{\boldsymbol{y}_t} = \underset{(n,\, 1),}{\boldsymbol{\varepsilon}_t} \tag{1}$$

where

$$\boldsymbol{A}(L) = \boldsymbol{I} + \boldsymbol{A}_1 L, \quad \boldsymbol{A}_1 \neq \boldsymbol{0} \tag{2}$$

is a linear polynomial in the lag operator L and $\boldsymbol{\varepsilon}_t$ *is a white noise process.*

Let the roots of the characteristic polynomial $\det \boldsymbol{A}(z)$ lie outside the unit circle except for a possibly multiple unit-root, and let the total effect matrix

$$\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{A}_1 \tag{3}$$

be of index $\upsilon \leq 2$.

Further, let $\boldsymbol{B}_\upsilon$, $\boldsymbol{C}_\upsilon$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{S}$ be full column-rank matrices defined as in Theorem 1 in the Appendix.

Then the following closed-form representation holds for the solution of equation (1):

$$\boldsymbol{y}_t = \boldsymbol{N}_\upsilon \boldsymbol{\omega}_1 t^{\upsilon-1} + \sum_{j=1}^{\upsilon} \boldsymbol{N}_j(\boldsymbol{\eta}_{jt} + \boldsymbol{\omega}_j) + \sum_{j=0}^{\infty} \boldsymbol{M}_j \boldsymbol{\varepsilon}_{t-j}, \tag{4}$$

where the $\boldsymbol{M}_j$'s are coefficient matrices with exponentially decreasing entries, $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ denote arbitrary vectors,

$$\boldsymbol{\eta}_{1t} = \sum_{\tau \leq t} \boldsymbol{\varepsilon}_\tau, \quad \boldsymbol{\eta}_{2t} = \sum_{\tau \leq t} \boldsymbol{\eta}_{1\tau} \tag{5}$$

are first and second order random walks, respectively,

$$\boldsymbol{N}_\upsilon = \boldsymbol{N} - \boldsymbol{N}_{\upsilon-1}, \tag{6}$$

$$\boldsymbol{N}_{\upsilon-1} = \begin{cases} \boldsymbol{N}(\boldsymbol{I} + \boldsymbol{A}) & \text{if } \upsilon = 2, \\ \boldsymbol{0} & \text{if } \upsilon = 1, \end{cases} \tag{7}$$

$$\boldsymbol{N} = \boldsymbol{C}_{\upsilon\perp}(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp})^{-1}\boldsymbol{B}'_{\upsilon\perp}. \tag{8}$$

The vector $\boldsymbol{y}_t$ given by (4) is an integrated (I) as well as cointegrated (CI) process, for which the following statements hold true:

(a) $\boldsymbol{y}_t \sim I(\upsilon)$, (9)

(b) $(\boldsymbol{C}_{\upsilon\perp})'_\perp \boldsymbol{y}_t \sim I(0) \Rightarrow \boldsymbol{y}_t \sim CI(\upsilon, \upsilon)$, $cr = r(\boldsymbol{A}^\upsilon)$, (10)

where *cr* stands for cointegration rank. Trivially $\boldsymbol{C}_\upsilon$ is one choice of the cointegrating matrix $(\boldsymbol{C}_{\upsilon\perp})_\perp$, which can more conveniently be specified as follows:

$$(\boldsymbol{C}_{\upsilon\perp})_{\perp} = \boldsymbol{C}(\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})_{\perp} \text{ if } \upsilon = 2 \text{ and } \boldsymbol{A}^{2} \neq \boldsymbol{0}, \tag{11}$$

$$(\boldsymbol{C}_{\upsilon\perp})_{\perp} \text{ is an empty matrix if } \upsilon = 2 \text{ and } \boldsymbol{A}^{2} = \boldsymbol{0}, \tag{12}$$

$$(\boldsymbol{C}_{\upsilon\perp})_{\perp} = \boldsymbol{C} \text{ if } \upsilon = 1, \tag{13}$$

(c) $\boldsymbol{B}'_{\perp}\boldsymbol{y}_t \sim I(\upsilon - 1) \Rightarrow \boldsymbol{y}_t \sim CI(\upsilon, \upsilon - 1), cr = n - r(\boldsymbol{A}),$ under $\upsilon = 2.$

(14)

Proof. The VAR model (1) is nothing but a constant-coefficient linear difference equation. Its solution consists – transient components apart – of a particular solution of the complete equation and of the complementary solution ascribable to unit roots (see, e.g., Faliva and Zoia [12, p. 26]).

In operator form, a particular solution of (1) is given by

$$\bar{\boldsymbol{y}}_t = \boldsymbol{A}^{-1}(L)\boldsymbol{\varepsilon}_t, \tag{15}$$

where

$$\boldsymbol{A}^{-1}(L) = \sum_{j=1}^{\upsilon} \boldsymbol{N}_j \nabla^{-j} + \boldsymbol{M}(L). \tag{16}$$

The latter result ensues from Theorem 4 in the Appendix, thanks to the isomorphism between polynomial algebras of complex variables and lag operator (see, e.g., Dhrymes [5, p. 23]), and to the sum calculus identities

$$\frac{1}{(I - L)^2} = \nabla^{-2} = \sum_{\tau \le t} \sum_{\vartheta \le \tau}, \quad \frac{1}{(I - L)} = \nabla^{-1} = \sum_{\tau \le t}. \tag{17}$$

Resorting to the said theorem – by taking $\upsilon = 2$ and $\upsilon = 1$ in turn – it yields the expressions of $\boldsymbol{N}_{\upsilon-1}$ and $\boldsymbol{N}_{\upsilon}$. This eventually leads to the following expression for $\bar{\boldsymbol{y}}_t$:

$$\bar{\boldsymbol{y}}_t = \sum_{j=1}^{\upsilon} \boldsymbol{N}_j \boldsymbol{\eta}_{jt} + \sum_{j=0}^{\infty} \boldsymbol{M}_j \boldsymbol{\varepsilon}_{t-j}, \tag{18}$$

where the $\boldsymbol{N}_j$'s are as specified in (6) and (7).

Likewise, the complementary solution is expressible as

$$\bar{\bar{\boldsymbol{y}}}_t = \boldsymbol{A}^{-1}(L)\boldsymbol{0} \tag{19}$$

and, resorting to Theorem 3, p. 27 in Faliva and Zoia [12], the following closed-form expression can be established for its permanent component:

$$\bar{\bar{\boldsymbol{y}}}_t = \sum_{j=1}^{\upsilon} \boldsymbol{N}_j\boldsymbol{\omega}_j + \boldsymbol{N}_\upsilon\boldsymbol{\omega}_1\boldsymbol{t}^{\upsilon-1}. \tag{20}$$

Adding $\bar{\boldsymbol{y}}_t$ and $\bar{\bar{\boldsymbol{y}}}_t$ gives the solution (4).

As far as results (a)-(c) are concerned, their proofs rest on the following considerations.

Result (a). By inspection of (4), we deduce that under $\upsilon = 2$, $\boldsymbol{y}_t$ is the resultant of a drift component $\sum_{j=1}^{2} \boldsymbol{N}_j\boldsymbol{\omega}_j$, of a deterministic linear trend component $\boldsymbol{N}_2\boldsymbol{\omega}_1 t$, of both a first and a second-order stochastic trend components, $\boldsymbol{N}_1\sum_{\tau\le t}\boldsymbol{\varepsilon}_t$ and $\boldsymbol{N}_2\sum_{\tau\le t}\sum_{\vartheta\le\tau}\boldsymbol{\varepsilon}_\vartheta$, respectively, and of a VMA(∞) component in the white noise argument $\boldsymbol{\varepsilon}_t$. As a result the solution is an integrated process of order 2.

On the other hand, under $\upsilon = 1$, $\boldsymbol{y}_t$ is the resultant of a drift component $\boldsymbol{N}_1\boldsymbol{\omega}_1$, of a first order stochastic trend component, $\boldsymbol{N}_1\sum_{\tau\le t}\boldsymbol{\varepsilon}_\tau$, and of a VMA$(\infty)$ component in the white noise argument $\boldsymbol{\varepsilon}_t$. As a result the solution is an integrated process of order 1. This proves (9).

Result (b). First of all observe that, under $\upsilon = 2$, the solution (4) can be expressed as follows:

$$\boldsymbol{y}_t = [\boldsymbol{N}_2,\ \boldsymbol{N}_1]\begin{bmatrix} \boldsymbol{\eta}_{2t} + \boldsymbol{\omega}_1 t + \boldsymbol{\omega}_2 \\ \boldsymbol{\eta}_{1t} + \boldsymbol{\omega}_1 \end{bmatrix} + \sum_{j=0}^{\infty} \boldsymbol{M}_j\boldsymbol{\varepsilon}_{t-j}. \tag{21}$$

It is clear from statement (vii) of Theorem 1 in the Appendix that the columns of $(\boldsymbol{C}_{2\perp})_\perp$ span the row kernel of $[\boldsymbol{N}_2,\ \boldsymbol{N}_1]$. This in turn entails that, by premultiplying both sides of (21) by $(\boldsymbol{C}_{2\perp})'_{\perp}$, the term containing non-stationary components – namely stochastic and deterministic trends –, disappears and the following

$$(\boldsymbol{C}_{2\perp})'_{\perp}\boldsymbol{y}_t = (\boldsymbol{C}_{2\perp})'_{\perp}\sum_{j=0}^{\infty}\boldsymbol{M}_j\boldsymbol{\varepsilon}_{t-j} \rightarrow (\boldsymbol{C}_{2\perp})'_{\perp}\boldsymbol{y}_t \sim I(0) \rightarrow \boldsymbol{y}_t \sim CI(2,\, 2) \tag{22}$$

holds accordingly. The cointegration rank, i.e., the rank of the cointegration matrix $(\boldsymbol{C}_{2\perp})_{\perp}$, turns out to be equal to $r(\boldsymbol{A}^2)$ in light of (4) of Theorem 1 of the Appendix, upon noting that $\boldsymbol{C}_2$ is trivially a choice of $(\boldsymbol{C}_{2\perp})_{\perp}$. Statement (11) is established by choosing $(\boldsymbol{C}_{2\perp})_{\perp} = \boldsymbol{C}(\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})_{\perp}$ according to (16) of Theorem 1 in the Appendix, and result (12) ensues from statement (b) of the said theorem, upon noting that if $\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}$ is a square non-singular matrix, its orthogonal complement collapses into an empty matrix (see, e.g., Faliva and Zoia [12, p. 131]). Should it be the case, the cointegration relationships recovering stationarity would no longer exist insofar as the cointegration rank would drop to zero.

Passing now to the case $\upsilon = 1$, observe that the solution (4) can be rewritten as

$$\boldsymbol{y}_t = \boldsymbol{N}_1(\boldsymbol{\eta}_{1t} + 2\boldsymbol{\omega}_1) + \sum_{j=0}^{\infty}\boldsymbol{M}_j\boldsymbol{\varepsilon}_{t-j}, \tag{23}$$

where

$$\boldsymbol{N}_1 = \boldsymbol{N} = \boldsymbol{C}_{\perp}(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp})^{-1}\boldsymbol{B}'_{\perp} \tag{24}$$

in light of (6) and (7), upon keeping in mind that

$$\boldsymbol{C}_1 = \boldsymbol{C} \tag{25}$$

under $\upsilon = 1$.

It is therefore clear that the columns of $(\boldsymbol{C}_{\perp})_{\perp}$ span the row kernel of $\boldsymbol{N}_1$. Premultiplication of both sides of (23) by $(\boldsymbol{C}_{\perp})'_{\perp}$ leads to annihilate the term containing the non-stationary component – namely the stochastic trend – and the following

$$(\boldsymbol{C}_\perp)'_\perp \boldsymbol{y}_t = (\boldsymbol{C}_\perp)'_\perp \sum_{j=0}^{\infty} \boldsymbol{M}_j \boldsymbol{\varepsilon}_{t-j} \to (\boldsymbol{C}_\perp)'_\perp \boldsymbol{y}_t \sim I(0) \to \boldsymbol{y}_t \sim CI(1, 1) \quad (26)$$

holds accordingly. The cointegration rank is equal to $r(\boldsymbol{A})$ in light of (5) of Theorem 1 in the Appendix, upon noting that $\boldsymbol{C}$ is trivially a choice of $(\boldsymbol{C}_\perp)_\perp$. This argument establishes (13) as well. The proof is now complete.

Result (c). First of all observe that, under $\upsilon = 2$, (4) can be rewritten as follows:

$$\boldsymbol{y}_t = \boldsymbol{N}_2(\boldsymbol{\eta}_{2t} + \boldsymbol{\omega}_1 t + \boldsymbol{\omega}_2) + \boldsymbol{N}_1(\boldsymbol{\eta}_{1t} + \boldsymbol{\omega}_1) + \sum_{j=0}^{\infty} \boldsymbol{M}_j \boldsymbol{\varepsilon}_{t-j}. \quad (27)$$

Then observe that in light of (19) of Theorem 1 in the Appendix, the columns of $(\boldsymbol{BG}_\perp)_\perp$ span the row kernel of $\boldsymbol{N}_2$ and one choice of $(\boldsymbol{BG}_\perp)_\perp$ is the partitioned matrix $[\boldsymbol{B}_\perp, (\boldsymbol{C}_{2\perp})_\perp]$ as per formula (18) of the said theorem. The columns of the latter block span the row kernel of $[\boldsymbol{N}_2, \boldsymbol{N}_1]$ (see proof of Result (b)), and the columns of the former block span the subspace of the row kernel of $\boldsymbol{N}_2$ not intersecting with the row kernel of $[\boldsymbol{N}_2, \boldsymbol{N}_1]$, respectively.

Hence, by premultiplying both sides of (27) by $\boldsymbol{B}'_\perp$, the non-stationarities due to second-order random walks and deterministic trend are removed whereas the non-stationarity due to first-order random walks is not, and the following

$$\boldsymbol{B}'_\perp \boldsymbol{y}_t = \boldsymbol{B}'_\perp \boldsymbol{N}_1(\boldsymbol{\eta}_{1t} + \boldsymbol{\omega}_1) + \boldsymbol{B}'_\perp \sum_{j=0}^{\infty} \boldsymbol{M}_j \boldsymbol{\varepsilon}_{t-j},$$

$$\boldsymbol{B}'_\perp \boldsymbol{y}_t \sim I(1) \to \boldsymbol{y}_t \sim CI(2, 1) \quad (28)$$

hold accordingly. The cointegration rank is equal to $n - r(\boldsymbol{A})$ in light of (5) of Theorem 1 of the Appendix. The proof is now complete.

So far we have considered one-lag VAR models; however multi-lag dynamic specifications happen to be the rule in econometric modelling, whence a stimulus to bridge the gap between simple and multi-lag analysis. In this connection, a companion-form representation of a multi-lag model (see, e.g., Banerjee et al. [2, p. 143]) turns out to provide the way-out to tailor the foregoing analysis to general dynamic models along the guidelines drawn below.

Consider to this end a one-lag n-dimensional VAR model, satisfying the hypothesis of Theorem 3.1, specified as follows:

$$\underset{(n,\,1)}{\boldsymbol{y}_t} + \underset{(n,\,n)}{\boldsymbol{A}_1}\boldsymbol{y}_{t-1} = \underset{(n,\,1)}{\boldsymbol{\varepsilon}_t} \tag{29}$$

and let the coefficient matrix $\boldsymbol{A}_1$ and the vector $\boldsymbol{y}_t$ be partitioned as

$$\underset{(n,1)}{\boldsymbol{A}_1} = \begin{bmatrix} \boldsymbol{P}_1 & \vdots & \boldsymbol{P}_2 & \boldsymbol{P}_3 & .. & \boldsymbol{P}_q \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ -\boldsymbol{I} & \vdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \vdots & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ .. & \vdots & .. & .. & .. & .. \\ \boldsymbol{0} & \vdots & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \tag{30}$$

$$\underset{(n,1)}{\boldsymbol{y}_t} = \begin{bmatrix} \tilde{\boldsymbol{y}}_t \\ \tilde{\boldsymbol{y}}_{t-1} \\ \vdots \\ \tilde{\boldsymbol{y}}_{t-q+1} \end{bmatrix}, \tag{31}$$

where the q^2 blocks of $\boldsymbol{A}_1$ are square matrices of order m, the q blocks of $\boldsymbol{y}_t$ are $m \times 1$ vectors and n equals mq.

Further, let the right-hand side vector $\boldsymbol{\varepsilon}_t$ be specified as

$$\boldsymbol{\varepsilon}_t = \boldsymbol{J}'\tilde{\boldsymbol{\varepsilon}}_t, \tag{32}$$

where $\tilde{\boldsymbol{\varepsilon}}_t \sim WN_{(m)}$ and

$$\boldsymbol{J} = [\boldsymbol{I}, \quad \boldsymbol{0}, \quad \ldots, \quad \boldsymbol{0}] \tag{33}$$

is a selection matrix whose q blocks are square matrices of order m.

By premultiplying both sides of (29) by $\boldsymbol{J}$ and by resorting to the companion-form matrix (30), (31) and (32), an isomorphic q-lag model

$$\underset{(m,1)}{\tilde{\boldsymbol{y}}_t} + \sum_{k=1}^{q} \boldsymbol{P}_k \tilde{\boldsymbol{y}}_{t-k} = \underset{(m,1)}{\tilde{\boldsymbol{\varepsilon}}_t} \tag{34}$$

arises from the parent one-lag model (29) as simple computations show.

Then, the solution of equation (34) can be recovered from that of equation (29), and cointegration analysis can be run by spanning the row kernels of the matrices $\boldsymbol{J}[\boldsymbol{N}_\upsilon, \boldsymbol{N}_{\upsilon-1}]$ and $\boldsymbol{J}\boldsymbol{N}_\upsilon$ as the following corollary shows.

Corollary 3.1.1. *Consider an m-dimensional q-lag VAR model specified as*

$$\underset{(m,1)}{\tilde{\boldsymbol{y}}_t} + \sum_{k=1}^{q} \boldsymbol{P}_k \tilde{\boldsymbol{y}}_{t-k} = \underset{(m,1)}{\tilde{\boldsymbol{\varepsilon}}_t} \tag{35}$$

and its companion-form reparametrization

$$\boldsymbol{y}_t + \boldsymbol{A}_1 \boldsymbol{y}_{t-1} = \boldsymbol{\varepsilon}_t \tag{36}$$

for which the hypotheses of Theorem 3.1 *are maintained. Here,* $\boldsymbol{A}_1$, $\boldsymbol{y}_t$ *and* $\boldsymbol{\varepsilon}_t$ *are as defined in* (30), (31) *and* (32), *respectively.*

The following closed-form representation holds for the solution of equation (35):

$$\tilde{\boldsymbol{y}}_t = \tilde{\boldsymbol{N}}_\upsilon \tilde{\boldsymbol{\omega}}_1 t^{\upsilon-1} + \sum_{j=1}^{\upsilon} \tilde{\boldsymbol{N}}_j (\tilde{\boldsymbol{\eta}}_{jt} + \tilde{\boldsymbol{\omega}}_j) + \sum_{j=0}^{\infty} \tilde{\boldsymbol{M}}_j L^j, \tag{37}$$

where the $\tilde{\boldsymbol{M}}_j$'s are coefficient matrices with exponentially decreasing entries, $\tilde{\boldsymbol{\omega}}_1$ and $\tilde{\boldsymbol{\omega}}_2$ denote arbitrary vectors,

$$\tilde{\boldsymbol{\eta}}_{1t} = \sum_{\tau \le t} \tilde{\boldsymbol{\varepsilon}}_\tau, \quad \tilde{\boldsymbol{\eta}}_{2t} = \sum_{\tau \le t} \tilde{\boldsymbol{\eta}}_{1\tau} \tag{38}$$

are first and second order random walks, respectively,

$$\tilde{\boldsymbol{N}}_\upsilon = \tilde{\boldsymbol{N}} - \tilde{\boldsymbol{N}}_{\upsilon-1}, \tag{39}$$

$$\widetilde{\boldsymbol{N}}_{\upsilon-1} = \begin{cases} \boldsymbol{JN}(\boldsymbol{I} + \boldsymbol{A})\boldsymbol{J}' & \text{if } \upsilon = 2, \\ \boldsymbol{0} & \text{if } \upsilon = 1, \end{cases} \tag{40}$$

$$\widetilde{\boldsymbol{N}} = \boldsymbol{JNJ}', \tag{41}$$

$\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{A}_1$ and $\boldsymbol{N}$ is the matrix (8) of Theorem 3.1.

The vector $\widetilde{\boldsymbol{y}}_t$ given by (37) is an integrated (I) as well as cointegrated (CI) process, for which the following statements hold:

(a) $\widetilde{\boldsymbol{y}}_t \sim I(\upsilon)$, (42)

(b) $(\widetilde{\boldsymbol{C}}_{\upsilon\perp})'_{\perp}\widetilde{\boldsymbol{y}}_t \sim I(0) \Rightarrow \widetilde{\boldsymbol{y}}_t \sim CI(\upsilon, \upsilon)$, (43)

where $\widetilde{\boldsymbol{C}}_{\upsilon\perp}$ is written for $\boldsymbol{JC}_{\upsilon\perp}$ and the rank qualification $r(\widetilde{\boldsymbol{B}}^{+}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp}) = r(\widetilde{\boldsymbol{S}}_{\perp})$ is adopted for $\upsilon = 2$. Trivially $\widetilde{\boldsymbol{C}}_{\upsilon}$ is one choice of $(\widetilde{\boldsymbol{C}}_{\upsilon\perp})_{\perp}$, which can be more conveniently specified as follows:

$$(\widetilde{\boldsymbol{C}}_{\upsilon\perp})_{\perp} = -\widetilde{\boldsymbol{C}}(\widetilde{\boldsymbol{B}}^{-}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp})_{\perp} \text{ if } \upsilon = 2 \text{ and } r(\boldsymbol{A}^2) > (q-1)m, \tag{44}$$

$$(\widetilde{\boldsymbol{C}}_{\upsilon\perp})_{\perp} \text{ is an empty matrix if } \upsilon = 2 \text{ and } r(\boldsymbol{A}^2) = (q-1)m, \tag{45}$$

$$(\widetilde{\boldsymbol{C}}_{\upsilon\perp})_{\perp} = \widetilde{\boldsymbol{C}} \text{ if } \upsilon = 1, \tag{46}$$

(c) $\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{y}}_t \sim I(\upsilon-1) \Rightarrow \widetilde{\boldsymbol{y}}_t \sim CI(\upsilon, \upsilon-1)$, under $\upsilon = 2$. (47)

Insofar as the rank assumptions

$$r(\widetilde{\boldsymbol{R}}'_{\perp}\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\boldsymbol{P}^{+}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp}) = r(\widetilde{\boldsymbol{S}}_{\perp}) \text{ and } r(\dot{\boldsymbol{P}}'\widetilde{\boldsymbol{B}}_{\perp}) = r(\widetilde{\boldsymbol{B}}_{\perp}) \tag{48}$$

are adopted, propositions (b) and (c) provide a full characterization of the cointegration properties of the solution.

To prove (37), observe that

(i) a particular solution of (35) can be obtained from that of (36) – namely $\overline{\boldsymbol{y}}_t$ of formula (18) – by premultiplication by $\boldsymbol{J}$, that is,

$$\overline{\widetilde{\boldsymbol{y}}}_t = \boldsymbol{J}\overline{\boldsymbol{y}}_t = \sum_{j=1}^{\upsilon}\boldsymbol{JN}_j\boldsymbol{\eta}_{jt} + \sum_{j=0}^{\infty}\boldsymbol{JM}_j\boldsymbol{\varepsilon}_{t-j}$$

$$= \sum_{j=1}^{\upsilon} \boldsymbol{J N}_j \boldsymbol{J}' \tilde{\boldsymbol{\eta}}_{jt} + \sum_{j=0}^{\infty} \boldsymbol{J M}_j \boldsymbol{J}' \tilde{\boldsymbol{\varepsilon}}_{t-j}$$

$$= \sum_{j=1}^{\upsilon} \tilde{\boldsymbol{N}}_j \tilde{\boldsymbol{\eta}}_{jt} + \sum_{j=0}^{\infty} \tilde{\boldsymbol{M}}_j \tilde{\boldsymbol{\varepsilon}}_{t-j} \tag{49}$$

keeping in mind (32) and its by-product

$$\boldsymbol{\eta}_{jt} = \boldsymbol{J}' \tilde{\boldsymbol{\eta}}_{jt}.$$

(ii) The permanent component of the complementary solution, namely

$$\tilde{\tilde{\bar{\boldsymbol{y}}}}_t = \sum_{j=1}^{\upsilon} \tilde{\boldsymbol{N}}_j \tilde{\boldsymbol{\omega}}_j + \tilde{\boldsymbol{N}}_{\upsilon} \tilde{\boldsymbol{\omega}}_1 t^{\upsilon-1} \tag{50}$$

can be obtained likewise from formula (20).

By adding (49) and (50), we get (37).

For what concerns Result (a) the proof is the same as in Theorem 3.1.

Proofs of subsequent results develop along the same lines as in Theorem 3.1, with Theorem 2 in the Appendix providing the algebraic support once offered by Theorem 1 in the Appendix.

Indeed, the row kernels of $\tilde{\boldsymbol{N}}$, for $\upsilon = 1$, and of $[\tilde{\boldsymbol{N}}_2, \tilde{\boldsymbol{N}}_1]$ and $\tilde{\boldsymbol{N}}_2$, for $\upsilon = 2$ under the rank conditions (45), turn out to be spanned by the columns of the matrices $\tilde{\boldsymbol{C}}$, and

$$(\tilde{\boldsymbol{C}}_{2\perp})_{\perp} = -\tilde{\boldsymbol{C}}(\tilde{\boldsymbol{B}}^{+} \dot{\boldsymbol{P}} \tilde{\boldsymbol{C}}_{\perp} \tilde{\boldsymbol{S}}_{\perp})_{\perp}, \tag{51}$$

$$(\boldsymbol{JBG})_{\perp} = [\dot{\boldsymbol{P}}' \tilde{\boldsymbol{B}}_{\perp}, \ (\tilde{\boldsymbol{C}}_{2\perp})_{\perp}], \tag{52}$$

respectively, according to (49) and (51) of Theorem 2 of the Appendix bearing in mind (52) and (53) of the said theorem. As a by-product, under $\upsilon = 2$, the columns of $\dot{\boldsymbol{P}}' \tilde{\boldsymbol{B}}_{\perp}$ turn out to span the subspace of the row kernel of $\tilde{\boldsymbol{N}}_2$ not intersecting with that of $[\tilde{\boldsymbol{N}}_2, \tilde{\boldsymbol{N}}_1]$.

Hence, by resorting to the same line of reasoning set forth in the proof of Theorem 3.1, the way is paved to prove (43), (44), (46) and (47), as well as (45), with (54) of Theorem 2 in the Appendix playing the same role formerly played by (9) of Theorem 1 in the Appendix.

Should either the rank qualification $r(\widetilde{\boldsymbol{B}}^{+}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp}) = r(\widetilde{\boldsymbol{S}}_{\perp})$ or the rank assumptions (48) fail to hold, then the previous arguments should be restated accordingly. The way would be paved, should we resort to the analytical set-up formerly devised by Faliva and Zoia [11, 12]. This is nevertheless beyond the scope of the present paper.

Appendix

Definition 1. Let $\boldsymbol{C}$ be an n-row matrix of full column-rank. An n-row matrix $\boldsymbol{C}_{\perp}$ of full column-rank is said to be an *orthogonal complement* of $\boldsymbol{C}$ if

$$\boldsymbol{C}'\boldsymbol{C}_{\perp} = \mathbf{0}, \quad r(\boldsymbol{C}_{\perp}) = n - r(\boldsymbol{C}). \tag{1}$$

Obviously $\boldsymbol{C}_{\perp}$ is not unique and trivially a choice of $(\boldsymbol{C}_{\perp})_{\perp}$ is $\boldsymbol{C}$ itself.

Note also that $\boldsymbol{C}_{\perp}$ is reduced to an empty matrix when $\boldsymbol{C}$ is square (see, e.g., Faliva and Zoia [12, p. 131]). We shall henceforth write

$$\boldsymbol{C}_{\perp} = \boldsymbol{K} \tag{2}$$

to indicate that $\boldsymbol{K}$ is one choice of $\boldsymbol{C}_{\perp}$.

Definition 2. Let $\boldsymbol{A}$ be a square matrix. The index of $\boldsymbol{A}$, written $\operatorname{ind}(\boldsymbol{A})$, is the least non-negative integer υ for which

$$r(\boldsymbol{A}^{\upsilon}) = r(\boldsymbol{A}^{\upsilon+1}). \tag{3}$$

Should $\boldsymbol{A}$ be non-singular, then $\operatorname{ind}(\boldsymbol{A}) = 0$, whereas when $\boldsymbol{A}$ is a null matrix, then $\operatorname{ind}(\boldsymbol{A}) = 1$ (Campbell and Meyer [3, p. 121]).

Theorem 1. *Let A be a non-null square matrix of order n and index* $\upsilon \le 2$ *and let*

$$\boldsymbol{A}^{\upsilon} = \boldsymbol{B}_{\upsilon}\boldsymbol{C}'_{\upsilon}, \quad r(\boldsymbol{A}^{\upsilon}) = r(\boldsymbol{B}_{\upsilon}) = r(\boldsymbol{C}_{\upsilon}), \tag{4}$$

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}', \quad r(\boldsymbol{A}) = r(\boldsymbol{B}) = r(\boldsymbol{C}), \tag{5}$$

$$\boldsymbol{C}'\boldsymbol{B} = \boldsymbol{F}\boldsymbol{G}', \quad r(\boldsymbol{C}'\boldsymbol{B}) = r(\boldsymbol{F}) = r(\boldsymbol{G}), \tag{6}$$

$$\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp} = \boldsymbol{R}\boldsymbol{S}', \quad r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}) = r(\boldsymbol{R}) = r(\boldsymbol{S}) \tag{7}$$

be rank factorizations of $\boldsymbol{A}^{\upsilon}$, $\boldsymbol{A}$, $\boldsymbol{C}'\boldsymbol{B}$ *and* $\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}$, *respectively, where* $\boldsymbol{B}_{\upsilon}$, $\boldsymbol{C}_{\upsilon}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *are full column-rank matrices.*

Then the following hold:

(a) $r(\boldsymbol{A}) - r(\boldsymbol{A}^2) = r(\boldsymbol{C}_{\perp}) - r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp})$, (8)

(b) $r(\boldsymbol{B}) = r(\boldsymbol{S}_{\perp})$ *if* $\boldsymbol{A}^2 = \boldsymbol{0}$, (9)

(c) $\det(\boldsymbol{C}'_{\upsilon}\boldsymbol{B}_{\upsilon}) \neq 0$, $\det(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp}) \neq 0$, (10)

(d) $\boldsymbol{N} = \boldsymbol{C}_{\upsilon\perp}(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp})^{-1}\boldsymbol{B}'_{\upsilon\perp}$ (11)

is invariant for any choice of $\boldsymbol{B}_{\upsilon\perp}$ *and* $\boldsymbol{C}_{\upsilon\perp}$,

(e) $\boldsymbol{N} = \boldsymbol{I}$ *if* $\boldsymbol{A}^{\upsilon} = \boldsymbol{0}$.

Further, the following hold for $\upsilon = 2$ *and* $\boldsymbol{A}^2 \neq \boldsymbol{0}$:

(i) $\det(\boldsymbol{G}'\boldsymbol{F}) \neq 0$, $\det(\boldsymbol{F}'_{\perp}\boldsymbol{G}_{\perp}) \neq 0$, (12)

(ii) $\boldsymbol{G}_{\perp} = \boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}$, $\boldsymbol{F}_{\perp} = \boldsymbol{C}^{+}\boldsymbol{B}_{\perp}\boldsymbol{R}_{\perp}$, (13)

where $\boldsymbol{B}^{+} = (\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'$ *and* $\boldsymbol{C}^{+} = (\boldsymbol{C}'\boldsymbol{C})^{-1}\boldsymbol{C}'$ *denote the Moore-Penrose inverses of* $\boldsymbol{B}$ *and* $\boldsymbol{C}$, *respectively.*

(iii) $\boldsymbol{C}_{2\perp} = [\boldsymbol{C}_{\perp}, \boldsymbol{A}_r^{-}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}]$, (14)

where

$$\boldsymbol{A}_r^{-} = (\boldsymbol{C}')^{-}\boldsymbol{B}^{+} \tag{15}$$

is a reflexive generalized inverse of $\boldsymbol{A}$ *and* $(\boldsymbol{C}')^{-}$ *is an arbitrary right-inverse of* $\boldsymbol{C}'$.

(iv) $(\boldsymbol{C}_{2\perp})_{\perp} = \boldsymbol{C}(\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})_{\perp}$, (16)

(v) $$(\boldsymbol{B}\boldsymbol{G}_{\perp})_{\perp} = (\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})_{\perp} \tag{17}$$

$$= [\boldsymbol{B}_{\perp},\ (\boldsymbol{C}_{2\perp})_{\perp}], \tag{18}$$

(vi) $$\boldsymbol{N}_2 = \boldsymbol{B}\boldsymbol{G}_{\perp}(\boldsymbol{F}'_{\perp}\boldsymbol{G}_{\perp})^{-1}\boldsymbol{F}'_{\perp}\boldsymbol{C}' \tag{19}$$

$$= -\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}(\boldsymbol{R}'_{\perp}\boldsymbol{B}'_{\perp}\boldsymbol{A}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})^{-1}\boldsymbol{R}'_{\perp}\boldsymbol{B}'_{\perp}, \tag{20}$$

where $\boldsymbol{N}_2 = -\boldsymbol{N}\boldsymbol{A}$,

(vii) $$[\boldsymbol{N}_2,\ \boldsymbol{N}_1] = \boldsymbol{C}_{2\perp}\boldsymbol{\Phi}, \tag{21}$$

where $\boldsymbol{N}_1 = \boldsymbol{N} - \boldsymbol{N}_2$ *and* $\boldsymbol{\Phi}$ *is a full row-rank matrix.*

Proof of (a). Resorting to Theorem 19 of Marsaglia and Styan [19] and bearing in mind the identities (see, e.g., Rao and Mitra [20, p. 156])

$$\boldsymbol{B}\boldsymbol{B}^{+} + (\boldsymbol{B}'_{\perp})^{+}\boldsymbol{B}'_{\perp} = \boldsymbol{I}, \quad \boldsymbol{C}\boldsymbol{C}^{+} + (\boldsymbol{C}'_{\perp})^{+}\boldsymbol{C}'_{\perp} = \boldsymbol{I} \tag{22}$$

the twin rank equalities

$$r[\boldsymbol{B},\ \boldsymbol{C}_{\perp}] = r(\boldsymbol{B}) + r((\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}^{+})\boldsymbol{C}_{\perp}) = r(\boldsymbol{B}) + r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}) = r(\boldsymbol{A}) + r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}), \tag{23}$$

$$r[\boldsymbol{B},\ \boldsymbol{C}_{\perp}] = r(\boldsymbol{C}_{\perp}) + r([\boldsymbol{I} - (\boldsymbol{C}'_{\perp})^{+}\boldsymbol{C}'_{\perp}]\boldsymbol{B}) = r(\boldsymbol{C}_{\perp}) + r(\boldsymbol{C}'\boldsymbol{B}) = r(\boldsymbol{C}_{\perp}) + r(\boldsymbol{A}^2) \tag{24}$$

are easily established. Equating the right-hand sides of (23) and (24) yields (8).

Proof of (b). Under $\boldsymbol{A}^2 = \boldsymbol{0}$, equality (8) takes the form

$$r(\boldsymbol{A}) = r(\boldsymbol{C}_{\perp}) - r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}) \tag{25}$$

whence (9) follows upon reminding (5) and (7) and noting that

$$r(\boldsymbol{C}_{\perp}) - r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}) = r(\boldsymbol{C}_{\perp}) - r(\boldsymbol{S}) = r(\boldsymbol{S}_{\perp}). \tag{26}$$

Proof of (c). As $\operatorname{ind}(\boldsymbol{A}^{\upsilon}) = 1$, bearing in mind (4) and restating (8) with $\boldsymbol{A}^{\upsilon}$ as an argument, the following prove true:

$$r(\boldsymbol{B}_\upsilon \boldsymbol{C}'_\upsilon) = r(\boldsymbol{B}_\upsilon \boldsymbol{C}'_\upsilon \boldsymbol{B}_\upsilon \boldsymbol{C}'_\upsilon) \to \det(\boldsymbol{C}'_\upsilon \boldsymbol{B}_\upsilon) \neq 0,$$

$$r(\boldsymbol{A}^\upsilon) - r(\boldsymbol{A}^{2\upsilon}) = 0 \to r(\boldsymbol{C}_{\upsilon\perp}) - r(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp}) = 0 \to \det(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp}) \neq 0. \quad (27)$$

Proof of (d). In order to prove the claimed invariance, reference can be made to Theorem 5, p. 5 in Faliva and Zoia [12].

Proof of (e). Should $\boldsymbol{A}^\upsilon$ be a null matrix, then $\boldsymbol{B}_\upsilon$ and $\boldsymbol{C}_\upsilon$ would be empty matrices, and $\boldsymbol{B}_{\upsilon\perp}$ and $\boldsymbol{C}_{\upsilon\perp}$ would be arbitrary non-singular matrices (see, e.g., Faliva and Zoia [12, p. 131], Chipman and Rao [4]), whence the equality $\boldsymbol{N} = \boldsymbol{I}$ would follow as a by-product.

Proof of (i). As $\operatorname{ind}(\boldsymbol{A}) = 2$, then $\operatorname{ind}(\boldsymbol{G}'\boldsymbol{F}) = 1$, and (10) applies accordingly with $\boldsymbol{F}$ and $\boldsymbol{G}$ in place of $\boldsymbol{B}_\upsilon$ and $\boldsymbol{C}_\upsilon$.

Proof of (ii). Reminding (6), (7) and (22) and upon noting that $\boldsymbol{F}^-\boldsymbol{F} = \boldsymbol{I}$, it is easy to check that

$$\boldsymbol{B}\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp = [\boldsymbol{I} - (\boldsymbol{B}'_\perp)^+\boldsymbol{B}'_\perp]\boldsymbol{C}_\perp\boldsymbol{S}_\perp = \boldsymbol{C}_\perp\boldsymbol{S}_\perp,$$

$$\boldsymbol{G}'\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp = \boldsymbol{F}^-\boldsymbol{F}\boldsymbol{G}'\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp = \boldsymbol{F}^-\boldsymbol{C}'\boldsymbol{B}\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp = \boldsymbol{F}^-\boldsymbol{C}'\boldsymbol{C}_\perp\boldsymbol{S}_\perp = \boldsymbol{0} \quad (28)$$

whence the conclusions that

$$r(\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp) = r(\boldsymbol{B}\boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp) = r(\boldsymbol{C}_\perp\boldsymbol{S}_\perp) = r(\boldsymbol{S}_\perp), \quad (29)$$

$$r[\boldsymbol{G},\ \boldsymbol{B}^+\boldsymbol{C}_\perp\boldsymbol{S}_\perp] = r(\boldsymbol{G}) + r(\boldsymbol{S}_\perp) \quad (30)$$

are easily drawn.

Further, observe that the following hold as $\upsilon = 2$:

$$r(\boldsymbol{A}^2) = r(\boldsymbol{C}'\boldsymbol{B}) = r(\boldsymbol{G}), \quad r(\boldsymbol{A}) - r(\boldsymbol{A}^2) = r(\boldsymbol{G}_\perp) \quad (31)$$

which in turn entails the equality

$$r(\boldsymbol{G}_\perp) = r(\boldsymbol{S}_\perp) \quad (32)$$

in light of (8) and (26).

Since both the orthogonality and the rank conditions of Definition 1 are satisfied, $\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}$ provides one choice of $\boldsymbol{G}_{\perp}$. The same conclusion about $\boldsymbol{C}^{+}\boldsymbol{B}_{\perp}\boldsymbol{R}_{\perp}$ with respect to $\boldsymbol{F}_{\perp}$ is drawn likewise.

Proof of (iii). As $\boldsymbol{C}_2 = \boldsymbol{C}\boldsymbol{G}$, Theorem 6, p. 7 in Faliva and Zoia [12] applies, yielding

$$\boldsymbol{C}_{2\perp} = [\boldsymbol{C}_{\perp}, (\boldsymbol{C}')^{-}\boldsymbol{G}_{\perp}] \tag{33}$$

which in turn leads to (14) by resorting to (13) and (15).

Proof of (iv). Formula (16) follows from backward application to (14) of the said Faliva and Zoia's theorem, by keeping in mind (15).

Proof of (v). Result (17) is easily established on the basis of (13) and (28).

Moving to (18), observe first that applying Theorem 6, p. 7 in Faliva and Zoia [12], to the matrix $(\boldsymbol{B}\boldsymbol{G}_{\perp})_{\perp}$ yields

$$(\boldsymbol{B}\boldsymbol{G}_{\perp})_{\perp} = [\boldsymbol{B}_{\perp}(\boldsymbol{B}')^{+}(\boldsymbol{G}_{\perp})_{\perp}]. \tag{34}$$

Premultiplying the latter block in the right-hand side by $\boldsymbol{C}\boldsymbol{B}'$ and resorting to (13) and (16), leads to

$$[\boldsymbol{B}_{\perp}, \boldsymbol{C}(\boldsymbol{G}_{\perp})_{\perp}] = [\boldsymbol{B}_{\perp}, \boldsymbol{C}(\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})_{\perp}] = [\boldsymbol{B}_{\perp}, (\boldsymbol{C}_{2\perp})_{\perp}]$$

which proves to be a choice of $(\boldsymbol{B}\boldsymbol{G}_{\perp})_{\perp}$ in light of the results below,

$$\boldsymbol{G}'_{\perp}\boldsymbol{B}'[\boldsymbol{B}_{\perp}, \boldsymbol{C}(\boldsymbol{G}_{\perp})_{\perp}] = [\boldsymbol{0}, \boldsymbol{G}'_{\perp}\boldsymbol{B}'\boldsymbol{C}(\boldsymbol{G}_{\perp})_{\perp}] = [\boldsymbol{0}, \boldsymbol{G}'_{\perp}\boldsymbol{G}\boldsymbol{F}'(\boldsymbol{G}_{\perp})_{\perp}] = [\boldsymbol{0}, \boldsymbol{0}],$$

$$\begin{aligned} r([\boldsymbol{B}_{\perp}, (\boldsymbol{C}_{2\perp})_{\perp}]) &= r(\boldsymbol{C}_{2\perp})_{\perp} + r(\boldsymbol{C}'_{2\perp}\boldsymbol{B}_{\perp}) = r(\boldsymbol{C}_2) + r([\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}, \boldsymbol{B}'_{\perp}(\boldsymbol{C}')^{-}\boldsymbol{G}_{\perp}]) \\ &= r(\boldsymbol{C}_2) + r(\boldsymbol{R}) + r(\boldsymbol{R}'_{\perp}\boldsymbol{B}'_{\perp}(\boldsymbol{C}')^{+}\boldsymbol{G}_{\perp}) = r(\boldsymbol{C}_2) + r(\boldsymbol{S}) + r(\boldsymbol{F}'_{\perp}\boldsymbol{G}_{\perp}) \\ &= r(\boldsymbol{A}^2) + r(\boldsymbol{S}) + r(\boldsymbol{G}_{\perp}) = r(\boldsymbol{A}^2) + r(\boldsymbol{S}) + r(\boldsymbol{S}_{\perp}) = r(\boldsymbol{A}^2) + r(\boldsymbol{C}_{\perp}) \\ &= r(\boldsymbol{A}^2) + n - r(\boldsymbol{A}) = n - r(\boldsymbol{G}_{\perp}) = r(\boldsymbol{B}\boldsymbol{G}_{\perp})_{\perp}. \end{aligned}$$

The rank equalities above have been obtained by making use of (7), (12), (13), (14), (31), (32) and (33), by choosing $(\boldsymbol{C}')^{+}$ as a generalized inverse of

$\boldsymbol{C}'$, by reminding the noteworthy equality $\boldsymbol{A}^{+} = (\boldsymbol{C}')^{+}\boldsymbol{B}^{+}$, and by resorting twice to the usual Marsaglia and Styan's theorem.

Proof of (vi). By making use of the identity

$$\boldsymbol{D}(\boldsymbol{V}'\boldsymbol{D})^{-1}\boldsymbol{V}' + \boldsymbol{V}_{\perp}(\boldsymbol{D}'_{\perp}\boldsymbol{V}_{\perp})^{-1}\boldsymbol{D}'_{\perp} = \boldsymbol{I}, \tag{35}$$

where $\boldsymbol{D}$ and $\boldsymbol{V}$ are full column-rank matrices such that $[\boldsymbol{D}, \boldsymbol{V}_{\perp}]$ is non-singular (see, e.g., Faliva and Zoia [12, p. 9]), by bearing in mind that $\boldsymbol{B}_2 = \boldsymbol{BF}$ and $\boldsymbol{C}_2 = \boldsymbol{CG}$, and by resorting to (6), (11), (12), (13) and (22), check that

$$\begin{aligned}
\boldsymbol{N}_2 &= -\boldsymbol{NA} = -\boldsymbol{C}_{2\perp}(\boldsymbol{B}'_{2\perp}\boldsymbol{C}_{2\perp})^{-1}\boldsymbol{B}'_{2\perp}\boldsymbol{A} = -[\boldsymbol{I} - \boldsymbol{B}_2(\boldsymbol{C}'_2\boldsymbol{B}_2)^{-1}\boldsymbol{C}'_2]\boldsymbol{A} \\
&= -(\boldsymbol{BC}' - \boldsymbol{BF}(\boldsymbol{G}'\boldsymbol{C}'\boldsymbol{BF})^{-1}\boldsymbol{G}'\boldsymbol{C}'\boldsymbol{BC}') = -(\boldsymbol{BC}' - \boldsymbol{BF}(\boldsymbol{G}'\boldsymbol{F})^{-2}\boldsymbol{G}'\boldsymbol{FG}'\boldsymbol{C}') \\
&= -\boldsymbol{B}[\boldsymbol{I} - \boldsymbol{F}(\boldsymbol{G}'\boldsymbol{F})^{-1}\boldsymbol{G}']\boldsymbol{C}' = -\boldsymbol{BG}_{\perp}(\boldsymbol{F}'_{\perp}\boldsymbol{G}_{\perp})^{-1}\boldsymbol{F}'_{\perp}\boldsymbol{C}' \\
&= -\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}(\boldsymbol{R}'_{\perp}\boldsymbol{B}'_{\perp}\boldsymbol{A}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp})^{-1}\boldsymbol{R}'_{\perp}\boldsymbol{B}'_{\perp}.
\end{aligned}$$

Proof of (vii). Upon noting that $[\boldsymbol{N}_2, \boldsymbol{N}_1] = [-\boldsymbol{NA}, \boldsymbol{N}(\boldsymbol{I} + \boldsymbol{A})]$ and that the block matrix $[-\boldsymbol{A}, (\boldsymbol{I} + \boldsymbol{A})]$ is of full row-rank, the conclusion that

$$r([\boldsymbol{N}_2, \boldsymbol{N}_1]) = r(\boldsymbol{N})$$

is easily drawn and the factorization (21) follows accordingly, in light of (11) by taking $\upsilon = 2$ and $\boldsymbol{\Phi} = (\boldsymbol{B}'_{2\perp}\boldsymbol{C}_{2\perp})^{-1}\boldsymbol{B}'_{2\perp}[-\boldsymbol{A}, \boldsymbol{I} + \boldsymbol{A}]$.

Theorem 2. *Let* $\boldsymbol{A}$ *be a square matrix of order* n *and index* $\upsilon \le 2$ *partitioned as follows*:

$$\underset{(n,n)}{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{bmatrix} = \left[\begin{array}{c|ccccc} \boldsymbol{I}_m + \boldsymbol{P}_1 & \boldsymbol{P}_2 & \boldsymbol{P}_3 & .. & \boldsymbol{P}_q \\ \hline -\boldsymbol{I}_m & \boldsymbol{I}_m & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_m & \boldsymbol{I}_m & \boldsymbol{0} & \boldsymbol{0} \\ .. & .. & .. & .. & .. \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m & \boldsymbol{I}_m \end{array}\right], \tag{36}$$

where $n = mq$, $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_q$ *are square matrices of order* m, *and let* $\boldsymbol{P}$

denote the Schur complement of $\mathbf{\Lambda}_{22}$, *namely*

$$\boldsymbol{P} = \mathbf{\Lambda}_{11} - \mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}_{21} = \boldsymbol{I} + \boldsymbol{P}_1 + \boldsymbol{P}_2 + \cdots + \boldsymbol{P}_q. \tag{37}$$

Further let $\boldsymbol{A}^{\upsilon}$, $\boldsymbol{A}$ *and* $\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}$ *be factorized as in Theorem* 1, *and*

$$\boldsymbol{P} = \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{C}}', \quad r(\boldsymbol{P}) = r(\widetilde{\boldsymbol{B}}) = r(\widetilde{\boldsymbol{C}}), \tag{38}$$

$$\widetilde{\boldsymbol{C}}'\widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{F}}\widetilde{\boldsymbol{G}}', \quad r(\widetilde{\boldsymbol{C}}'\widetilde{\boldsymbol{B}}) = r(\widetilde{\boldsymbol{F}}) = r(\widetilde{\boldsymbol{G}}), \tag{39}$$

$$\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp} = \widetilde{\boldsymbol{R}}\widetilde{\boldsymbol{S}}', \quad r(\widetilde{\boldsymbol{R}}) = r(\widetilde{\boldsymbol{S}}) = r(\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}) \tag{40}$$

be rank factorizations of $\boldsymbol{P}$, $\widetilde{\boldsymbol{C}}'\widetilde{\boldsymbol{B}}$ *and* $\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}$, *respectively, where*

$$\dot{\boldsymbol{P}} = \sum_{k=1}^{q} k\boldsymbol{P}_k. \tag{41}$$

Besides, put

$$\boldsymbol{J} = [\boldsymbol{I}_m \;\; \mathbf{0}_{m,\, m(q-1)}]. \tag{42}$$

Then the following hold:

(a) *a reflexive generalized inverse of* $\boldsymbol{A}$ *is given by*

$$\boldsymbol{A}_r^- = \begin{bmatrix} \boldsymbol{P}^+ & -\boldsymbol{P}^+\mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1} \\ -\mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}_{21}\boldsymbol{P}^+ & \mathbf{\Lambda}_{22}^{-1} + \mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}_{21}\boldsymbol{P}^+\mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1} \end{bmatrix}, \tag{43}$$

where

$$\boldsymbol{P}^+ = (\widetilde{\boldsymbol{C}}')^+\widetilde{\boldsymbol{B}}^+ \tag{44}$$

is the Moore-Penrose inverse of $\boldsymbol{P}$,

(b) $\boldsymbol{C}_{\perp} = \boldsymbol{u}_q \otimes \widetilde{\boldsymbol{C}}_{\perp}$, $\boldsymbol{J}\boldsymbol{C}_{\perp} = \widetilde{\boldsymbol{C}}_{\perp}$, (45)

$$\boldsymbol{B}'_{\perp} = [\widetilde{\boldsymbol{B}}'_{\perp}, -\widetilde{\boldsymbol{B}}'_{\perp}(\boldsymbol{P}_2 + \boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q), -\widetilde{\boldsymbol{B}}'_{\perp}(\boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q), \ldots, -\widetilde{\boldsymbol{B}}'_{\perp}\boldsymbol{P}_q],$$

$$\boldsymbol{J}\boldsymbol{B}_{\perp} = \widetilde{\boldsymbol{B}}_{\perp}, \tag{46}$$

where $\boldsymbol{u}_q$ *is a* $q \times 1$ *vector of* 1's *and* $\otimes$ *denotes the Kronecker product.*

Further, the following hold under $\upsilon = 2$:

(i) $\boldsymbol{A}_r^- \boldsymbol{C}_\perp \boldsymbol{S}_\perp = (\boldsymbol{C}')^- \boldsymbol{B}^+ \boldsymbol{C}_\perp \boldsymbol{S}_\perp$, (47)

(ii) $\boldsymbol{J}\boldsymbol{C}_{2\perp} = [\widetilde{\boldsymbol{C}}_\perp, -\boldsymbol{P}^+ \dot{\boldsymbol{P}} \widetilde{\boldsymbol{C}}_\perp \widetilde{\boldsymbol{S}}_\perp]$, (48)

(iii) $(\boldsymbol{J}\boldsymbol{C}_{2\perp})_\perp = -\widetilde{\boldsymbol{C}}(\widetilde{\boldsymbol{B}}^+ \dot{\boldsymbol{P}} \widetilde{\boldsymbol{C}}_\perp \widetilde{\boldsymbol{S}}_\perp)_\perp$ *provided* $r(\widetilde{\boldsymbol{B}}^+ \dot{\boldsymbol{P}} \widetilde{\boldsymbol{C}}_\perp \widetilde{\boldsymbol{S}}_\perp) = r(\widetilde{\boldsymbol{S}}_\perp)$, (49)

(iv) $(\boldsymbol{J}\boldsymbol{B}\boldsymbol{G}_\perp)_\perp = (\widetilde{\boldsymbol{C}}_\perp \widetilde{\boldsymbol{S}}_\perp)_\perp$ *in general* (50)

$$= [\dot{\boldsymbol{P}}' \widetilde{\boldsymbol{B}}_\perp, (\boldsymbol{J}\boldsymbol{C}_{2\perp})_\perp] \text{ if } r(\widetilde{\boldsymbol{R}}'_\perp \widetilde{\boldsymbol{B}}'_\perp \dot{\boldsymbol{P}} \boldsymbol{P}^+ \dot{\boldsymbol{P}} \widetilde{\boldsymbol{C}}_\perp \widetilde{\boldsymbol{S}}_\perp) = r(\widetilde{\boldsymbol{S}}_\perp)$$

and

$$r(\dot{\boldsymbol{P}}' \widetilde{\boldsymbol{B}}_\perp) = r(\widetilde{\boldsymbol{B}}_\perp), \tag{51}$$

(v) $\boldsymbol{J}\boldsymbol{N}_2 = \boldsymbol{J}\boldsymbol{B}\boldsymbol{G}_\perp (\boldsymbol{F}'_\perp \boldsymbol{G}_\perp)^{-1} \boldsymbol{F}'_\perp \boldsymbol{C}'$, (52)

(vi) $\boldsymbol{J}[\boldsymbol{N}_2, \boldsymbol{N}_1] = \boldsymbol{J}\boldsymbol{C}_{2\perp}\boldsymbol{\Phi}$, (53)

where $\boldsymbol{N}_2$, $\boldsymbol{N}_1$ *and* $\boldsymbol{\Phi}$ *are defined as in Theorem* 1,

(vi) $r(\widetilde{\boldsymbol{B}}) = r(\widetilde{\boldsymbol{S}}_\perp)$ *if* $r(\boldsymbol{A}^2) = (q-1)m$. (54)

Proof of (a). The proof of (43) follows along the same line of reasoning as in Theorem 15, p. 232 in Faliva [8]. The reflexivity property $\boldsymbol{A}_r^- \boldsymbol{A} \boldsymbol{A}_r^- = \boldsymbol{A}_r^-$ is easily checked.

Proof of (b). To prove (45) and (46) observe that by inspection of (5), the conclusion that $\boldsymbol{B}'_\perp$ and $\boldsymbol{C}_\perp$ are full rank solutions of the homogeneous equations

$$\boldsymbol{\Xi}'\boldsymbol{A} = \boldsymbol{0}, \tag{55}$$

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}, \tag{56}$$

respectively, is easily drawn.

Besides, upon noticing that $\boldsymbol{A}$ can be factorized as follows:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Lambda}_{12}\boldsymbol{\Lambda}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{0} & \boldsymbol{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{\Lambda}_{22}^{-1}\boldsymbol{\Lambda}_{21} & \boldsymbol{I} \end{bmatrix}, \tag{57}$$

equations (55) and (56) can be more conveniently rewritten in partitioned form as

$$[\boldsymbol{\Xi}_1', \boldsymbol{\Xi}_2']\begin{bmatrix} \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{C}}' & \boldsymbol{\Lambda}_{12} \\ \mathbf{0} & \boldsymbol{\Lambda}_{22} \end{bmatrix} = [\mathbf{0}', \mathbf{0}'], \tag{58}$$

$$\begin{bmatrix} \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{C}}' & \mathbf{0} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{59}$$

Solving (58) yields

$$\boldsymbol{\Xi}_1' = \widetilde{\boldsymbol{B}}_\perp',$$

$$\boldsymbol{\Xi}_2' = [-\widetilde{\boldsymbol{B}}_\perp'(\boldsymbol{P}_2 + \boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q), -\widetilde{\boldsymbol{B}}_\perp'(\boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q), \ldots, -\widetilde{\boldsymbol{B}}_\perp'\boldsymbol{P}_q], \tag{60}$$

whereas solving (59) yields

$$\boldsymbol{X}_1 = \widetilde{\boldsymbol{C}}_\perp, \quad \boldsymbol{X}_2 = \boldsymbol{u}_{q-1} \otimes \widetilde{\boldsymbol{C}}_\perp \tag{61}$$

as simple computations show.

Proof of (i). First of all observe that by making use of (37), (41), (45) and (46), some computations give

$$\begin{aligned} \boldsymbol{B}_\perp'\boldsymbol{C}_\perp &= \widetilde{\boldsymbol{B}}_\perp'\widetilde{\boldsymbol{C}}_\perp - \widetilde{\boldsymbol{B}}_\perp'(\boldsymbol{P}_2 + \boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q + \boldsymbol{P}_3 + \cdots + \boldsymbol{P}_q + \cdots + \boldsymbol{P}_q)\widetilde{\boldsymbol{C}}_\perp \\ &= \widetilde{\boldsymbol{B}}_\perp'\widetilde{\boldsymbol{C}}_\perp - \widetilde{\boldsymbol{B}}_\perp'\sum_{k=2}^{q}(k-1)\boldsymbol{P}_k\widetilde{\boldsymbol{C}}_\perp \\ &= \widetilde{\boldsymbol{B}}_\perp'\widetilde{\boldsymbol{C}}_\perp - \widetilde{\boldsymbol{B}}_\perp'\left(\sum_{k=1}^{q}k\boldsymbol{P}_k - \sum_{k=1}^{q}\boldsymbol{P}_k\right)\widetilde{\boldsymbol{C}}_\perp \\ &= \widetilde{\boldsymbol{B}}_\perp'(\boldsymbol{P} - \dot{\boldsymbol{P}})\widetilde{\boldsymbol{C}}_\perp = -\widetilde{\boldsymbol{B}}_\perp'\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_\perp. \end{aligned} \tag{62}$$

Thus any rank factorization of $\boldsymbol{B}_\perp'\boldsymbol{C}_\perp$ is also a rank factorization of $-\widetilde{\boldsymbol{B}}_\perp'\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_\perp$ and vice versa, which entails in particular that

$$\boldsymbol{S} = \widetilde{\boldsymbol{S}}, \quad \boldsymbol{S}_\perp = \widetilde{\boldsymbol{S}}_\perp. \tag{63}$$

Then, keeping in mind (36), (37), (41), (43) and (45), notice that

$$\boldsymbol{\Lambda}_{12}\boldsymbol{\Lambda}_{22}^{-1}(\boldsymbol{u}_{q-1} \otimes \widetilde{\boldsymbol{C}}_{\perp}) = [\boldsymbol{P}_2, \boldsymbol{P}_3, ..., \boldsymbol{P}_q]\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{I} & \boldsymbol{I} & \boldsymbol{I} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \widetilde{\boldsymbol{C}}_{\perp} \\ \widetilde{\boldsymbol{C}}_{\perp} \\ \cdots \\ \widetilde{\boldsymbol{C}}_{\perp} \end{bmatrix}$$

$$= \sum_{k=2}^{q}(k-1)\boldsymbol{P}_k\widetilde{\boldsymbol{C}}_{\perp} = \left(\sum_{k=2}^{q} k\boldsymbol{P}_k - \sum_{k=2}^{q}\boldsymbol{P}_k\right)\widetilde{\boldsymbol{C}}_{\perp}$$

$$= \left(\sum_{k=1}^{q} k\boldsymbol{P}_k - \sum_{k=1}^{q}\boldsymbol{P}_k\right)\widetilde{\boldsymbol{C}}_{\perp} = (\dot{\boldsymbol{P}} - \boldsymbol{P} + \boldsymbol{I})\widetilde{\boldsymbol{C}}_{\perp}$$

$$= (\dot{\boldsymbol{P}} + \boldsymbol{I})\widetilde{\boldsymbol{C}}_{\perp}, \tag{64}$$

$$(\boldsymbol{I} - \boldsymbol{P}\boldsymbol{P}^{+})\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} = (\boldsymbol{I} - \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{B}}^{+})\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} = \widetilde{\boldsymbol{B}}_{\perp}(\widetilde{\boldsymbol{B}}'_{\perp}\widetilde{\boldsymbol{B}}_{\perp})^{-1}\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} = \boldsymbol{0},$$

$$\boldsymbol{A}\boldsymbol{A}_r^{-}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp} = \begin{bmatrix} \boldsymbol{P}\boldsymbol{P}^{+} & (\boldsymbol{I} - \boldsymbol{P}\boldsymbol{P}^{+})\boldsymbol{\Lambda}_{12}\boldsymbol{\Lambda}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I}_{q-1} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \\ (\boldsymbol{u}_{q-1} \otimes \widetilde{\boldsymbol{C}}_{\perp})\widetilde{\boldsymbol{S}}_{\perp} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{P}\boldsymbol{P}^{+}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} + (\boldsymbol{I} - \boldsymbol{P}\boldsymbol{P}^{+})(\dot{\boldsymbol{P}} + \boldsymbol{I})\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \\ (\boldsymbol{u}_{q-1} \otimes \widetilde{\boldsymbol{C}}_{\perp})\widetilde{\boldsymbol{S}}_{\perp} \end{bmatrix}$$

$$= \begin{bmatrix} (\boldsymbol{P}\boldsymbol{P}^{+} + \boldsymbol{I} - \boldsymbol{P}\boldsymbol{P}^{+})\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \\ (\boldsymbol{u}_{q-1} \otimes \widetilde{\boldsymbol{C}}_{\perp})\widetilde{\boldsymbol{S}}_{\perp} \end{bmatrix},$$

$$(\boldsymbol{u}_q \otimes \widetilde{\boldsymbol{C}}_{\perp})\widetilde{\boldsymbol{S}}_{\perp} = \boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}, \tag{65}$$

$$(\boldsymbol{C}')^{-}\boldsymbol{B}^{+}\boldsymbol{A}\boldsymbol{A}_r^{-} = (\boldsymbol{C}')^{-}\boldsymbol{C}'\boldsymbol{A}_r^{-} = \boldsymbol{A}_r^{-} \tag{66}$$

for some $(\boldsymbol{C}')^{-}$ in view of Lemma 2.5.2, p. 28 in Rao and Mitra [20].

Hence, equality (47) ensues from (65) and (66) via premultiplication of the former by $(\boldsymbol{C}')^{-}\boldsymbol{B}^{+}$.

Proof of (ii). Resorting to (43), (45), (63), (64) and (14) premultiplying the latter by $\boldsymbol{J}$, yields

$$\begin{aligned} \boldsymbol{J}\boldsymbol{C}_{2\perp} &= [\boldsymbol{J}\boldsymbol{C}_{\perp},\ \boldsymbol{J}\boldsymbol{A}_r^{-}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp}] \\ &= [\widetilde{\boldsymbol{C}}_{\perp},\ [\boldsymbol{P}^{+},\ -\boldsymbol{P}^{+}\boldsymbol{\Lambda}_{12}\boldsymbol{\Lambda}_{22}^{-1}]]\cdot\begin{bmatrix} \widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \\ (\boldsymbol{u}_{q-1}\otimes\boldsymbol{I}_m)\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \end{bmatrix} \\ &= [\widetilde{\boldsymbol{C}}_{\perp},\ \boldsymbol{P}^{+}(\boldsymbol{I}-\dot{\boldsymbol{P}}-\boldsymbol{I})\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp}] \\ &= [\widetilde{\boldsymbol{C}}_{\perp},\ -\boldsymbol{P}^{+}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp}]. \end{aligned}$$

Proof of (iii). Result (49) can be obtained from (48) following the same line of reasoning used to deduce (16) from (14) in Theorem 3.1, by making use of (44). The rank condition is required for $(\widetilde{\boldsymbol{B}}^{+}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp})_{\perp}$ to be a meaningful expression.

Proof of (iv). Result (50), which is the mirror image of (17), is easily established upon noting that

$$\boldsymbol{J}\boldsymbol{B}\boldsymbol{G}_{\perp} = \boldsymbol{J}\boldsymbol{B}\boldsymbol{B}^{+}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp} = \boldsymbol{J}\boldsymbol{C}_{\perp}\boldsymbol{S}_{\perp} = \widetilde{\boldsymbol{C}}_{\perp}\widetilde{\boldsymbol{S}}_{\perp} \tag{67}$$

in light of (13), (28), (45) and (63).

Bearing in mind (67), the proof of (51) rests essentially on the same line of arguments set forth to prove (18). The rank conditions are needed for the columns of the matrix in the right-hand side to provide a basis for the row kernel of $\boldsymbol{J}\boldsymbol{B}\boldsymbol{G}_{\perp}$.

Proof of (v) and (vi). Results (52) and (53) follow from (19) and (21), respectively, through premultiplication by $\boldsymbol{J}$.

Proof of (vii). Keeping in mind (8), (45) and (62), we can write

$$r(\boldsymbol{A}) - r(\boldsymbol{A}^2) = r(\boldsymbol{C}_{\perp}) - r(\boldsymbol{B}'_{\perp}\boldsymbol{C}_{\perp}) = r(\widetilde{\boldsymbol{C}}_{\perp}) - r(\widetilde{\boldsymbol{B}}'_{\perp}\dot{\boldsymbol{P}}\widetilde{\boldsymbol{C}}_{\perp}) = r(\widetilde{\boldsymbol{S}}_{\perp}). \tag{68}$$

Further, resorting to the rank equality

$$r(\boldsymbol{A}) = r(\boldsymbol{\Lambda}_{22}) + r(\boldsymbol{P}),$$

noting that

$$r(\boldsymbol{\Lambda}_{22}) = m(q-1), \quad r(\boldsymbol{P}) = r(\widetilde{\boldsymbol{B}})$$

and substituting into (68), we eventually get the equality

$$r(\widetilde{\boldsymbol{B}}) + m(q-1) - r(\boldsymbol{A}^2) = r(\widetilde{\boldsymbol{S}}_{\perp}). \tag{69}$$

By inspection of (69), it follows that

$$r(\widetilde{\boldsymbol{B}}) = r(\widetilde{\boldsymbol{S}}_{\perp}) \leftrightarrow r(\boldsymbol{A}^2) = m(q-1).$$

We will now present a useful decomposition of a square matrix into a component of index one, known as core component (Campbell and Meyer [3, p. 127]), and a nilpotent term.

Theorem 3. *A square matrix* $\boldsymbol{A}$ *with index* υ *has a unique decomposition*

$$\boldsymbol{A} = \boldsymbol{K} + \boldsymbol{H} \tag{70}$$

with the properties

(i) $\operatorname{ind}(\boldsymbol{K}) = 1,$ (71)

(ii) $\boldsymbol{H}^{\upsilon} = 0,$ (72)

(iii) $\boldsymbol{HK} = \boldsymbol{KH} = \boldsymbol{0},$ (73)

(iv) $r(\boldsymbol{A}^k) = r(\boldsymbol{H}^k) + r(\boldsymbol{K}), \quad k = 1, 2, \ldots,$ (74)

(v) $\boldsymbol{A}^{\upsilon} = \boldsymbol{K}^{\upsilon},$ (75)

(vi) $\boldsymbol{C}_{\upsilon\perp}(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp})^{-1}\boldsymbol{B}'_{\upsilon\perp} = \overline{\boldsymbol{C}}_{\perp}(\overline{\boldsymbol{B}}'_{\perp}\overline{\boldsymbol{C}}_{\perp})^{-1}\overline{\boldsymbol{B}}'_{\perp},$ (76)

where $\boldsymbol{B}_{\upsilon}$, $\boldsymbol{C}_{\upsilon}$ *are as defined in* (4) *and* $\overline{\boldsymbol{B}}$ *and* $\overline{\boldsymbol{C}}$ *are full column-rank matrices obtained by a rank factorization of* $\boldsymbol{K}$, *that is*,

$$\boldsymbol{K} = \overline{\boldsymbol{B}}\overline{\boldsymbol{C}}', \quad r(\boldsymbol{K}) = r(\overline{\boldsymbol{B}}) = r(\overline{\boldsymbol{C}}). \tag{77}$$

Proof. For a proof of (i)-(v) see Rao and Mitra [20, p. 93] and Campbell and Meyer [3, p. 121]. For what concerns (vi) observe first that

$$(\overline{\boldsymbol{B}}\overline{\boldsymbol{C}}')^{\upsilon} = \overline{\boldsymbol{B}}(\overline{\boldsymbol{C}}'\overline{\boldsymbol{B}})^{\upsilon-1}\overline{\boldsymbol{C}}' = \boldsymbol{B}_{\upsilon}\boldsymbol{C}'_{\upsilon} \tag{78}$$

because of (75) and (77). Hence, $\overline{\boldsymbol{B}}_{\perp}$ and $\overline{\boldsymbol{C}}_{\perp}$ turn out to play the role of orthogonal complements of $\boldsymbol{B}_\upsilon$ and $\boldsymbol{C}_\upsilon$ and equality (76) ensues from Theorem 5, p. 5 in Faliva and Zoia [12].

The lemma and the theorem below provide useful matrix-function inversion formulas about a pole on the basis of the core-nilpotent decomposition above.

Lemma. *Consider the matrix functions*

$$\boldsymbol{H}(z) = \boldsymbol{I} + \overline{\boldsymbol{H}} \frac{1}{(1-z)}, \tag{79}$$

$$\boldsymbol{K}(z) = (1-z)(\boldsymbol{I} - \boldsymbol{K}) + \boldsymbol{K}, \tag{80}$$

$$\overline{\boldsymbol{H}} = (\boldsymbol{I} - \boldsymbol{H})^{-1}\boldsymbol{H}, \tag{81}$$

where $\boldsymbol{H}$ *and* $\boldsymbol{K}$ *are as in the foregoing theorem, and* $\det \boldsymbol{K}(z)$ *has all its roots outside the unit circle, except for a possibly multiple unit-root.*

The following Laurent expansions hold for $\boldsymbol{H}^{-1}(z)$ *and* $\boldsymbol{K}^{-1}(z)$ *in a deleted neighbourhood of* $z = 1$:

$$\text{(i)}\ \boldsymbol{H}^{-1}(z) = \boldsymbol{I} + \sum_{i=1}^{\upsilon-1} \frac{(-1)^i}{(1-z)^i} \overline{\boldsymbol{H}}^i, \tag{82}$$

$$\text{(ii)}\ \boldsymbol{K}^{-1}(z) = \frac{1}{(1-z)} \boldsymbol{N} + \widetilde{\boldsymbol{M}}(z), \tag{83}$$

where

$$\boldsymbol{N} = \boldsymbol{C}_{\upsilon\perp}(\boldsymbol{B}'_{\upsilon\perp}\boldsymbol{C}_{\upsilon\perp})^{-1}\boldsymbol{B}'_{\upsilon\perp}, \tag{84}$$

$$\widetilde{\boldsymbol{M}}(z) = \sum_{i=0}^{\infty} \widetilde{\boldsymbol{M}}_i z^i, \tag{85}$$

$$\widetilde{\boldsymbol{M}}(1) = \overline{\boldsymbol{B}}(\overline{\boldsymbol{C}}'\overline{\boldsymbol{B}})^{-2}\overline{\boldsymbol{C}}' \tag{86}$$

and the $\widetilde{\boldsymbol{M}}_i$*'s are matrices with exponentially decreasing entries.*

Proof. To prove (i), observe that $\boldsymbol{H}$ and $(\boldsymbol{I} - \boldsymbol{H})^{-1}$ commute, the matrix $\overline{\boldsymbol{H}}$ enjoys the same nilpotency property as $\boldsymbol{H}$ as a by-product and the expansion in the right-hand side of (82) holds accordingly.

The proof of (ii) can be obtained resorting to Theorem 1, p. 37 in Faliva and Zoia [12], upon noting that

$$\dot{\boldsymbol{K}} = \boldsymbol{K} - \boldsymbol{I} \Rightarrow \boldsymbol{B}_{\perp}'\dot{\boldsymbol{K}}\boldsymbol{C}_{\perp} = -\boldsymbol{B}_{\perp}'\boldsymbol{C}_{\perp}, \tag{87}$$

$$\operatorname{ind}(\boldsymbol{K}) = 1 \Rightarrow \det(\boldsymbol{B}_{\perp}'\boldsymbol{C}_{\perp}) \neq 0, \tag{88}$$

where the dot stands for derivatives and reference is made to (10). Results (83), (84) and (85) hold accordingly. Finally, formula (86) proves true in light of the said theorem upon noting that

$$\ddot{\boldsymbol{K}} = \boldsymbol{0} \tag{89}$$

and resorting to identity (35).

Theorem 4. *Consider the matrix polynomial*

$$\boldsymbol{A}(z) = (1-z)\boldsymbol{I} + z\boldsymbol{A}, \quad \boldsymbol{A} = \boldsymbol{H} + \boldsymbol{K} \tag{90}$$

which can be factorized as

$$\boldsymbol{A}(z) = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{H}(z)\boldsymbol{K}(z), \tag{91}$$

where the symbols have the same meaning as in the previous lemma.

Then, the following Laurent expansion

$$\boldsymbol{A}^{-1}(z) = \sum_{i=1}^{\upsilon} \frac{1}{(1-z)^{i}} \boldsymbol{N}_i + \sum_{i=0}^{\infty} \boldsymbol{M}_i z^i \tag{92}$$

holds in a deleted neighbourhood of $\boldsymbol{z} = 1$.

The following closed-form expressions hold for the coefficient matrices $\boldsymbol{N}_{\upsilon}$ *and* $\boldsymbol{N}_{\upsilon-1}$:

$$\boldsymbol{N}_{\upsilon} = (-1)^{\upsilon-1}\boldsymbol{N}\boldsymbol{H}^{\upsilon-1}, \tag{93}$$

$$\boldsymbol{N}_{\upsilon-1} = (-1)^{\upsilon-2}\boldsymbol{N}\boldsymbol{H}^{\upsilon-2} + (-1)^{\upsilon-2}(\upsilon-1)\boldsymbol{N}\boldsymbol{H}^{\upsilon-1} \tag{94}$$

adopting the convention that $\boldsymbol{H}^0 = \boldsymbol{I}$.

Proof. The proof of (92) is straightforward upon noting that

$$\boldsymbol{A}^{-1}(z) = \boldsymbol{K}^{-1}(z)\boldsymbol{H}^{-1}(z)(\boldsymbol{I} - \boldsymbol{H})^{-1} \tag{95}$$

and by replacing the inverses of the matrix functions appearing in the right-hand sides with their Laurent expansions given by formulae (82) and (83) of the Lemma.

Formulas (93) and (94) follow from (82), (83) and (86) by making use of the first matrix coefficient $\widetilde{\boldsymbol{M}}(1)$ in the expansion of $\widetilde{\boldsymbol{M}}(z)$ about $z = 1$. Indeed, simple computations yield

$$\boldsymbol{N}_{\upsilon} = (-1)^{\upsilon-1}\boldsymbol{N}\boldsymbol{H}^{\upsilon-1}(\boldsymbol{I} - \boldsymbol{H})^{-\upsilon}, \tag{96}$$

$$\boldsymbol{N}_{\upsilon-1} = (-1)^{\upsilon-2}\boldsymbol{N}\boldsymbol{H}^{\upsilon-2}(\boldsymbol{I} - \boldsymbol{H})^{-\upsilon+1} + (-1)^{\upsilon-1}\widetilde{\boldsymbol{M}}(1)\boldsymbol{H}^{\upsilon-1}(\boldsymbol{I} - \boldsymbol{H})^{-\upsilon}. \tag{97}$$

Now, applying the binomial theorem to $(\boldsymbol{I} - \boldsymbol{H})^{-\upsilon}$ and $(\boldsymbol{I} - \boldsymbol{H})^{-\upsilon+1}$, formulas (96) and (97) simplify into formulas (93) and (94), bearing in mind the nilpotency of $\boldsymbol{H}$ and that

$$\widetilde{\boldsymbol{M}}(1)\boldsymbol{H} = \boldsymbol{0} \tag{98}$$

in light of formulas (86) and (73).

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