# AN ALGORITHM FOR FUZZY RELATION EQUATIONS WITH MAX-PRODUCT COMPOSITION 

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#### Abstract

In this paper, we study the solution procedure of fuzzy relation system with max-product composition. We first simplify the system and get its standard form. Then we define the standard form's minimal complete set suite and chained-set suite, and reveal the relations among these concepts and minimal solutions. Finally, by utilizing these concepts and results, we obtain an algorithm for the fuzzy relation system with max-product composition.


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## 1. Introduction

The object of this paper is to study the following fuzzy relation equations with max-product composition:

$$
\left\{\begin{array}{l}
a_{11} \cdot x_{1} \vee a_{12} \cdot x_{2} \vee \ldots \vee a_{1 n} \cdot x_{n}=b_{1}  \tag{I}\\
a_{21} \cdot x_{1} \vee a_{22} \cdot x_{2} \vee \ldots \vee a_{2 n} \cdot x_{n}=b_{2} \\
\ldots \cdot \ldots \\
\ldots
\end{array} \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,\right.
$$

where $a_{i j}, x_{j}, b_{i} \in[0,1](i=1,2, \ldots, m, j=1,2, \ldots, n)$, ' $'$ represents the ordinary multiplication and $a \vee b=\max \{a, b\}$.

Model (I)'s various applications have been found in finite fuzzy machines (as inference engine) and some optimization problems [4-8, 16].

The notion of fuzzy relation equations with the max-min composition was first investigated by Sanchez [17]. Since then the fuzzy relation equations have been extended to the fuzzy relation equations with the $t$-norm composition, in which the max-product composition is a member [2, 10, 13]. Several studies [3, 11, 12, 18, 19,21 ] have shown that the max-min operator may not always be the most desirable fuzzy relational composition and in fact the max-product operator was superior in these instances. Some outlines for selecting an appropriate operator of fuzzy relation have been provided by Yager [20].

Bourke and Fisher [1] studied fuzzy relation equations with max-product composition and provided some theoretical results for the solution set of (I). Their results showed that when the solution set of (I) is not empty, it can be completely determined by one unique maximum solution and a finite number of minimal solutions.

Loetamonphong and Fang [6, 7] explored also the solution set of (I). They studied the feasible domain of (I) and got some characteristics of the solution set of (I). They pointed out that since the total number of minimal solutions of (I) has a combinatorial nature in terms of the problem size, an efficient solution procedure is always in demand.

Markovskii [9] discussed the relation between equations with max-product
composition and the covering problem, which belongs to category of NP-hard problems. By means of a straightforward exhaustive search of this NP-hard problem, Markovskii obtained an algorithm to solve (I). The method may involve heavy and complicated work.

Peeva and Kyosev [16] obtained an algorithm to solve (I) by extending the methodology developed for max-min fuzzy relational equations [14, 15] for the case of max-product composition.

We can see from [16] that the involved workload of the algorithm is still great.
The aim of this paper is to search for an efficient and convenient method of algebra to solve (I). The algorithm only relates to the structure of (I) and consists of some simple and direct calculation procedures.

## 2. Preliminaries

Let $\boldsymbol{I}=\{1,2, \ldots, m\}$ and $\boldsymbol{J}=\{1,2, \ldots, n\}$ be two index sets.
System (I) can be described by matrix notation:

$$
\begin{equation*}
A \circ x^{T}=b^{T} \tag{I}
\end{equation*}
$$

where $A=\left(a_{i j}\right)_{m \times n}, x=\left(x_{1}, x_{2}, \ldots, x_{n}\right), b=\left(b_{1}, b_{2}, \ldots, b_{m}\right), a_{i j}, x_{j}, b_{i} \in[0,1]$, $\forall i \in \boldsymbol{I}, \forall j \in \boldsymbol{J}$, and the operation " $\circ$ " represents the max-product composition.

Definition 1 [1]. For system (I), let $\hat{x}=\left(\hat{x}_{1}, \hat{x}_{2}, \ldots, \hat{x}_{n}\right)$, in which

$$
\hat{x}_{j}=\stackrel{n}{\wedge_{i=1}} a_{i j} \odot b_{i}, \quad \forall j \in J
$$

where $a \wedge b=\min \{a, b\}, a_{i j} \odot b_{i}= \begin{cases}1, & \text { if } a_{i j} \leq b_{i}, \\ b_{i} / a_{i j}, & \text { if } a_{i j}>b_{i} .\end{cases}$
We call $\hat{x}$ the criterion vector of (I).
Theorem 1 [1]. If (I) is solvable, then $\hat{x}$ is the maximum solution of (I).
Lemma 1. For (I), denote $D_{i}=\left\{j \mid a_{i j} \neq 0, j \in \boldsymbol{J}\right\}, i \in \boldsymbol{I}$. Let (I) be solvable and $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be any solution of (I). If in (I), there is $b_{i}=0$, then

$$
\begin{equation*}
x_{j}=0, \quad \forall j \in D_{i} \tag{1}
\end{equation*}
$$

Proof. Substituting $x$ into (I), the $i$ th equation of (I) is

$$
a_{i 1} x_{1} \vee a_{i 2} x_{2} \vee \ldots \vee a_{i n} x_{n}=b_{i}=0 .
$$

For any $j \in D_{i}$, we have $a_{i j} x_{j}=0, a_{i j} \neq 0$, so $x_{j}=0$.
If there is $b_{i}=0$ in (I), then using Lemma 1, we can simplify (I) as follows:

1. Substituting the result (1) into (I), the ith equation of (I) is satisfied, no matter what the other components of $x$ are. Hence, in further solving process, we do not need to consider the $i$ th equation any more. So, we can delete the $i$ th equation of (I).
2. For each $j \in D_{i}$, substituting the result (1) into (I), the $j$ th column of (I) becomes a zero column. In further solving process, the zero columns will be useless, moreover we have had the result: $x_{j}=0$. Hence, we do not need to consider the zero columns any more. So, we can delete the $j$ th column of (I) for each $j \in D_{i}$.

After we delete the $i$ th row and the $j$ th column $\left(\forall j \in D_{i}\right)$, we get a simplified system (II) in which all the constant terms are not zero. Clearly, if (I) is solvable, then (II) is solvable.

Hereinafter, we always assume that (I) is solvable.
We call (II) the standard form of (I).
Clearly, once we get all the minimal solutions of (II), we can combine them with the fact " $x_{j}=0, \forall j \in D_{i}$ " to form all the minimal solutions of (I).

So, from now on, our discussion is to focus on the standard form (II). In order to avoid fussy notations, we still use system (I) to express system (II). In (II), we only change the supposition of (I) " $b_{i} \in[0,1], \forall i \in \boldsymbol{I}$ " into " $b_{i} \in(0,1], \forall i \in \boldsymbol{I}$.". That is to say, system (II) is

$$
\begin{equation*}
A \circ x^{T}=b^{T}, \tag{II}
\end{equation*}
$$

where $A=\left(a_{i j}\right)_{m \times n}, \quad x=\left(x_{1}, x_{2}, \ldots, x_{n}\right), \quad b=\left(b_{1}, b_{2}, \ldots, b_{m}\right), \quad a_{i j}, x_{j} \in[0,1]$, $b_{i} \in(0,1], \forall i \in I, \forall j \in J$, and the operation " $\circ$ " represents the max-product composition.

We denote the solution set of (II) by

$$
X(A, b)=\left\{\left(x_{1}, \ldots, x_{n}\right) \mid x_{j} \in[0,1], j \in \boldsymbol{J} \text { and } A \circ x^{T}=b^{T}\right\}
$$

Define $X=[0,1]^{n}$. For $x^{1}, x^{2} \in X$, we say $x^{1} \leq x^{2}$ if and only if $x_{j}^{1} \leq x_{j}^{2}$, $\forall j \in \boldsymbol{J}$. In this way, the operator " $\leq$ " forms a partial order relation on $X$ and $(X, \leq)$ becomes a lattice.
$\hat{x} \in X(A, b)$ is called a maximum solution if $x \leq \hat{x}$ for all $x \in X(A, b)$. Also, $\breve{x} \in X(A, b)$ is called a minimal solution if $x \leq \breve{x}$, for any $x \in X(A, b)$, implies $x=\breve{x}$.

Denote the set of all minimal solutions of (II) by $\bar{X}(A, b)$. Then it is clear that

$$
X(A, b)=\bigcup_{\widetilde{x} \in \bar{X}(A, b)}\{x \in X \mid \widetilde{x} \leq x \leq \hat{x}\}
$$

## 3. Minimal Solutions and Minimal Complete Set Suite

Hereinafter, we always assume that (II) is solvable.
For (II), let $\hat{x}=\left(\hat{x}_{1}, \hat{x}_{2}, \ldots, \hat{x}_{n}\right)$, in which $\hat{x}_{j}=\widehat{i=1}_{n}^{n} a_{i j} \odot b_{i}, \forall j \in \boldsymbol{J}$.
Since (II) is a special form of general system (I), according to Theorem 1, we know that $\hat{x}$ is the maximum solution of (II).

Note 1. Because $b_{i} \in(0,1], \forall i \in \boldsymbol{I}$, then $\hat{x}_{j}=\stackrel{n}{\wedge_{i=1}} a_{i j} \odot b_{i}>0, \forall j \in \boldsymbol{J}$.
For (II), $\forall j \in \boldsymbol{J}$ denote $I_{j}=\left\{i \mid a_{i j} \hat{x}_{j}=b_{i}, i \in \boldsymbol{I}\right\}[6]$ and $\bar{I}_{j}=\left\{i \mid a_{i j} \hat{x}_{j} \neq b_{i}\right.$, $i \in I\}$.

From the definitions of $I_{j}$ and $\bar{I}_{j}$, we can easily get the following results:
Lemma 2. $\forall i \in \boldsymbol{I}, \forall j \in \boldsymbol{J}, a_{i j} \hat{x}_{j} \leq b_{i}$.
Lemma 3. $\bar{I}_{j}=\left\{i \mid a_{i j} \hat{x}_{j}<b_{i}, i \in \boldsymbol{I}\right\}$.
Lemma 4. Let $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a solution of (II). If $a_{i j} x_{j}=b_{i}$, then $x_{j}=\hat{x}_{j}$.

Lemma 5. Let $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a minimal solution of (II). If $x_{j} \neq 0$ $(1 \leq j \leq n)$, then $x_{j}=\hat{x}_{j}$.

Proof. Substitute $x$ into (II) and we have

$$
\begin{equation*}
a_{i 1} x_{1} \vee a_{i 2} x_{2} \vee \ldots \vee a_{i j} x_{j} \vee \ldots \vee a_{i n} x_{n}=b_{i}, \quad i \in \boldsymbol{I} . \tag{2}
\end{equation*}
$$

By (2), we have $\forall i \in I, a_{i j} x_{j} \leq b_{i}$.

If $\forall i \in \boldsymbol{I}, a_{i j} x_{j}<b_{i}$, then utilizing $x$, we can get one new vector $x^{\prime}$ by the method: in $x$, let $x_{j}=0$ and keep all the other components intact. Clearly by (2), $x^{\prime}$ is still a solution of (II), a contradiction to the hypothesis that $x$ is a minimal solution of (II) in which $x_{j} \neq 0$.

Hence, $\exists i_{0} \in I$ such that $a_{i_{0} j} x_{j}=b_{i_{0}}$. So, by Lemma 4, $x_{j}=\hat{x}_{j}$.

Note 2. If $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is a minimal solution of (II), then $x \neq 0$. In fact, if $x=0$, then substituting $x=0$ into (II), it would be $b_{1}=b_{2}=\ldots=b_{m}=0$, a contradiction to the supposition: $b_{i} \in(0,1], \forall i \in I$.

For $j_{1}, j_{2}, \ldots, j_{k} \in \boldsymbol{J}$, we denote $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, in which $x_{j_{1}}=\hat{x}_{j_{1}}, x_{j_{2}}=\hat{x}_{j_{2}}, \ldots, x_{j_{k}}=\hat{x}_{j_{k}}, \quad x_{j}=0, \quad \forall j \in \boldsymbol{J}-\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$, where $1 \leq k \leq n$.

For example, $x\{1,2, \ldots, n\}=\left(\hat{x}_{1}, \hat{x}_{2}, \ldots, \hat{x}_{n}\right)=\hat{x}$, it is the maximum solution of (II).

By Lemma 5, we have
Theorem 2. Let $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a minimal solution of (II), in which $x_{j_{1}}, x_{j_{2}}, \ldots, x_{j_{k}} \neq 0, x_{j}=0, \forall j \in J-\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$, then $x=x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ where $1 \leq k \leq n$.

Definition 2. If there are $j_{1}, j_{2}, \ldots, j_{k} \in J$ such that $\bigcup_{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}} I_{j}=I$, then we call $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ a complete set suite, where $1 \leq k \leq n$.

Theorem 3. $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a complete set suite if and only if $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ is a solution of (II).

Proof. Let $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ be a solution of (II). For each $i \in \boldsymbol{I}$, substituting $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ into (II), the $i$ th equation of (II) is

$$
a_{i j_{1}} \hat{x}_{j_{1}} \vee a_{i j_{2}} \hat{x}_{j_{2}} \vee \ldots \vee a_{i j_{k}} \hat{x}_{j_{k}}=b_{i}
$$

According to the above equation, there is some $j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ such that $a_{i j} \hat{x}_{j}=b_{i}$, then $i \in I_{j}$, hence $\boldsymbol{I} \subseteq I_{j_{1}} \cup I_{j_{2}} \cup \ldots \cup I_{j_{k}}$. Hence $I_{j_{1}} \cup I_{j_{2}}$ $\cup \ldots \cup I_{j_{k}}=\boldsymbol{I}$.

If $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a complete set suite, then $I_{j_{1}} \cup I_{j_{2}} \cup \ldots \cup I_{j_{k}}=I$. Denying the thesis, let us assume that $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ is not a solution of (II). Then substituting $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ into (II), there must be some $i \in \boldsymbol{I}$ such that $a_{i j_{1}} \hat{x}_{j_{1}} \vee a_{i j_{2}} \hat{x}_{j_{2}} \vee \ldots \vee a_{i j_{k}} \hat{x}_{j_{k}} \neq b_{i}$. By Lemma 2, in fact, the above is $a_{i j_{1}} \hat{x}_{j_{1}}$ $\vee a_{i j_{2}} \hat{x}_{j_{2}} \vee \ldots \vee a_{i j_{k}} \hat{x}_{j_{k}}<b_{i}$. So, $a_{i j_{t}} \hat{x}_{j_{t}}<b_{i}, t=1,2, \ldots, k$. Hence, by Lemma 3, $i \in \bar{I}_{j_{t}}, t=1,2, \ldots, k$, then $i \notin I_{j_{1}} \cup I_{j_{2}} \cup \ldots \cup I_{j_{k}}$, thus $I_{j_{1}} \cup I_{j_{2}} \cup \ldots \cup I_{j_{k}} \neq \boldsymbol{I}$, a contradiction to the hypothesis.

Definition 3. Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be a complete set suite. If for each $t \in\{1,2, \ldots, k\}, \quad \underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}-\left\{j_{t}\right\}}{ } I_{j} \neq \boldsymbol{I}$, then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is called a minimal complete set suite, where $2 \leq k \leq n$. Specially, if there exists a $I_{j}$ such that $I_{j}=\boldsymbol{I}$, then $I_{j}$ is called a minimal complete set suite also.

Theorem 4. $x=x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ is a minimal solution of (II) if and only if $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite.

Proof. If $x$ is a minimal solution of (II), then by Theorem 3, $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a complete set suite.

For each $j_{t} \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$, according to Definition 3, we need to prove that $\underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}-\left\{j_{t}\right\}}{ } I_{j} \neq \boldsymbol{I}$. Denying the thesis, let us assume that
$\underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}-\left\{j_{t}\right\}}{ } I_{j}=\boldsymbol{I}$, then by Theorem 3, $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right)=x\left\{j_{1}\right.$, $\left.j_{2}, \ldots, j_{t-1}, j_{t+1}, \ldots, j_{k}\right\}$ is a solution of (II). But $x^{*} \leq x$ and $x_{j_{t}}^{*}=0$ $<x_{j_{t}}=\hat{x}_{j_{t}} \neq 0 \quad($ see Note 1), a contradiction to the hypothesis that $x=x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ is a minimal solution.

Now, we prove the sufficiency. Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be a minimal complete set suite, then by Theorem 3, $x$ is a solution of (II). In the following, we will prove that $x$ is a minimal solution of (II).

Denying the thesis, let us assume that $x$ is not a minimal solution of (II), then there must exist a minimal solution of (II): $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right)$ such that $x^{*} \leq x$ and $x_{j^{*}}^{*}<x_{j^{*}}$, for some $j^{*} \in J$.

Since $x^{*} \leq x$ and $x_{j}=0, \forall j \in \boldsymbol{J}-\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$, we have that

$$
\begin{equation*}
x_{j}^{*}=0, \quad \forall j \in \boldsymbol{J}-\left\{j_{1}, j_{2}, \ldots, j_{k}\right\} \tag{3}
\end{equation*}
$$

Since $x_{j^{*}}^{*}<x_{j^{*}}, \quad x_{j^{*}} \neq 0$, so $j^{*} \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$. Without loss of generality, we may assume that $j^{*}=j_{k}$,

$$
\begin{equation*}
x_{j_{k}}^{*}<x_{j_{k}}=\hat{x}_{j_{k}} . \tag{4}
\end{equation*}
$$

If $x_{j_{k}}^{*} \neq 0$, then by Lemma $5, x_{j_{k}}^{*}=\hat{x}_{j_{k}}$, a contradiction to (4). Thus $x_{j_{k}}^{*}=0$. By connecting the fact with (3), we get $x_{j}^{*}=0, \forall j \in \boldsymbol{J}-\left\{j_{1}, j_{2}, \ldots\right.$, $\left.j_{k-1}\right\}$. So, the nonzero components of $x^{*}$ are among $x_{j_{1}}^{*}, x_{j_{2}}^{*}, \ldots, x_{j_{k-1}}^{*}$.

Without loss of generality, we may assume that the set of the nonzero components of $x^{*}$ is $\left\{x_{j_{1}}^{*}, x_{j_{2}}^{*}, \ldots, x_{j_{s}}^{*}\right\}\left(\right.$ here, $\left.\left\{j_{1}, j_{2}, \ldots, j_{s}\right\} \subseteq\left\{j_{1}, j_{2}, \ldots, j_{k-1}\right\}\right)$. By Theorem 2, $x^{*}=x\left\{j_{1}, j_{2}, \ldots, j_{s}\right\}$, hence by Theorem 3, we have $I_{j_{1}} \cup I_{j_{2}}$ $\cup \cdots \cup I_{j_{s}}=\boldsymbol{I}$. Because $\left\{j_{1}, j_{2}, \ldots, j_{s}\right\} \subseteq\left\{j_{1}, j_{2}, \ldots, j_{k-1}\right\}$, then $I_{j_{1}} \cup I_{j_{2}}$ $\cup \cdots \cup I_{j_{k-1}}=\boldsymbol{I}$, a contradiction to the hypothesis that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite.

Let $\boldsymbol{S}$ be the set of all minimal complete set suites of (II).
By Theorems 2 and 4, we get the following corollary:
Corollary 1. For each $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\} \in \boldsymbol{S}$, let $\sigma:\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ $\mapsto x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$, then $\sigma$ is a one-to-one correspondence from $\boldsymbol{S}$ to $\breve{X}(A, b)$, where $1 \leq k \leq n$.

## 4. Chained-set Suites and Independent-set Suites

Denote $\boldsymbol{F}=\left\{I_{1}, I_{2}, \ldots, I_{n}\right\}, \boldsymbol{F}^{*}=\left\{I_{j} \mid 1 \in I_{j}, I_{j} \in \boldsymbol{F}\right\}$.
Definition 4. For $I_{j_{1}} \in \boldsymbol{F}^{*}$, if in $\boldsymbol{F}$ there exists a complete set suite: $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ such that
(1) $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \neq \boldsymbol{I}$;
(2) For each $t \in\{2,3, \ldots, k\}, i_{t} \in I_{j_{t}}$,
where $i_{t}$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{t-1}}}, t=2,3, \ldots, k$;
(3) For each $t \in\{2,3, \ldots, k\}, I_{j_{s}} \not \subset I_{j_{t}}, s=1,2, \ldots, t-1$.

Then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is called a chained-set suite (expanded by $I_{j_{1}}$ ).
Specially, if there exists a $I_{j}$ such that $I_{j}=I$, then we call $I_{j}$ a chained-set suite also.

Theorem 5. If $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite, then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a chained-set suite.

Proof. Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be a minimal complete set suite.
If $k=1$, then $I_{j_{1}}=I$, so $I_{j_{1}}$ is a chained-set suite. In the following, let $k>1$.

By Definition 3, we have

$$
\begin{equation*}
I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \cup I_{j_{k}}=\boldsymbol{I} \tag{5}
\end{equation*}
$$

From (5), we know that 1 must belong to some $I_{j_{s}}, 1 \leq s \leq k$. Without loss of generality, we may assume that $1 \in I_{j_{1}}$, i.e., $I_{j_{1}} \in \boldsymbol{F}^{*}$.

Now, we first prove that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ satisfies (1), (2), (3) of Definition 4.
a. We first prove that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ satisfies Definition 4(2):

If $k=2$, then by Definition 3, we have $I_{j_{1}} \neq \boldsymbol{I}$, hence $\bar{I}_{j_{1}} \neq \phi$. Let $i_{2}$ be the smallest element of $\bar{I}_{j_{1}}$. By equation (5), $i_{2} \in I_{j_{2}}$. So, the thesis is true.

Let $k$ be a natural number which $>2$ (Next, we use mathematical induction to $t)$.

By Definition 3, we have $I_{j_{1}} \neq \boldsymbol{I}$, hence $\bar{I}_{j_{1}} \neq \phi$. Let $i_{2}$ be the smallest element of $\bar{I}_{j_{1}}$. By equation (5), $i_{2}$ must belong to some $I_{j_{s}}, 2 \leq s \leq k$. Without loss of generality, we may assume that $i_{2} \in I_{j_{2}}$. That is to say, when $t=2, i_{t} \in I_{j_{t}}$.

By Definition 3, $I_{j_{1}} \cup I_{j_{2}} \neq \boldsymbol{I}$. Let $i_{3}$ be the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}}}$. By equation (5), $i_{3}$ must belong to some $I_{j_{s}}, 3 \leq s \leq k$. Without loss of generality, we may assume that $i_{3} \in I_{j_{3}}$. That is to say, when $t=3, i_{t} \in I_{j_{t}}$.

Suppose that for each $t \in\{2,3, \ldots, k-1\}, i_{t} \in I_{j_{t}}$,
where $i_{t}$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{t-1}}}, t=2,3, \ldots, k-1$.
Because $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite, by its definition, we have $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \neq \boldsymbol{I}$. Let $i_{k}$ be the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}}}$. By equation (5), $i_{k}$ must belong to $I_{j_{k}}$. That is to say, when $t=k, i_{t} \in I_{j_{t}}$. End.
b. Clearly, by Definition 3, $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ satisfies Definition 4(1).
c. Now, we prove that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ satisfies Definition 4(3). Denying the thesis, let us assume that there are some $t \in\{2,3, \ldots, k\}$ and some $s \in\{1,2, \ldots, t-1\}$ such that $I_{j_{s}} \subset I_{j_{t}}$.

Because $I_{j_{s}} \subset I_{j_{t}}$ and $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{s-1}} \cup I_{j_{s}} \cup I_{j_{s+1}} \cup \cdots \cup I_{j_{t}} \cup \cdots$ $\cup I_{j_{k}}=I$ (by equation (5)), then $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{s-1}} \cup I_{j_{s+1}} \cup \cdots \cup I_{j_{t}} \cup \cdots$ $\cup I_{j_{k}}=\boldsymbol{I}$, a contradiction to Definition 3. Hence, $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ satisfies Definition 4(3).

Note 3. A chained-set suite is not always a minimal complete set suite. For example:

Let $I=\{1,2,3,4,5,6\}, \quad I_{1}=\{1,2,6\}, \quad I_{2}=\{3,6\}, \quad I_{3}=\{3,4,5\}$. Clearly, $I_{1}, I_{2}, I_{3}$ is a chained-set suite but not a minimal complete set suite (since $\left.I_{1} \cup I_{3}=\boldsymbol{I}\right)$.

Definition 5. Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be a complete set suite $(2 \leq k \leq n)$. For each $t \in\{1,2, \ldots, k\}$, denote $I_{j_{t}}^{*}=I_{j_{t}}-\underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}-\left\{j_{t}\right\}}{U} I_{j}$. If $I_{j_{t}}^{*} \neq \phi$, then we call the elements of $I_{j_{t}}^{*}$ the independent elements of $I_{j_{t}}$ and call $I_{j_{t}}^{*}$ the set of independent elements of $I_{j_{t}}$.

If $\forall t \in\{1,2, \ldots, k\}, I_{j_{t}}^{*} \neq \phi$, then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is called an independentset suite.

Specially, if there exists a $I_{j}$ such that $I_{j}=\boldsymbol{I}$, then we also call $I_{j}$ an independent-set suite.

Theorem 6. $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite if and only if $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is an independent-set suite.

Proof. If $k=1$, then $I_{j_{1}}=\boldsymbol{I}$, and the thesis is true. In the following, let $k>1$.

Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be a minimal complete set suite. Denying the thesis, let us assume that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is not an independent-set suite. That is to say, there exists some $t \in\{1,2, \ldots, k\}$ such that $I_{j_{t}}^{*}=\phi$. Then according to Definition 5, we have $I_{j_{t}} \subseteq I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{t-1}} \cup I_{j_{t+1}} \cup I_{j_{t+2}} \cup \cdots \cup I_{j_{k}}$, thus $I_{j_{1}} \cup I_{j_{2}}$
$\cup \cdots \cup I_{j_{t-1}} \cup I_{j_{t+1}} \cup I_{j_{t+2}} \cup \cdots \cup I_{j_{k}}=I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{t-1}} \cup I_{j_{t}} \cup I_{j_{t+1}} \cup \cdots$ $\cup I_{j_{k}}=I$, a contradiction to the hypothesis that $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite.

Let $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ be an independent-set suite. Then it must be $\underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}\left\{\left\{j_{t}\right\}\right.}{ } I_{j} \neq \boldsymbol{I}, \forall t \in\{12, \ldots, k\}$. In fact, if there exists $t \in\{1,2, \ldots, k\}$ such that $\underset{j \in\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}-\left\{\left\{_{j_{t}}\right\}\right.}{\cup} I_{j}=\boldsymbol{I}$, then $I_{j_{t}}^{*}=I_{j_{t}}-\boldsymbol{I}=\phi$, a contradiction to the hypothesis that $I_{j_{t}}^{*} \neq \phi$. So, by Definition $3,\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is a minimal complete set suite.

## 5. An Algorithm for Fuzzy Relation Equations with Max-Product Composition

Based on the concepts and results discussed above, we present now an algorithm for finding all the solutions of (I).

Process 1. (Preparation) Check (I)'s feasibility, get (I)'s standard form (II) and then compute (II)'s maximum solution and correlative index sets.

Step 1. According to Definition 1 , compute (I)'s criterion vector $x^{*}$. If $x^{*}$ is a solution of (I), then $x^{*}$ is the maximum solution of (I). Go to Step 2. Otherwise, (I) has no solution, stop.

Step 2. Check (I)'s constant terms. If there exists $b_{i}=0$, then let

$$
\begin{equation*}
x_{j}=0, \quad \forall j \in D_{i}=\left\{j \mid a_{i j} \neq 0, j \in \boldsymbol{J}\right\} \tag{6}
\end{equation*}
$$

and then in (I), delete the $i$ th row and the $j$ th column $\left(\forall j \in D_{i}\right)$. After this kind of simplification, we obtain (I)'s standard form (II).

Step 3. Compute (II)'s maximum solution: $\hat{x}=\left(\hat{x}_{1}, \hat{x}_{2}, \ldots, \hat{x}_{n}\right)$, in which $\hat{x}_{j}=\hat{i=1} \hat{\wedge}_{n}^{n} a_{i j} \odot b_{i}, \forall j \in J$.

Step 4. Compute $I_{j}=\left\{i \mid a_{i j} \hat{x}_{j}=b_{i}, i \in \boldsymbol{I}\right\}, \forall j \in \boldsymbol{J}$.

Let $\boldsymbol{F}=\left\{I_{1}, I_{2}, \ldots, I_{n}\right\}, \boldsymbol{F}^{*}=\left\{I_{j} \mid 1 \in I_{j}, I_{j} \in \boldsymbol{F}\right\}$.
Process 2. Search for chained-set suites of (II), and then obtain minimal complete set suites of (II).

Step 1. For each $I_{j_{1}} \in \boldsymbol{F}^{*}$, consider the following two situations:

1. If $I_{j_{1}}=\boldsymbol{I}$, then $I_{j_{1}}$ is a chained-set suite. Then draw $\operatorname{sign} \sqrt{ }$ behind $I_{j_{1}}$. Go to Step $\mathrm{k}+1$.
2. If $I_{j_{1}} \neq \boldsymbol{I}$, then in $\boldsymbol{F}$, make a search for $I_{j}$, which satisfies:

$$
\begin{equation*}
i_{2} \in I_{j} \text { and } I_{j_{1}} \not \subset I_{j} \tag{7}
\end{equation*}
$$

where $i_{2}$ is the smallest element of $\bar{I}_{j_{1}}$.
(1) If in $\boldsymbol{F}$ there is no $I_{j}$ satisfying (7), then $I_{j_{1}}$ cannot be expanded into a chained-set suite. Draw sign $\times$ behind $I_{j_{1}}$. Go to Step $\mathrm{k}+1$.
(2) If in $F$ there is some $I_{j}$ satisfying (7), then denote $\boldsymbol{I}^{(1)}=\left\{I_{j} \in \boldsymbol{F} \mid i_{2} \in I_{j}\right.$, $\left.I_{j_{1}} \not \subset I_{j}\right\}=\left\{I_{1}^{1}, I_{2}^{1}, \ldots, I_{l_{1}}^{1}\right\}$, and by using $I_{j_{1}}$ and $I^{(1)}$, draw a tree diagram: $I_{j_{1}} \rightarrow I_{d}^{1}, d=1,2, \ldots, l_{1}$ (see Figure 1, Tree diagram 1). Go to Step 2.


Fig.1. Tree diagram 1

Step 2. For each $I_{j_{2}} \in I^{(1)}$, consider the following two situations:

1. If $I_{j_{1}} \cup I_{j_{2}}=I$, then $\left\{I_{j_{1}}, I_{j_{2}}\right\}$ is a chained-set suite expanded by $I_{j_{1}}$. In Tree diagram 1, draw sign $\sqrt{ }$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}}$. Go to Step $\mathrm{k}+1$.
2. If $I_{j_{1}} \cup I_{j_{2}} \neq \boldsymbol{I}$, then in $\boldsymbol{F}$, make a search for $I_{j}$, which satisfies:

$$
\begin{equation*}
i_{3} \in I_{j} \text { and } I_{j_{s}} \not \subset I_{j}, s=1,2 \tag{8}
\end{equation*}
$$

where $i_{3}$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}}}$.
(1) If in $\boldsymbol{F}$ there is no $I_{j}$ satisfying (8), then $I_{j_{1}}$ cannot be expanded into a chained-set suite. Draw sign $\times$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}}$. Go to Step $\mathrm{k}+1$.
(2) If in $\boldsymbol{F}$ there is some $I_{j}$ satisfying (8), then denote $\boldsymbol{I}^{(2)}=\left\{I_{j} \in\right.$ $\left.\boldsymbol{F} \mid i_{3} \in I_{j}, I_{j_{s}} \not \subset I_{j}, s=1,2\right\}=\left\{I_{1}^{2}, I_{2}^{2}, \ldots, I_{l_{2}}^{2}\right\}$. In Tree diagram 1, by using $I^{(2)}$, expand the tree branch $I_{j_{1}} \rightarrow I_{j_{2}}$ into $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow I_{d}^{2}, d=1,2, \ldots, I_{2}$. Then we get Tree diagram 2 (see Figure 2.). Go to Step 3.

Continue in order, according to the above-mentioned manner. In general, suppose that from Step $\mathrm{k}-2$, we obtain

$$
\begin{aligned}
\boldsymbol{I}^{(k-2)} & =\left\{I_{j} \in \boldsymbol{F} \mid i_{k-1} \in I_{j}, I_{j_{s}} \not \subset I_{j}, s=1,2, \ldots, k-2\right\} \\
& =\left\{I_{1}^{k-2}, I_{2}^{k-2}, \ldots, I_{l_{k-2}}^{k-2}\right\}
\end{aligned}
$$

where $\quad I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-2}} \neq \boldsymbol{I}, \quad i_{k-1} \quad$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \bigcup I_{j_{k-2}}}$, and the relevant tree diagram is Tree diagram $k-2$ (see Figure 3). Now, we consider Step k-1.

Step $\mathbf{k}-\mathbf{1}$. For each $I_{j_{k-1}} \in I^{(k-2)}$

1. If $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}}=\boldsymbol{I}$, then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k-1}}\right\}$ is a chained-set suite expanded by $I_{j_{1}}$. In Tree diagram $\mathrm{k}-2$, draw $\sqrt{ }$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow \cdots \rightarrow I_{j_{k-2}} \rightarrow I_{j_{k-1}}$. Go to Step $\mathrm{k}+1$.
2. If $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \neq \boldsymbol{I}$, then in $\boldsymbol{F}$, make a search for $I_{j}$, which satisfies:

$$
\begin{equation*}
i_{k} \in I_{j} \text { and } I_{j_{s}} \not \subset I_{j}, \quad s=1,2, \ldots, k-1 \tag{9}
\end{equation*}
$$

where $i_{k}$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \bigcup I_{j_{k-1}}}$.
(1) If in $\boldsymbol{F}$ there is no $I_{j}$ satisfying (9), then $I_{j_{1}}$ cannot be expanded into a chained-set suite. In Tree diagram $\mathrm{k}-2$, draw sign $\times$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow \cdots \rightarrow I_{j_{k-1}}$. Go to Step $\mathrm{k}+1$.


Figs. Tree diagramk
(2) If in $\boldsymbol{F}$ there is some $I_{j}$ satisfying (9), then denote

$$
\boldsymbol{I}^{(k-1)}=\left\{I_{j} \in \boldsymbol{F} \mid i_{k} \in I_{j}, I_{j_{s}} \not \subset I_{j}, s=1,2, \ldots, k-1\right\}=\left\{I_{1}^{k-1}, I_{2}^{k-1}, \ldots, I_{l_{k-1}}^{k-1}\right\}
$$

In Tree diagram $\mathrm{k}-2$, by using $\boldsymbol{I}^{(k-1)}$, expand the tree branch $I_{j_{1}} \rightarrow I_{j_{2}}$ $\rightarrow \cdots \rightarrow I_{j_{k-2}}$ into $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow \cdots \rightarrow I_{j_{k-2}} \rightarrow I_{d}^{k-1}, d=1,2, \ldots, I_{k-1}$. Then we get Tree diagram k-1 (see Figure 4). Go to Step k.

Step $\mathbf{k}$. Because the amount of elements of $\boldsymbol{F}$ is a finite number $n$, the process of search for a chained-set suite expanded by $I_{j_{1}}$ can stop at Step $\mathrm{k}(1 \leq k \leq n)$. At this time, for each $I_{j_{k}} \in I^{(k-1)}$, there may exist two situations:

1. $I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \cup I_{j_{k}} \neq \boldsymbol{I}$, moreover, in $\boldsymbol{F}$ there is no such $I_{j}$, which satisfies the following condition:

$$
i_{k+1} \in I_{j}, \quad I_{j_{s}} \not \subset I_{j}, \quad s=12, \ldots, k
$$

where $i_{k+1}$ is the smallest element of $\overline{I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k}}}$.

Then $I_{j_{1}}$ cannot be expanded into a chained-set suite. In Tree diagram $\mathrm{k}-1$, draw $\times$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow \cdots \rightarrow I_{j_{k}}$. Go to Step $\mathrm{k}+1$.
2. $\quad I_{j_{1}} \cup I_{j_{2}} \cup \cdots \cup I_{j_{k-1}} \cup I_{j_{k}}=I$, then $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k-1}}, I_{j_{k}}\right\}$ is a chained-set suite. In Tree diagram $\mathrm{k}-1$, draw $\sqrt{ }$ behind the tree branch $I_{j_{1}} \rightarrow I_{j_{2}}$ $\rightarrow \cdots \rightarrow I_{j_{k-1}} \rightarrow I_{j_{k}}$. Go to Step $\mathrm{k}+1$.

After Step $k$, we get Tree diagram $k$ (see Figure 5).
Step $\mathbf{k}+1$. For each $I_{j_{1}} \in \boldsymbol{F}^{*}$, check the tree diagram generated by $I_{j_{1}}$, say Tree diagram k (see Figure 5). In the tree diagram, for each tree branch marked with $\sqrt{ }$, say $I_{j_{1}} \rightarrow I_{j_{2}} \rightarrow \cdots \rightarrow I_{j_{k}}, ~$, we can get a chained-set suite (expanded by $I_{j_{1}}$ ): $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ (see Definition 4). Go to Step $\mathrm{k}+2$.

Let $\boldsymbol{L}$ be the set of all chained-set suites of (II).
Let $\boldsymbol{S}$ be the set of all minimal complete set suites of (II).
Step $\mathbf{k}+2$. Check chained-set suite $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$. If $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\}$ is an independent-set suite (see Definition 5), then it is a minimal complete set suite (by Theorem 6), and then let us retain it in $\boldsymbol{L}$. Otherwise, it is not a minimal complete set suite (by Theorem 6), and then let us delete it in $\boldsymbol{L}$.

After the above operations, all the retained elements in $\boldsymbol{L}$ compose $\boldsymbol{S}$.
Process 3. Get all minimal solutions of (II).
According to Corollary 1, $\sigma:\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\} \mapsto x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$ is a one-to-one correspondence from $\boldsymbol{S}$ to $\bar{X}(A, b)$. So for each $\left\{I_{j_{1}}, I_{j_{2}}, \ldots, I_{j_{k}}\right\} \in \boldsymbol{S}$, we can get a minimal solution $x\left\{j_{1}, j_{2}, \ldots, j_{k}\right\}$. Thus, with the method we can get $\breve{X}(A, b)$ from $S$.

Process 4. Expand the minimal solutions of (II) into the minimal solutions of (I).

We can combine each minimal solution of (II) with the fact (6) (i.e., $x_{j}=0$, $\forall j \in D_{i}$ ) to form all the minimal solutions of (I).

Process 5. Get all solutions of (I).
Let $X^{\prime}(A, b)$ be the set of all minimal solutions of $(\mathrm{I}), X^{\prime \prime}(A, b)$ be the set of all solutions of (I) and $x^{*}$ be the maximum solution of (I). Then $X^{\prime \prime}(A, b)=$ $\bigcup_{x^{\prime} \in X^{\prime}(A, b)}^{\bigcup}\left\{x \in X \mid x^{\prime} \leq x \leq x^{*}\right\}$. End.

## 6. An Example

In this section, an example is provided to display our algorithm. The model cited in the example comes from [9] and [7].

Example 1. Let $A \circ x^{T}=b^{T}$, i.e.,

$$
\left(\begin{array}{cccccccccc}
0.5 & 0.7 & 0.7 & 0.8 & 0.2 & 0.9 & 0.1 & 0.5 & 0.9 & 0.4  \tag{I}\\
0.5 & 0.4 & 0.4 & 0.1 & 0.4 & 0.9 & 0.2 & 0.8 & 0.9 & 0.6 \\
0.8 & 0.4 & 0.5 & 0.2 & 0.1 & 0.8 & 0.4 & 0.8 & 0.6 & 0.6 \\
0.8 & 0.4 & 0.7 & 0.3 & 0.7 & 0.8 & 0.5 & 0.8 & 0.2 & 0.8 \\
0.3 & 0.8 & 0.7 & 0.1 & 0.3 & 0.5 & 0.5 & 0.8 & 0.4 & 0.6 \\
0.2 & 0.5 & 0.9 & 0.4 & 0.3 & 0.6 & 0.6 & 0.8 & 0.5 & 0.3 \\
0.6 & 0.4 & 0.1 & 0.8 & 0.6 & 0.5 & 0.1 & 0.3 & 0.2 & 0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0.4
\end{array}\right) \circ\left(\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4} \\
x_{5} \\
x_{6} \\
x_{7} \\
x_{8} \\
x_{9} \\
x_{10}
\end{array}\right)=\left(\begin{array}{c}
0.72 \\
0.72 \\
0.64 \\
0.64 \\
0.56 \\
0.56 \\
0.48 \\
0
\end{array}\right) .
$$

Solution. By Definition 1, we have $\hat{x}=(0.8,0.7,0.62,0.6,0.8,0.8,0,0.7$, $0.8,0$ ). It is easy to test that $\hat{x}$ is a solution of (I), so by Theorem $1, \hat{x}$ is (I)'s maximum solution.

Because in (I), there is $b_{8}=0$, then by Lemma $1, x_{7}=x_{10}=0$. By deleting the 8th row, the 7th column and the 10th column, we get (I)'s standard form (II):

$$
\begin{equation*}
B \circ y^{T}=\bar{b}^{T} \tag{II}
\end{equation*}
$$

where $y=\left(y_{1}, y_{2}, \ldots, y_{8}\right), \bar{b}=(0.72,0.72,0.64,0.64,0.56,0.56,0.48)$, i.e.,

$$
\left(\begin{array}{llllllll}
0.5 & 0.7 & 0.7 & 0.8 & 0.2 & 0.9 & 0.5 & 0.9  \tag{II}\\
0.5 & 0.4 & 0.4 & 0.1 & 0.4 & 0.9 & 0.8 & 0.9 \\
0.8 & 0.4 & 0.5 & 0.2 & 0.1 & 0.8 & 0.8 & 0.6 \\
0.8 & 0.4 & 0.7 & 0.3 & 0.7 & 0.8 & 0.8 & 0.2 \\
0.3 & 0.8 & 0.7 & 0.1 & 0.3 & 0.5 & 0.8 & 0.4 \\
0.2 & 0.5 & 0.9 & 0.4 & 0.3 & 0.6 & 0.8 & 0.5 \\
0.6 & 0.4 & 0.1 & 0.8 & 0.6 & 0.5 & 0.3 & 0.2
\end{array}\right) \circ\left(\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3} \\
y_{4} \\
y_{5} \\
y_{6} \\
y_{7} \\
y_{8}
\end{array}\right)=\left(\begin{array}{l}
0.72 \\
0.72 \\
0.64 \\
0.64 \\
0.56 \\
0.56 \\
0.48
\end{array}\right) .
$$

Here,

$$
y_{1}=x_{1}, y_{2}=x_{2}, y_{3}=x_{3}, y_{4}=x_{4}, y_{5}=x_{5}, y_{6}=x_{6}, y_{7}=x_{8}, y_{8}=x_{9} .
$$

For (II), we have $\boldsymbol{I}=\{1,2,3,4,5,6,7\}, \boldsymbol{J}=\{1,2,3,4,5,6,7,8\}$.
Compute (II)'s maximum solution: $\hat{y}=\left(\hat{y}_{1}, \hat{y}_{2}, \hat{y}_{3}, \hat{y}_{4}, \hat{y}_{5}, \hat{y}_{6}, \hat{y}_{7}, \hat{y}_{8}\right)$ in which $\hat{y}_{1}=0.8, \hat{y}_{2}=0.7, \hat{y}_{3}=0.62, \hat{y}_{4}=0.6, \quad \hat{y}_{5}=0.8, \quad \hat{y}_{6}=0.8, \quad \hat{y}_{7}=0.7$, $\hat{y}_{8}=0.8$.

According to formula $I_{j}=\left\{i \mid a_{i j} \hat{y}_{j}=b_{i}, i \in \boldsymbol{I}\right\}, \forall j \in \boldsymbol{J}$, we have $I_{1}=\{3,4,7\}, \quad I_{2}=\{5\}, \quad I_{3}=\{6\}, \quad I_{4}=\{7\}, \quad I_{5}=\{7\}, \quad I_{6}=\{1,2,3,4\}$, $I_{7}=\{5,6\}, I_{8}=\{1,2\}$.

$$
\text { So, } \boldsymbol{F}=\left\{I_{1}, I_{2}, \ldots, I_{8}\right\}, \boldsymbol{F}^{*}=\left\{I_{j} \mid 1 \in I_{j}, I_{j} \in \boldsymbol{F}\right\}=\left\{I_{6}, I_{8}\right\} .
$$

For $I_{6}, I_{8} \in \boldsymbol{F}^{*}$, the tree diagram generated by $I_{6}$ or $I_{8}$ is as follows (see Figure 6).


Fig. 6 The tree diagram generate d by $I$ or $I$

From Figure 6, we get $\boldsymbol{L}$, which consists of the following 8 chained-set suites:

$$
\begin{aligned}
& \left\{I_{6}, I_{2}, I_{3}, I_{1}\right\},\left\{I_{6}, I_{2}, I_{3}, I_{4}\right\},\left\{I_{6}, I_{2}, I_{3}, I_{5}\right\} ;\left\{I_{6}, I_{7}, I_{1}\right\} \\
& \left\{I_{6}, I_{7}, I_{4}\right\},\left\{I_{6}, I_{7}, I_{5}\right\} ;\left\{I_{8}, I_{1}, I_{2}, I_{3}\right\},\left\{I_{8}, I_{1}, I_{7}\right\}
\end{aligned}
$$

It is easy to test that each element of $\boldsymbol{L}$ is an independent-set suite (For example: for $\left\{I_{6}, I_{2}, I_{3}, I_{1}\right\}$, it is easy to get that $I_{6}^{*}=\{1,2\}, I_{2}^{*}=\{5\}, I_{3}^{*}=\{6\}$ and $I_{1}^{*}=\{7\}$. By Definition 5, $\left\{I_{6}, I_{2}, I_{3}, I_{1}\right\}$ is an independent-set suite.). So, by Theorem 6, each element of $\boldsymbol{L}$ is a minimal complete set suite. So, $\boldsymbol{L}=\boldsymbol{S}$.

By the one-to-one relation between $S$ and $X(B, \bar{b})$, (Corollary 1), we get $\breve{X}(B, \bar{b})$, which consists of the following 8 minimal solutions of (II):

$$
\begin{aligned}
& y^{1}=\left(\hat{y}_{1}, \hat{y}_{2}, \hat{y}_{3}, 0,0, \hat{y}_{6}, 0,0\right)=(0.8,0.7,0.62,0,0,0.8,0,0) \\
& y^{2}=\left(0, \hat{y}_{2}, \hat{y}_{3}, \hat{y}_{4}, 0, \hat{y}_{6}, 0,0\right)=(0,0.7,0.62,0.6,0,0.8,0,0) \\
& y^{3}=(0,0.7,0.62,0,0.8,0.8,0,0), y^{4}=(0.8,0,0,0,0,0.8,0.7,0) \\
& y^{5}=(0,0,0,0.6,0,0.8,0.7,0), y^{6}=(0,0,0,0,0.8,0.8,0.7,0) \\
& y^{7}=(0.8,0.7,0.62,0,0,0,0,0.8), y^{8}=(0.8,0,0,0,0,0,0.7,0.8)
\end{aligned}
$$

Further, expand the minimal solutions of (II) into the minimal solutions of (I) (Note that $x_{7}=x_{10}=0$ and (8), we have):

$$
\begin{aligned}
x^{1} & =(0.8,0.7,0.62,0,0,0.8,0,0,0,0), x^{2}=(0,0.7,0.62,0.6,0,0.8,0,0,0,0), \\
x^{3} & =(0,0.7,0.62,0,0.8,0.8,0,0,0,0), x^{4}=(0.8,0,0,0,0,0.8,0,0.7,0,0), \\
x^{5} & =(0,0,0,0.6,0,0.8,0,0.7,0,0), x^{6}=(0,0,0,0,0.8,0.8,0,0.7,0,0), \\
x^{7} & =(0.8,0.7,0.62,0,0,0,0,0,0.8,0), x^{8}=(0.8,0,0,0,0,0,0,0.7,0.8,0) .
\end{aligned}
$$

Thus, the set of all the solutions of (I) is

$$
X^{\prime \prime}(A, b)=\bigcup_{x^{\prime} \in X^{\prime}(A, b)}\left\{x \in X \mid x^{\prime} \leq x \leq \hat{x}\right\}
$$

where $X^{\prime}(A, b)=\left\{x^{1}, x^{2}, x^{3}, x^{4}, x^{5}, x^{6}, x^{7}, x^{8}\right\}, X=[0,1]^{10}$.

## 7. Conclusions

In this paper, we obtain an effective algorithm to solve (I). Its main processes are clear: First, through simple computations, we can check (I)'s feasibility and get (I)'s standard form (II) and correlative index sets. And then by observing these index sets, we can obtain all the minimal complete set suites of (II). Lastly, with the one-to-one correspondence relation between the obtained set suites and the minimal solutions of (II), we can easily get all the minimal solutions of (II).

The algorithm only relates to the structure of (I), and consists of some simple and direct calculation procedures.

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