JP Jour. Geometry & Topology **6**(1) (2006), 45-54

AN EXPLICIT CONSTRUCTION OF IMMERSIONS BETWEEN COMPLEX PROJECTIVE SPACES

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Abstract

In this paper we construct explicitly an analytic immersion $\boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ of degree $d > 2$ in case when $n+1$ is a prime. The proof uses special values of minors of Vandermonde's determinant.

1. Introduction and Statement of Results

The existence problem of embeddings or immersions between given differentiable manifolds is a fundamental problem in differential topology. It has been mainly treated in case when the target manifold is an Euclidean space. When the target manifolds are some other manifolds such as real or complex projective spaces or lens spaces, some results have been obtained in ([1], [4], [5], [6], [7], [8], [9]). In this paper we treat analytic immersions between complex projective spaces.

Let $\boldsymbol{CP}^n$ denote the complex projective space of dimension n. For a continuous map $f : \boldsymbol{CP}^n \to \boldsymbol{CP}^m$ with $n \leq m$, the degree of f which we denote by $\deg(f)$ is the integer determined by the induced

2000 Mathematics Subject Classification: Primary 57R42; Secondary 15A03, 15A15.

Keywords and phrases: immersions, complex projective spaces, degree of a map.

Received September 17, 2005

homomorphism $f^{*} : H^2(\boldsymbol{CP}^m, \boldsymbol{Z}) \cong \boldsymbol{Z} \to H^2(\boldsymbol{CP}^n, \boldsymbol{Z}) \cong \boldsymbol{Z}$. It is well known that the homotopy class of f is completely characterized by $\deg(f)$ ([1], [8]).

On the existence of an analytic embedding or immersion $\boldsymbol{CP}^n \to \boldsymbol{CP}^m$ of given degree, the following facts are known:

Fact 1. For any $d > 0$ there exists an analytic embedding $\boldsymbol{CP}^n \to \boldsymbol{CP}^{2n+1}$ of degree d and an analytic immersion $\boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ of degree d [1, Theorem 1.2].

Fact 2. If $f : \boldsymbol{CP}^n \to \boldsymbol{CP}^m$ is an analytic immersion and $m < 2n$, then $\deg(f) = 1$ [1, Theorem 2.1].

Fact 3. A map $f : \boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ is homotopic to an analytic embedding if and only if $\deg(f) = 1$ or $\deg(f) = 2$ [1, Theorem 2.2].

An explicit construction of an analytic embedding $\boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ of degree 2 is given in [2]. In this paper we give an explicit construction of an analytic immersion $\boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ of given degree $d > 2$ in case when $n + 1$ is a prime.

Our theorem is the following:

Theorem 1.1. *Let d be an integer such that $d > 2$. We define homogeneous polynomials $f_i(z) = f_i(z_0, z_1, \ldots, z_n) \in \boldsymbol{C}[z_0, z_1, \ldots, z_n]$ $(i = 0, 1, \ldots, 2n)$, by*

$$f_i(z) = \begin{cases} z_i^d, & \text{if } 0 \le i \le n, \\ \sum_{j=0}^{n} z_j z_{\overline{j+i-n}}^{d-1}, & \text{if } n < i \le 2n+1, \end{cases}$$

where $\overline{k}$ is the residue of k divided by $n + 1$.

Then $f = (f_0, \ldots, f_{2n})$ induces an analytic map $\bar{f} : \boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ defined by $\bar{f}([z_0, z_1, \ldots, z_n]) = [f_0(z), f_1(z), \ldots, f_{2n}(z)]$, where $[z_0, z_1, \ldots, z_n]$ is the point of $\boldsymbol{CP}^n$ whose homogeneous coordinate is $(z_0, z_1, \ldots, z_n)$. If $n + 1$ is a prime, then $\bar{f}$ is an immersion of degree d.

The proof of Theorem 1.1 is reduced to the following:

Theorem 1.2. *For any point* $(a_0, \ldots, a_n) \in \boldsymbol{C}^{n+1} - \{0\}$, *we define the* $(n+1) \times (2n+1)$ *matrix J as follows*:

$$J = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & 0 & \cdots & 0 & a_2 & a_3 & \cdots & a_0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & a_n & a_0 & \cdots & a_{n-2} \\ 0 & 0 & \cdots & 0 & a_n & a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix}.$$

Then $\operatorname{rank} J = n+1$, *if* $n+1$ *is a prime.*

To prove Theorem 1.2, we use the following proposition.

Proposition 1.3. *Let p be a prime and* $\zeta = \exp(2\pi\sqrt{-1}/p)$. *We define the* $p \times p$ *matrix P by*

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(p-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{p-1} & \zeta^{2(p-1)} & \cdots & \zeta^{(p-1)^2} \end{pmatrix}.$$

Then for any integer r $(1 \leq r \leq p)$, *any r-th minor of P is not equal to zero.*

2. Proofs of Theorems

In this section we prove Theorems 1.1 and 1.2 assuming Proposition 1.3. Proposition 1.3 will be proved in the next section.

In general, if a map $\bar{f} : \boldsymbol{CP}^n \to \boldsymbol{CP}^m$ is defined by $f = (f_0, \ldots, f_m) : \boldsymbol{C}^{n+1} - \{0\} \to \boldsymbol{C}^{m+1} - \{0\}$, where f_j $(j = 0, \ldots, m)$ are homogeneous polynomials of degree d, then $\deg(\bar{f}) = d$ (see for example [1]), and if f is an immersion, then $\bar{f}$ is an immersion. Thus to prove Theorem 1.1, it suffices to prove that the Jacobian matrix $J(f)$ of $f : \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{2n+1}$ has maximal rank for any $(z_0, \ldots, z_n) \in \boldsymbol{C}^{n+1} - \{0\}$, if $n+1$ is a prime.

The Jacobian matrix $J(f)$ of the map f in Theorem 1.1 is

$$\begin{pmatrix} dz_0^{d-1} & 0 & 0 & \cdots & 0 & z_1^{d-1} + (d-1)z_n z_0^{d-2} & z_2^{d-1} + (d-1)z_{n-1}z_0^{d-2} & \cdots \\ 0 & dz_1^{d-1} & 0 & \cdots & 0 & z_2^{d-1} + (d-1)z_0 z_1^{d-2} & z_3^{d-1} + (d-1)z_n z_1^{d-2} & \cdots \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & z_n^{d-1} + (d-1)z_{n-2}z_{n-1}^{d-2} & z_0^{d-1} + (d-1)z_{n-3}z_{n-1}^{d-2} & \cdots \\ 0 & 0 & \cdots & 0 & dz_n^{d-1} & z_0^{d-1} + (d-1)z_{n-1}z_n^{d-2} & z_1^{d-1} + (d-1)z_{n-2}z_n^{d-2} & \cdots \end{pmatrix}.$$

If we put $z_i^{d-1} = a_i$ $(i = 0, 1, \ldots, n)$ and define the $(n+1)\times(2n+1)$ matrix J by

$$J = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & 0 & \cdots & 0 & a_2 & a_3 & \cdots & a_0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & a_n & a_0 & \cdots & a_{n-2} \\ 0 & 0 & \cdots & 0 & a_n & a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix},$$

then it is easily seen that $\operatorname{rank} J(f) = \operatorname{rank} J$ for $(z_0, \ldots, z_n) \in \boldsymbol{C}^{n+1} - \{0\}$. Thus the proof of Theorem 1.1 is reduced to Theorem 1.2.

Now assuming that Proposition 1.3 holds, we prove Theorem 1.2. Let V be the vector space over $\boldsymbol{C}$ consisting of $n+1$ dimensional complex column vectors, and $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard unit vectors in V. We denote $\boldsymbol{v} = a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + \cdots + a_n\boldsymbol{e}_n$, and define the $(n+1)\times(n+1)$ matrix T by

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then the column vectors of the matrix J are

$$a_0\boldsymbol{e}_0,\ a_1\boldsymbol{e}_1,\ \ldots,\ a_n\boldsymbol{e}_n,\ T\boldsymbol{v},\ T^2\boldsymbol{v},\ \ldots,\ T^n\boldsymbol{v},$$

and the eigenvalues of T are $1, \zeta, \zeta^2, \ldots, \zeta^n$, where $\zeta = \exp(2\pi\sqrt{-1}/(n+1))$, and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^n \end{pmatrix}, \begin{pmatrix} 1 \\ \zeta^2 \\ \vdots \\ \zeta^{2n} \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ \zeta^n \\ \vdots \\ \zeta^{n^2} \end{pmatrix}$$

which we denote by $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ respectively.

We define the subspaces W_1 and W_2 of V by

$$W_1 = \langle a_0\boldsymbol{e}_0, a_1\boldsymbol{e}_1, \ldots, a_n\boldsymbol{e}_n \rangle, \quad W_2 = \langle \boldsymbol{v}, T\boldsymbol{v}, \ldots, T^n\boldsymbol{v} \rangle.$$

We put $\dim W_2 = r$.

Lemma 2.1. *There exist r integers $i_1, \ldots, i_r$ $(0 \le i_1 < \cdots < i_r \le n)$ such that $W_2 = \langle \boldsymbol{p}_{i_1}, \ldots, \boldsymbol{p}_{i_r} \rangle$ and $\boldsymbol{v} = \sum_{k=1}^{r} \alpha_{i_k}\boldsymbol{p}_{i_k}$, where $\alpha_{i_k} \neq 0$.*

Proof. Since W_2 is a T-invariant subspace of V and T is diagonalizable, W_2 is a direct sum of eigenspaces of T. Therefore $W_2 = \langle \boldsymbol{p}_{i_1}, \ldots, \boldsymbol{p}_{i_r} \rangle$ for some $i_1, \ldots, i_r$. Hence $\boldsymbol{v} = \sum_{k=1}^{t} \alpha_{j_k}\boldsymbol{p}_{j_k}$, where $\alpha_{j_k} \neq 0$ for some $\{j_1, \ldots, j_t\} \subset \{i_1, \ldots, i_r\}$ because $\boldsymbol{v} \in W_2$. Let $\langle \boldsymbol{p}_{j_1}, \ldots, \boldsymbol{p}_{j_t} \rangle = W_2'$. Then W_2' is a T-invariant subspace containing $\boldsymbol{v}$ and $W_2' \subset W_2$. Since W_2 is the minimal T-invariant subspace containing $\boldsymbol{v}$, we have $W_2' = W_2$ and therefore $\{j_1, \ldots, j_t\} = \{i_1, \ldots, i_r\}$. This completes the proof.

Note that $\operatorname{rank} J = \dim(W_1 + W_2)$ since $\boldsymbol{v} \in W_1$. If we denote by s the number of zeroes in $a_0, \ldots, a_n$, then $\dim W_1 = n + 1 - s$.

We denote $\boldsymbol{p}_i = \sum_{j=0}^{n} p_{ji}\boldsymbol{e}_j$, $p_{ji} = \zeta^{ji}$ for $0 \le i \le n$.

Now we assume that $n + 1 = p$ is a prime. If $s = 0$, then $W_1 = V$. Hence we assume that $s \ge 1$. Then there exist integers $j_1, \ldots, j_s$ such that $a_{j_1} = \cdots = a_{j_s} = 0$, and hence $W_1 = \{\boldsymbol{y} = {}^t(y_0, \ldots, y_n) \mid y_{j_1} = \cdots = y_{j_s} = 0\}$. We denote the $s \times r$ matrix $(p_{j_\mu i_\lambda})_{1 \le \mu \le s, 1 \le \lambda \le r}$ by P'. Since by Lemma 2.1

any vector $\boldsymbol{x} \in W_2$ is written as $\boldsymbol{x} = \sum_{\lambda=1}^{r} x_\lambda \boldsymbol{p}_{i_\lambda}$, the condition that $\boldsymbol{x} \in W_1 \cap W_2$ is described as

$$P' \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \boldsymbol{C}^s.$$

By Proposition 1.3 the rank of P' is maximal. Since $\boldsymbol{v} \in W_1 \cap W_2$, we have $W_1 \cap W_2 \neq 0$ and therefore $s < r$ and $\operatorname{rank} P' = s$. Hence $\dim W_1 \cap W_2 = r - s$. Thus $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 =$ $n + 1 - s + r - \dim W_1 \cap W_2 = n + 1$. This completes the proof of Theorem 1.2.

Remark 1. If $n + 1 = lm$, where l, m are integers with $1 < l, m < n + 1$, then $p_{lm} = \zeta^{lm} = 1$. Hence if we put $\boldsymbol{v} = \boldsymbol{p}_0 - \boldsymbol{p}_m = a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + \cdots + a_n\boldsymbol{e}_n$, then we have $a_0 = a_l = 0$, $W_2 = \langle \boldsymbol{p}_0, \boldsymbol{p}_m \rangle$ and $W_1 \cap W_2 \neq 0$. Therefore $\dim W_1 \leq n + 1 - 2$, $\dim W_2 = 2$ and $\dim(W_1 \cap W_2) \geq 1$. Hence $\operatorname{rank} J = \dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \leq n + 1 - 2 + 2 - 1$ $= n$. Thus if $n + 1$ is not a prime, then the map $\bar{f} : \boldsymbol{CP}^n \to \boldsymbol{CP}^{2n}$ in Theorem 1.1 is not an immersion.

3. Proof of Proposition 1.3

Proposition 1.3 is equivalent to the following proposition:

Proposition 3.1. *Let p be a prime and $\zeta = \exp(2\pi\sqrt{-1}/p)$. For any integer r such that $1 \leq r \leq p$ and any sequence of integers $k_1, k_2, \ldots, k_r$ such that $0 \leq k_1 < k_2 < \cdots < k_r < p$, we define the polynomial $D(x_1, x_2, \ldots, x_r)$ $\in \boldsymbol{Z}[x_1, x_2, \ldots, x_r]$ by*

$$D(x_1, x_2, \ldots, x_r) = \begin{vmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_r^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_r^{k_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_r} & x_2^{k_r} & \cdots & x_r^{k_r} \end{vmatrix}.$$

Then for any sequence of integers $l_1, l_2, \ldots, l_r$ *such that* $0 \le l_1 < l_2 < \cdots < l_r < p$, $D(\zeta^{l_1}, \zeta^{l_2}, \ldots, \zeta^{l_r}) \neq 0$.

In what follows, we prove Proposition 3.1 supposing that the assumption in Proposition 3.1 holds.

Since $D(x_1, x_2, \ldots, x_r)$ is an alternating polynomial in $x_1, x_2, \ldots, x_r$, we have

$$D(x_1, x_2, \ldots, x_r) = \prod_{1 \le i < j \le r} (x_j - x_i) D'(x_1, x_2, \ldots, x_r),$$

$$D'(x_1, x_2, \ldots, x_r) \in \boldsymbol{Z}[x_1, \ldots, x_r].$$

First we prepare following lemma:

Lemma 3.2. *If* $D(\zeta^{l_1}, \zeta^{l_2}, \ldots, \zeta^{l_r}) = 0$, *then* $D'(1, 1, \ldots, 1) \equiv 0 \bmod p$.

Proof. Since $\zeta^{l_i} \neq \zeta^{l_j}$ for $i \neq j$, $D(\zeta^{l_1}, \zeta^{l_2}, \ldots, \zeta^{l_r}) = 0$ implies $D'(\zeta^{l_1}, \zeta^{l_2}, \ldots, \zeta^{l_r}) = 0$. Let $D'(x^{l_1}, x^{l_2}, \ldots, x^{l_r}) = F(x) \in \boldsymbol{Z}[x]$. Then $F(\zeta) = 0$ and therefore the minimal polynomial of ζ divides $F(x)$ in $\boldsymbol{Z}[x]$. Thus we have

$$F(x) = Q(x)(x^{p-1} + \cdots + x + 1), \quad Q(x) \in \boldsymbol{Z}[x].$$

Hence $F(1) = Q(1)p \equiv 0 \bmod p$. This completes the proof.

Now we calculate $D'(1, 1, \ldots, 1)$. For an integer k and a positive integer j we define the polynomial $\sigma_{k,j}(x_1, \ldots, x_j)$ by

$$\sigma_{k,j}(x_1, \ldots, x_j) = \begin{cases} \sum_{\lambda_1 + \cdots + \lambda_j = k, \lambda_i \ge 0} x_1^{\lambda_1} \cdots x_j^{\lambda_j}, & \text{if } k \ge 0, \\ 0, & \text{if } k < 0. \end{cases}$$

Then we have the following lemma:

Lemma 3.3.

$$\sigma_{k,j+1}(x_1, \ldots, x_j, y) - \sigma_{k,j+1}(x_1, \ldots, x_j, z) = (y - z)\sigma_{k-1,j+2}(x_1, \ldots, x_j, y, z).$$

Proof.

$$\sigma_{k,\, j+1}(x_1,\, ...,\, x_j,\, y) - \sigma_{k,\, j+1}(x_1,\, ...,\, x_j,\, z)$$

$$= \sum\nolimits_{\lambda_1+\cdots+\lambda_j+\mu=k} x_1^{\lambda_1} \cdots x_j^{\lambda_j}(y^{\mu} - z^{\mu})$$

$$= (y-z)\sum\nolimits_{\lambda_1+\cdots+\lambda_j+\mu=k} x_1^{\lambda_1} \cdots x_j^{\lambda_j} \sum\nolimits_{\mu_1+\mu_2=\mu-1} y^{\mu_1} z^{\mu_2}$$

$$= (y-z)\sum\nolimits_{\lambda_1+\cdots+\lambda_j+\mu_1+\mu_2=k-1} x_1^{\lambda_1} \cdots x_j^{\lambda_j} y^{\mu_1} z^{\mu_2}$$

$$= (y-z)\sigma_{k-1,\, j+2}(x_1,\, ...,\, x_j,\, y,\, z).$$

From above lemma we get following:

Lemma 3.4.

$$D'(x_1,\, ...,\, x_r)$$

$$= \begin{vmatrix} x_1^{k_1} & \sigma_{k_1-1,2}(x_1,\, x_2) & \sigma_{k_1-2,3}(x_1,\, x_2,\, x_3) & \cdots & \sigma_{k_1-(r-1),\, r}(x_1,\, x_2,\, ...,\, x_r) \\ x_1^{k_2} & \sigma_{k_2-1,2}(x_1,\, x_2) & \sigma_{k_2-2,3}(x_1,\, x_2,\, x_3) & \cdots & \sigma_{k_2-(r-1),\, r}(x_1,\, x_2,\, ...,\, x_r) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{k_r} & \sigma_{k_r-1,2}(x_1,\, x_2) & \sigma_{k_r-2,3}(x_1,\, x_2,\, x_3) & \cdots & \sigma_{k_r-(r-1),\, r}(x_1,\, x_2,\, ...,\, x_r) \end{vmatrix}.$$

Proof. Using Lemma 3.3 we have

$$D(x_1,\, ...,\, x_r)$$

$$= \begin{vmatrix} x_1^{k_1} & x_2^{k_1} - x_1^{k_1} & \cdots & x_r^{k_1} - x_1^{k_1} \\ x_1^{k_2} & x_2^{k_2} - x_1^{k_2} & \cdots & x_r^{k_2} - x_1^{k_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_r} & x_2^{k_r} - x_1^{k_r} & \cdots & x_r^{k_r} - x_1^{k_r} \end{vmatrix}$$

$$= \prod\nolimits_{j=2}^{r}(x_j - x_1) \begin{vmatrix} x_1^{k_1} & \sigma_{k_1-1,\, 2}(x_1,\, x_2) & \cdots & \sigma_{k_1-1,\, 2}(x_1,\, x_r) \\ x_1^{k_2} & \sigma_{k_2-1,\, 2}(x_1,\, x_2) & \cdots & \sigma_{k_2-1,\, 2}(x_1,\, x_r) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_r} & \sigma_{k_r-1,\, 2}(x_1,\, x_2) & \cdots & \sigma_{k_r-1,\, 2}(x_1,\, x_r) \end{vmatrix}$$

$$= \prod_{j=2}^{r}(x_j - x_1)\prod_{j=3}^{r}(x_j - x_2)$$

$$\times \begin{vmatrix} x_1^{k_1} & \sigma_{k_1-1,2}(x_1, x_2) & \sigma_{k_1-2,3}(x_1, x_2, x_3) & \cdots & \sigma_{k_1-2,3}(x_1, x_2, x_r) \\ x_1^{k_2} & \sigma_{k_2-1,2}(x_1, x_2) & \sigma_{k_2-2,3}(x_1, x_2, x_3) & \cdots & \sigma_{k_2-2,3}(x_1, x_2, x_r) \\ \vdots & \vdots & \vdots & \vdots & \\ x_1^{k_r} & \sigma_{k_r-1,2}(x_1, x_2) & \sigma_{k_r-2,3}(x_1, x_2, x_3) & \cdots & \sigma_{k_r-2,3}(x_1, x_2, x_r) \end{vmatrix}$$

$$\vdots$$

$$= \prod_{1\le i<j\le r}(x_j - x_i)\begin{vmatrix} x_1^{k_1} & \sigma_{k_1-1,2}(x_1, x_2) & \cdots & \sigma_{k_1-(r-1),r}(x_1, x_2, \ldots, x_r) \\ x_1^{k_2} & \sigma_{k_2-1,2}(x_1, x_2) & \cdots & \sigma_{k_2-(r-1),r}(x_1, x_2, \ldots, x_r) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_r} & \sigma_{k_r-1,2}(x_1, x_2) & \cdots & \sigma_{k_r-(r-1),r}(x_1, x_2, \ldots, x_r) \end{vmatrix}.$$

Now we can calculate $D'(1, 1, \ldots, 1)$ as follows:

Lemma 3.5. $2!3!\cdots(r-1)!\,D'(1, 1, \ldots, 1) = \prod_{1\le i<j\le r}(k_j - k_i)$.

Proof. If $l \ge 0$, $\sigma_{l,j}(1, 1, \ldots, 1) = \binom{j+l-1}{l}$. Hence if $k \ge j-1$, $\sigma_{k-(j-1),j}(1, 1, \ldots, 1) = \binom{k}{k-(j-1)} = \frac{k(k-1)\cdots(k-(j-2))}{(j-1)!}$. This equality holds also for $k < j-1$. Thus by Lemma 3.4 we have

$$2!3!\cdots(r-1)!\,D'(1, 1, \ldots, 1)$$

$$= \begin{vmatrix} 1 & k_1 & k_1(k_1-1) & \cdots & k_1(k_1-1)\cdots(k_1-(r-2)) \\ 1 & k_2 & k_2(k_2-1) & \cdots & k_2(k_2-1)\cdots(k_2-(r-2)) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & k_r & k_r(k_r-1) & \cdots & k_r(k_r-1)\cdots(k_r-(r-2)) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & k_1 & k_1^2 & \cdots & k_1^{r-1} \\ 1 & k_2 & k_2^2 & \cdots & k_2^{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & k_r & k_r^2 & \cdots & k_r^{r-1} \end{vmatrix}$$

$$= \prod_{1\le i<j\le r}(k_j - k_i).$$

By Lemma 3.5 any prime factor of $D'(1, 1, \ldots, 1)$ is smaller than p. Hence $D'(1, 1, \ldots, 1)$ is not a multiple of p. Thus by Lemma 3.2, $D(\zeta^{l_1}, \zeta^{l_2}, \ldots, \zeta^{l_r}) \neq 0$. This completes the proof of Proposition 3.1.

Acknowledgement

The author expresses his gratitude to Professor Tsutomu Yasui of Kagoshima University for his kind advice and helpful discussion during the preparation of this manuscript.

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