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NEW GENERAL FORM OF HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract

A new form of Hardy-Hilbert's integral inequality in n-variables is given.

1. Introduction

Let $f,\ g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)dt < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(t)dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi\left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt\right)^{1/2}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [3]). Inequality (1) is well known as *Hilbert's integral inequality*. This inequality had been extended by Hardy [2] as follows:

If $p > 1,\ 1/p + 1/q = 1,\ f,\ g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)dt < \infty \quad \text{and} \quad \int_0^\infty g^q(t)dt < \infty,$$

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then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt \right)^{1/p} \left(\int_0^\infty g^q(t)dt \right)^{1/q}, \quad (2)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called *Hardy-Hilbert's integral inequality* and is important in analysis and application (cf. Mitrinovic et al. [4]).

Gradually, B. Yang gave the following extensions of (2) as follows:

Theorem A [5]**.** *If* $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$ *satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \quad and \quad \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)dt \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)dt \right)^{1/q}, \quad (3)$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right)$ *is the best possible, B is the beta function.*

Theorem B [6]**.** *If* $n \in N - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \geq 0$ *satisfy*

$$0 < \int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)dt < \infty \quad (i = 1, 2, \ldots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j \right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n$$

$$< \frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i} \right) \left(\int_0^\infty t^{p_i - 1 - \lambda} f_i^{p_i}(t)dt \right)^{1/p_i}, \quad (4)$$

where the constant factor $\frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i} \right)$ *is the best possible.*

The following result is needed for our aim:

Theorem C [1]. *Let f be a nonnegative integrable function. Define*

$$F(x) = \int_a^x f(t)dt.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p > 1. \tag{5}$$

2. New Result

The aim of this paper is to give a new general form of Hardy-Hilbert's integral inequality using simpler new method. In fact, we prove the following:

Theorem 1. *Let* $f_i \geq 0$, $p_i > 1$, $\lambda > n + p_i - 1$, $i = 1, \ldots, n$, $n \geq 2$, $\sum_{i=1}^n \frac{1}{p_i} = 1$. *Define*

$$F_i(x) = \int_0^x f_i(t)dt.$$

Then

$$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1)\cdots F_n(t_n)}{(t_1 + \cdots + t_n)^\lambda} dt_1 \cdots dt_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma(1 + \lambda - n - p_i)\int_0^\infty t^{n+p_i-\lambda-1} f_i^{p_i}(t)dt\right)^{1/p_i}. \tag{6}$$

Proof. Define

$$L_i(x) = \int_0^\infty e^{-tx} f_i(t)dt.$$

Observe that

$$\begin{aligned} L_i(x) &= \int_0^\infty e^{-tx} F_i'(t)dt \\ &= [e^{-tx}F_i(t)]_0^\infty + x\int_0^\infty e^{-tx}F_i(t)dt \\ &= x\int_0^\infty e^{-tx}F_i(t)dt, \end{aligned}$$

which implies that

$$\int_0^\infty s^{\lambda-n-1} L_1(s) \cdots L_n(s) ds$$

$$= \int_0^\infty s^{\lambda-n-1} \left(s\int_0^\infty e^{-t_1 s} F_1(t_1) dt_1 \right) \cdots \left(s\int_0^\infty e^{-t_n s} F_n(t_n) dt_n \right) ds$$

$$= \int_0^\infty \cdots \int_0^\infty F_1(t_1) \cdots F_n(t_n) dt_1 \cdots dt_n \int_0^\infty s^{\lambda-1} e^{-s(t_1+\cdots+t_n)} ds$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1) \cdots F_n(t_n)}{(t_1 + \cdots + t_n)^\lambda} dt_1 \cdots dt_n \int_0^\infty u^{\lambda-1} e^{-u} du, \quad (s(t_1 + \cdots + t_n) = u)$$

$$= \Gamma(\lambda) \int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1) \cdots F_n(t_n)}{(t_1 + \cdots + t_n)^\lambda} dt_1 \cdots dt_n. \tag{7}$$

Also, we have

$$\int_0^\infty s^{\lambda-n-1} L^p(s) ds = \int_0^\infty s^{\lambda-n-1} \left(\int_0^\infty e^{-ts} f(t) dt \right)^p ds$$

$$\le \int_0^\infty s^{\lambda-n-1} \int_0^\infty e^{-ts} f^p(t) dt \left(\int_0^\infty e^{-ts} dt \right)^{p-1}$$

$$= \int_0^\infty f^p(t) dt \int_0^\infty s^{\lambda-n-p} e^{-ts} ds$$

$$= \int_0^\infty t^{n+p-\lambda-1} f^p(t) dt \int_0^\infty z^{\lambda-n-p} e^{-z} dz \quad (ts = z)$$

$$= \Gamma(1 + \lambda - n - p) \int_0^\infty t^{n+p-\lambda-1} f^p(t) dt. \tag{8}$$

Making use of (7) and (8), we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1) \cdots F_n(t_n)}{(t_1 + \cdots + t_n)^\lambda} dt_1 \cdots dt_n$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-n-1} L_1(s) \cdots L_n(s) ds$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\frac{\lambda-n-1}{p_1}} L_1(s) \cdots s^{\frac{\lambda-n-1}{p_n}} L_n(s) ds$$

$$\leq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty s^{\lambda-n-1} L_1^{p_1}(s) ds \right)^{1/p_1} \cdots \left(\int_0^\infty s^{\lambda-n-1} L_n^{p_n}(s) ds \right)^{1/p_n}$$

$$\leq \frac{1}{\Gamma(\lambda)} \left(\prod_{i=1}^n \Gamma(1+\lambda-n-p_i) \int_0^\infty t^{n+p_i-\lambda-1} f_i^{p_i}(t) dt \right)^{1/p_i}.$$

This completes the proof of the theorem.

Theorem 2. *Let* $f_i \geq 0$, $q_i > 0$, $i = 1, \ldots, n$, $n \geq 2$, $p_i = 1 + q_i \left(\frac{\lambda+1}{n} \right)$, $\lambda + n + 1 > 0$, $\sum_{i=1}^n \frac{1}{p_i} = 1$. *Define*

$$M_{q_i}(x) = \int_0^\infty e^{-t^{q_i} x} f_i(t) dt.$$

Then

$$\int_0^\infty s^\lambda M_{q_1}^{p_1}(s) \cdots M_{q_n}^{p_n}(s) ds < \frac{\Gamma(\lambda+n+1)}{n^{\lambda+n+1}} \prod_{i=1}^n q_i \left(\frac{p_i}{p_i - 1} \right)^{p_i} \int_0^\infty f_i^{p_i}(t) dt. \quad (9)$$

Proof. Define

$$F(x) = \int_0^x f(t) dt.$$

Then we have

$$M_q(x) = \int_0^\infty e^{-t^q x} f(t) dt = \int_0^\infty e^{-t^q x} F'(t) dt$$

$$= [e^{-t^q x} F(t)]_0^\infty + \int_0^\infty qxt^{q-1} e^{-t^q x} F(t) dt$$

$$= qx \int_0^\infty t^{q-1} e^{-t^q x} F(t) dt,$$

which implies that

$$M_q^p(x) \le (qx)^p \int_0^\infty t^{q-1} e^{-t^q x} F^p(t)dt \left(\int_0^\infty t^{q-1} e^{-t^q x} dt \right)^{p-1}$$

$$= qx \int_0^\infty t^{q-1} e^{-t^q x} F^p(t)dt.$$

Making use of the inequality

$$\sqrt[n]{a_1 \cdots a_n} \le \frac{a_1 + \cdots + a_n}{n}, \quad a_i > 0,$$

$$\int_0^\infty s^\lambda M_{q_1}^{p_1}(s) \cdots M_{q_n}^{p_n}(s)ds$$

$$\le \int_0^\infty s^\lambda \left(q_1 s \int_0^\infty t_1^{q_1 - 1} e^{-t_1^{q_1} s} F_1^{p_1}(t_1)dt_1 \right) \cdots \left(q_n s \int_0^\infty t_n^{q_n - 1} e^{-t_n^{q_n} s} F_n^{p_n}(t_n)dt_n \right) ds$$

$$= q_1 \ldots q_n \int_0^\infty \cdots \int_0^\infty t_1^{q_1 - 1} F_1^{p_1}(t_1) \cdots t_n^{q_n - 1} F_n^{p_n}(t_n)dt_1$$

$$\cdots dt_n \int_0^\infty s^{\lambda + n} e^{-s(t_1^{q_1} + \cdots + t_n^{q_n})} ds$$

$$= q_1 \ldots q_n \int_0^\infty \cdots \int_0^\infty \frac{t_1^{q_1 - 1} F_1^{p_1}(t) \cdots t_n^{q_n - 1} F_n^{p_n}(t)}{(t_1^{q_1} + \cdots + t_n^{q_n})^{\lambda + n + 1}} dt_1$$

$$\cdots dt_n \int_0^\infty z^\lambda e^{-z} dz \quad (z = s(t_1^{q_1} + \cdots + t_n^{q_n}))$$

$$\le \frac{\Gamma(\lambda + n + 1)}{n^{\lambda + n + 1}} q_1 \ldots q_n \int_0^\infty \cdots \int_0^\infty \frac{t_1^{q_1 - 1} F_1^{p_1}(t_1) \cdots t_n^{q_n - 1} F_n^{p_n}(t)}{t_1^{q_1(\lambda + n + 1)} \cdots t_n^{q_n(\lambda + n + 1)}} dt_1 \cdots dt_n$$

$$= \frac{\Gamma(\lambda + n + 1)}{n^{\lambda + n + 1}} \prod_{i=1}^n q_i \int_0^\infty \left(\frac{F_i(t)}{t} \right)^{p_i} dt$$

$$< \frac{\Gamma(\lambda + n + 1)}{n^{\lambda + n + 1}} \prod_{i=1}^n q_i \left(\frac{p_i}{p_i - 1} \right)^{p_i} \int_0^\infty f_i^{p_i}(t)dt.$$

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