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SOME RESULTS CONCERNING PARTITIONS OF A GIVEN RANK

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Abstract

The rank of a partition is the largest part less the number of parts. We present several theorems concerning (i) the number of partitions with a given rank; (ii) the number of partitions into distinct parts with a given rank.

1. Introduction

For n is a natural number, let a partition, π, of n be given by: $n = n_1 + n_2 + n_3 + \cdots + n + n_r,$ where $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_r.$ The *rank* of π, denoted $\rho(\pi),$ is defined by

$$\rho(\pi) = n_1 - r.$$

(This definition was first given by Dyson in [2].)

Clearly, $|\rho(\pi)| \leq n - 1.$ Furthermore, if π^* is the partition that is conjugate to π, then $\rho(\pi^*) = -\rho(\pi).$ If $|m| \leq n - 1,$ let $N(m, n)$ denote the number of partitions of n whose rank is m, with $N(0, 0) = 0.$ (This is the notation used in [1].) Since $N(-m, n) = N(m, n),$ it suffices to consider $0 \leq m \leq n - 1.$ We will also use the notation: $\rho_m(n) = N(m, n).$ In this

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note, we present some results regarding the number of partitions of a given rank that are implicit in earlier works of others, as well as some new results.

2. Preliminaries

Definition 1. If $k \in Z$, then $\omega(k) = \dfrac{k(3k-1)}{2}$ (pentagonal numbers).

Definition 2. $p(n)$ is the number of partitions of n.

Definition 3. $q_0(n)$ is the number of self-conjugate partitions of n.

Definition 4. $\rho_m(n) = N(m, n)$ is the number of partitions of n that have rank m, where $0 \le m \le n-1$.

Definition 5. $\rho_+(n)$ is the number of partitions of n with positive rank.

Definition 6. $\rho_-(n)$ is the number of partitions of n with negative rank.

Definition 7. π^* is the partition that is conjugate to π.

Identities. Let $x \in C$, $|x| < 1$. Then

$$\rho(\pi^*) = -\rho(\pi), \tag{1}$$

$$\prod_{n\ge1}(1-x^n) = 1 + \sum_{n\ge1}(-1)^n\{x^{\omega(n)} + x^{\omega(-n)}\}, \tag{2}$$

$$\prod_{n\ge1}(1-x^n)^{-1} = \sum_{n\ge0} p(n)x^n, \tag{3}$$

$$p(n) \equiv q_0(n) \pmod 2 \text{ for all } n \ge 0, \tag{4}$$

$$p(n) + \sum_{k\ge1}(-1)^k\{p(n-\omega(k)) + p(n-\omega(-k))\} = 0 \text{ for all } n \ge 1, \tag{5}$$

$$\sum_{n=0}^{\infty}\rho_m(n)x^n = \sum_{n=1}^{\infty}(-1)^{n-1}x^{\omega(n)+mn}(1-x^n)\prod_{r=1}^{\infty}(1-x^r)^{-1}. \tag{6}$$

Remarks. (1) is self-evident; (2) through (5) are well known; (6) is Equation 2.12 in [1].

3. Unrestricted Partitions with a Given Rank

We begin with an explicit formula for $\rho_m(n)$ in terms of $p(n)$.

Theorem 1. *If* $0 \le m \le n-1$, *then*

$$\rho_m(n) = \sum_{k=1}^{\infty} (-1)^{k-1}\{p(n - \omega(k) - mk) - p(n - \omega(-k) - mk)\}.$$

Proof. This follows directly from (6).

In particular, we have

Corollary 1.

$$\rho_0(n) = \sum_{k=1}^{\infty} (-1)^{k-1}\{p(n - \omega(k)) - p(n - \omega(-k))\}.$$

Proof. This follows from Theorem 1, with $m = 0$.

The next theorem states that $\rho_0(n)$ has the same parity as $p(n)$.

Theorem 2. $\rho_0(n) \equiv p(n) \pmod 2$ *for all* $n \ge 1$.

Proof. If π is a self-conjugate partition of π, that is, if $\pi^* = \pi$, then (1) implies $\rho(\pi) = 0$. The partitions of n with rank 0 that are not self-conjugate (if any) occur in conjugate pairs. Therefore $\rho_0(n) \equiv q_0(n) \pmod 2$. The conclusion now follows from (4).

The next theorem gives a formula for the number of partitions with positive rank.

Theorem 3. *If* $n \ge 1$, *then*

$$\rho_+(n) = \sum_{k=1}^{\infty} (-1)^{k-1} p(n - \omega(-k)).$$

Proof. (1) implies $\rho_+(n) = \rho_-(n)$, so that $p(n) = \rho_0(n) + 2\rho_+(n)$, that is, $2\rho_+(n) = p(n) - \rho_0(n)$. Now Corollary 1 implies

$$2\rho_+(n) = p(n) + \sum_{k=1}^{\infty} (-1)^k \{p(n - \omega(k)) - p(n - \omega(-k))\},$$

whereas (5) implies

$$0 = p(n) + \sum_{k=1}^{\infty} (-1)^k \{p(n - \omega(k)) + p(n - \omega(-k))\}.$$

The conclusion now follows if we subtract and divide by 2.

Next, we present a recurrence for $\rho_m(n)$.

Theorem 4. *If* $0 \le m \le n - 1$, *then*

$$\rho_m(n) + \sum_{j=1}^{\infty} (-1)^j \{\rho_m(n - \omega(j)) + \rho_m(n - \omega(-j))\} = \begin{cases} (-1)^{k-1}, & \text{if } n = \omega(k) + mk, \\ (-1)^k, & \text{if } n = \omega(-k) + mk, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (6) implies

$$\left(\sum_{n=0}^{\infty} \rho_m(n) x^n\right) \prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=0}^{\infty} (-1)^{n-1} x^{\omega(n) + mn} (1 - x^n).$$

The conclusion now follows by invoking (2) and matching coefficients of like powers of x.

4. Partitions into Distinct Parts with a Given Rank

Theorem 5. *If* π *is a partition of* n *into distinct parts, then* $\rho(\pi) \ge 0$; *furthermore, the inequality is strict unless* n *is triangular, that is,* $n = k(k+1)/2$, *and* n *is partitioned as a sum of consecutive integers*:

$$n = k + (k - 1) + (k - 2) + \cdots + 3 + 2 + 1. \tag{7}$$

Proof. Let a partition, π, of n into distinct parts be given by;

$$n = n_1 + n_2 + n_3 + \cdots + n_k,$$

where $n_i > n_{i+1}$ for all i such that $1 \leq i \leq k-1$, hence $\rho(\pi) = n_1 - k$. Now

$$n_1 = \sum_{i=1}^{k-1} (n_i - n_{i+1}) + n_k \geq \sum_{i=1}^{k-1} 1 + 1 = k,$$

so that $\rho(\pi) = n_1 - k \geq 0$. The inequality is strict unless $n_k = 1$ and $n_i - n_{i+1} = 1$ for all i. This implies $n_i = k + 1 - i$ for all i, so $n = \sum_{i=1}^{k} i = k(k+1)/2$, that is, n is triangular, and is partitioned as in (7), so that $\rho(\pi) = k - k = 0$.

Lemma 1. *Let* $n = n_1 + n_2 + n_3 + \cdots + n_r$ *with* $n_1 > n_2 > n_3 > \cdots > n_r$. *If k is the unique integer such that* $k(k-1)/2 < n \leq k(k+1)/2$, *then* $r \leq k$. *Furthermore, if n is not triangular, then* $r < k$.

Proof.

$$\frac{r(r+1)}{2} = \sum_{i=1}^{r} i \leq \sum_{i=1}^{r} n_i = n \leq \frac{k(k+1)}{2} \to r \leq k.$$

Now suppose that $r = k$. If $n_1 \leq k-1$, then $n_i \leq k - i$ for all i, and in particular, $n_k \leq 0$, an impossibility. Therefore $n_1 \geq k$. Similarly,

$$n_2 \geq k-1, n_3 \geq k-2, ..., n_k \geq 1.$$

Now

$$n = \sum_{i=1}^{k} n_i \to n \geq \sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

But, since also $n \leq k(k+1)/2$, we have $n = k(k+1)/2$, that is, n is triangular.

Theorem 6. *If n is not triangular, then n has a unique partition*, π, *into distinct parts such that* $\rho(\pi) = 1$.

Proof. (Existence) If n is not triangular, let k be the unique integer such that $k(k-1)/2 < n < k(k+1)/2$. Therefore there exists a unique integer, j, such that $n = \frac{k(k+1)}{2} - j$ and $1 \leq j \leq k-1$. Thus n may be represented as a sum of $k-1$ consecutive integers, excluding j, of which the largest is k. It follows that $\rho(\pi) = k - (k-1) = 1$.

(Uniqueness) Let n be partitioned as in the statement of Lemma 1. By hypothesis, we have $n_1 - r = 1$. Lemma 1 and the hypothesis imply $r < k$, so that $r \leq k-1$. If $n_1 \geq k+1$, then $n_1 - r \geq 2$, an impossibility. Therefore $n_1 = k$.

Let $\rho_k^*(n)$ denote the number of partitions of n into distinct parts, having rank k. Then Theorems 5 and 6 may be restated as follows:

Theorem 5a.

$$\rho_0^*(n) = \begin{cases} 1, & \textit{if } n \textit{ is triangular}, \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 5b.

$$\rho_1^*(n) = \begin{cases} 0, & \textit{if } n \textit{ is triangular}, \\ 1, & \textit{otherwise.} \end{cases}$$

References

[1] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, Proc. London Math. Soc. (3) 4 (1954), 84-106.

[2] F. J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944), 10-15.