# APPELL POLYNOMIAL SEQUENCES: A LINEAR ALGEBRA APPROACH 

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#### Abstract

In this paper, we develop a new approach to studying Appell polynomial sequences via linear algebra. This approach offers a powerful tool for investigating the properties of Appell polynomials. Within a selfcontained theory, employing only matrix operations, we show how the derivation of some of old and new properties of Appell polynomials is greatly simplified.


## 1. Introduction

In the mathematical literature of the past few decades there has been a revival of interest in Appell polynomial sequences. Appell polynomial sequences arise in theoretical physics, chemistry, approximation theory, and numerous other fields. Di Bucchianico and Loeb recently summarized and documented more than five hundred old and new findings related to Appell polynomial sequences in [3]. Attention has centered on finding a novel representation of Appell polynomials and studying their properties (see [5] and [7]). For instance, in [5], Lehmer illustrated six different 2000 Mathematics Subject Classification: 15A15, 05A19, 11B68, 11B75.

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approaches to representing Bernoulli polynomial sequence, which is one of the Appell polynomial sequences.

The well-known mathematical tool, developed by G. Rota, in the study of Appell polynomials uses the algebra of functionals, which is not easily accessible by non-specialists. In light of the many polynomials rooted in physics and engineering, it is essential to develop an easily comprehensible mathematical tool, which is accessible to physicists and engineers. In this paper, we develop a novel approach - investigating Appell polynomials via linear algebra, using matrix operations as our main tool.

This paper is organized as follows. In Section 2, we present generalized Pascal functional matrices introduced in [8] and their properties, which play an important role in establishing the theoretical foundation of the linear algebra approach. In Section 3, we set up the representation of Appell polynomial sequences via generalized Pascal functional matrices, reprove the well-known Appell identity, and obtain the generalized Appell identity. In Section 4, we further the study of Appell polynomials' properties, such as connection constant theorem, inverse relation, and duplication formula. Section 5 is an extension of Section 3. In this section, we develop new and redevelop old identities of Appell polynomials. In Section 6, we obtain the differential equation and recurrence relation for Appell polynomials, which were presented in [4], and give differential equations and recurrence relations for Bernoulli, Euler, and Hermite polynomials (note the results in [4] are incorrect). We conclude the paper in Section 7 and state our future direction.

## 2. Generalized Pascal Functional Matrices and Wronskians

Let us state the definition of the generalized Pascal functional matrix of an analytic function, introduced in [8]. To avoid any unnecessary confusion, we use $f^{(k)}$ to stand for the $k$ th order derivative of $f$ and use $f^{k}$ to represent the $k$ th power of $f$ in the entire paper. In addition, $f^{(0)}=f$ and $f^{0}=1$.

Definition 2.1. Let $f(t)$ be an analytic function. Then the generalized Pascal functional matrix of $f(t)$, denoted by $\mathcal{P}_{n}[f(t)]$, is an $(n+1)$ by $(n+1)$ matrix and is defined as

$$
\left(\mathcal{P}_{n}[f(t)]\right)_{i, j}=\left\{\begin{array}{ll}
\binom{i}{j} f^{(i-j)}(t) & \text { if } i \geq j, \\
0 & \text { otherwise }
\end{array} \quad \text { for } \quad i, j=0,1,2, \ldots, n\right.
$$

The generalized Pascal functional matrices interact nicely with Wronskian vector. For clarity we define the Wronskian vector of an analytic function $f(t)$.

Definition 2.2. The $n t h$ order Wronskian vector of $f(t)$ is an $(n+1) \times 1$ matrix, denoted by $\mathcal{W}_{n}[f(t)]$, and is defined as

$$
\mathcal{W}_{n}[f(t)]=\left[\begin{array}{llll}
f(t) & f^{\prime}(t) & \cdots & f^{(n)}(t)
\end{array}\right]^{T}
$$

In what follows, we study the Pascal functional matrices and Wronskian functional vectors in a neighborhood of $t=0$. Hence when we mention analytic, we mean analytic near $t=0$.

Let us now state some properties of the generalized Pascal functional matrix and Wronskian functional vector.

Property 2.1. (a) If $f(t)=1$, then $\mathcal{P}_{n}[1]=I_{n+1}$ (identity matrix).
(b) If $f(t)=e^{t}$, then $\mathcal{P}_{n}\left[e^{t}\right]_{t=0}=P_{L}$ (regular lower triangular Pascal matrix defined in [1]).
(c) If $f(t)=e^{a t}$, then $\mathcal{P}_{n}\left[e^{a t}\right]_{t=0}=P_{L}^{a}=P_{L}[a]$ (generalized lower triangular matrix in [2]).
(d) $\mathcal{P}_{n}[\cdot]$ and $\mathcal{W}_{n}[\cdot]$ are linear, that is, for any constants $a$ and $b$, and any analytic functions $f(t)$ and $g(t)$,

$$
\begin{aligned}
& \mathcal{P}_{n}[a f(t)+b g(t)]=a \mathcal{P}_{n}[f(t)]+b \mathcal{P}_{n}[g(t)] \\
& \mathcal{W}_{n}[a f(t)+b g(t)]=a \mathcal{W}_{n}[f(t)]+b \mathcal{W}_{n}[g(t)]
\end{aligned}
$$

(e) For any analytic functions $f(t)$ and $g(t)$,

$$
\mathcal{P}_{n}[f(t)] \mathcal{P}_{n}[g(t)]=\mathcal{P}_{n}[g(t)] \mathcal{P}_{n}[f(t)]=\mathcal{P}_{n}[f(t) g(t)] .
$$

Furthermore, if $f(0) \neq 0$, then $\left(\mathcal{P}_{n}[f(t)]\right)^{-1}=\mathcal{P}_{n}\left[f^{-1}(t)\right]$, where $f^{-1}(t)$ denotes the multiplicative inverse of $f(t)$.

We will call $f(t)$ to be invertible if $f(0) \neq 0$. Thus the set of all invertible analytic functions forms a commutative group under the binary operator

$$
f * g=\mathcal{P}_{n}[f(t)] \mathcal{P}_{n}[g(t)] .
$$

(f) For any analytic functions $f(t)$ and $g(t)$,

$$
\mathcal{P}_{n}[f(t)] \mathcal{W}_{n}[g(t)]=\mathcal{P}_{n}[g(t)] \mathcal{W}_{n}[f(t)]=\mathcal{W}_{n}[(f g)(t)] .
$$

(g) For any analytic function $f(t)$,

$$
\mathcal{W}_{n}[f(a t)]_{t=0}=\operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{W}_{n}[f(t)]_{t=0}
$$

(h) For any analytic function $f(t), \mathcal{P}_{n}[f(a t)]_{t=0}$ and $\mathcal{P}_{n}[f(t)]_{t=0}$ are similar and

$$
\begin{equation*}
\mathcal{P}_{n}[f(a t)]_{t=0}=\operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{P}_{n}[f(t)]_{t=0} \operatorname{diag}\left[1, a^{-1}, \ldots, a^{-n}\right] . \tag{1}
\end{equation*}
$$

Proof. Properties (a)-(d) are trivial. The proof of Property (e) can be found in [8]. For Property (f), we note that $\mathcal{W}_{n}[f(t)]$ equals the first column of $\mathcal{P}_{n}[f(t)]$ and therefore Property (f) follows from Property (e). For Property (g), we note that $(f(a t))^{(k)}=a^{k} f^{(k)}(a t)$ and $\left.(f(a t))^{(k)}\right|_{t=0}=$ $a^{k} f^{(k)}(0)$. For Property (h), using Definition 2.1 and knowing $(f(a t))^{(k)}=$ $a^{k} f^{(k)}(a t)$, we have
$\left(\mathcal{P}_{n}[f(a t)]_{t=0}\right)_{i, j}=\left\{\begin{array}{ll}\binom{i}{j} a^{i-j} f^{(i-j)}(0) & \text { if } i \geq j, \\ 0 & \text { otherwise, }\end{array} \quad\right.$ for $i, j=0,1,2, \ldots, n$. (2)

By a simple matrix multiplication, we obtain

$$
\begin{align*}
& \left(\operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{P}_{n}[f(t)]_{t=0} \operatorname{diag}\left[1, a^{-1}, \ldots, a^{-n}\right]\right)_{i, j} \\
= & \left\{\begin{array}{ll}
\binom{i}{j} a^{i-j} f^{(i-j)}(0) & \text { if } i \geq j, \\
0 & \text { otherwise, }
\end{array} \text { for } i, j=0,1,2, \ldots, n .\right. \tag{3}
\end{align*}
$$

Equating Eqs. (2) and (3) yields Property (h).
Now consider the system of differential equations in $\mathbb{R}^{n+1}$,

$$
\begin{equation*}
\frac{d}{d x} \bar{y}(x)=\mathcal{P}_{n}[f(t)]_{t=0} \bar{y}(x), \quad-\infty<x<\infty \tag{4}
\end{equation*}
$$

with an initial value

$$
\begin{equation*}
\bar{y}(0)=\bar{y}_{0} \in \mathbb{R}^{n+1} \tag{5}
\end{equation*}
$$

where $\bar{y}(x) \in \mathbb{R}^{n+1}$ denotes a vector of functions.
Theorem 2.1. Let $f(t)$ be an analytic function with $f(0)=0$. Then the solution to the above initial value problem of Eqs. (4)-(5) is given by

$$
\begin{equation*}
\bar{y}(x)=\mathcal{P}_{n}\left[e^{f(t) x}\right]_{t=0} \bar{y}_{0} . \tag{6}
\end{equation*}
$$

Proof. The fundamental matrix of the system of Eq. (4) is

$$
\begin{equation*}
\Phi(x)=e^{\mathcal{P}_{n}[f(t)]_{t=0} x}=I_{n+1}+\sum_{k=1}^{\infty} \mathcal{P}_{n}^{k}[f(t)]_{t=0} \frac{x^{k}}{k!} . \tag{7}
\end{equation*}
$$

Since $f(0)=0, \quad\left(\mathcal{P}_{n}[f(t)]_{t=0}\right)_{i, j}=0$ for $i \leq j$ and hence $\mathcal{P}_{n}^{k}[f(t)]_{t=0}$ is a zero matrix for all $k \geq n+1$. Therefore the above sum is a finite sum and applying Properties (d) and (e), we get

$$
\Phi(x)=I_{n+1}+\mathcal{P}_{n}\left[\sum_{k=1}^{\infty} f^{k}(t) \frac{x^{k}}{k!}\right]_{t=0}=I_{n+1}+\mathcal{P}_{n}\left[e^{f(t) x}-1\right]_{t=0}
$$

By Properties (a) and (d), we obtain

$$
\Phi(x)=\mathcal{P}_{n}\left[e^{f(t) x}\right]_{t=0} .
$$

Thus the solution to the above initial value problem of Eqs. (4)-(5) is

$$
\begin{equation*}
\bar{y}(x)=\mathcal{P}_{n}\left[e^{f(t) x}\right]_{t=0} \bar{y}_{0} . \tag{8}
\end{equation*}
$$

This completes the proof.

## 3. Appell Polynomial Sequences and Appell Vectors

An Appell polynomial sequence is often defined by a generating function (see [7]).

Definition 3.1. Let $g(t)$ be an invertible analytic function. Then the sequence $\left\{s_{n}(x)\right\}$ is the Appell polynomial sequence for $g(t)$ if and only if

$$
\frac{1}{g(t)} e^{x t}=\sum_{k=0}^{\infty} \frac{s_{k}(x)}{k!} t^{k}
$$

Note 3.1. Since $g(t)$ is invertible, $\frac{1}{g(t)} e^{x t}$ is analytic, and by Taylor's Theorem,

$$
s_{k}(x)=\left.\left(\frac{d}{d t}\right)^{(k)} \frac{1}{g(t)} e^{x t}\right|_{t=0} .
$$

An alternative definition of an Appell polynomial sequence can be found in [7].

Definition 3.2. The sequence $\left\{s_{n}(x)\right\}$ is Appell polynomial sequence for an invertible analytic function $g(t)$ if and only if

$$
s_{n}^{\prime}(x)=n s_{n-1}(x)
$$

with the initial condition

$$
s_{k}(0)=\left.\left(\frac{d}{d t}\right)^{(k)}\left(\frac{1}{g(t)}\right)\right|_{t=0}
$$

The equivalence of Definitions 3.1 and 3.2 is obtained by taking the derivative with respect to $x$ of both sides of the expression in Definition 3.1.

In order to establish linear algebra approach to studying Appell polynomial sequences, we introduce Appell vector.

Definition 3.3. Let $\left\{s_{n}(x)\right\}$ be the Appell polynomial sequence for an invertible analytic function $g(t)$. Then $\bar{S}_{n}(x)=\left(s_{0}(x), \ldots, s_{n}(x)\right)^{T}$ is called the Appell vector for $g(t)$.

We now show Appell vector is a solution to the system of differential equations (4) with $f(t)=t$ and it has a simple representation as an Wronskian functional vector.

Theorem 3.1. Let $\bar{S}_{n}(x)=\left(s_{0}(x), \ldots, s_{n}(x)\right)^{T}$ be a vector of functions. Then $\bar{S}_{n}(x)$ is the Appell vector for $g(t)$ if and only if

$$
\begin{equation*}
\bar{S}_{n}(x)=\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \mathcal{W}_{n}\left[e^{x t}\right]_{t=0}=\mathcal{W}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0} \tag{9}
\end{equation*}
$$

Proof. Since

$$
\left(\mathcal{P}_{n}[t]_{t=0}\right)_{i, j}=\left\{\begin{array}{ll}
i & \text { if } i=j+1, \\
0 & \text { otherwise },
\end{array} \text { for } i, j=0,1,2, \ldots, n\right.
$$

Definition 3.2, in vector form, is equivalent to

$$
\begin{equation*}
\frac{d}{d x} \bar{S}_{n}(x)=\mathcal{P}_{n}[t]_{t=0} \bar{S}_{n}(x) \tag{10}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\bar{S}_{n}(0)=\mathcal{W}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \tag{11}
\end{equation*}
$$

Therefore, by Theorem 2.1 and using Property (f),

$$
\begin{aligned}
\bar{S}_{n}(x) & =\mathcal{P}_{n}\left[e^{x t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \mathcal{W}_{n}\left[e^{x t}\right]_{t=0} \\
& =\mathcal{W}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0}
\end{aligned}
$$

On the other hand, if $\bar{S}_{n}(x)$ satisfies Eq. (9), then

$$
s_{k}(x)=\left.\left(\frac{d}{d t}\right)^{(k)} \frac{1}{g(t)} e^{x t}\right|_{t=0} .
$$

Thus $\left\{s_{n}(x)\right\}$ is the Appell polynomial sequence for $g(t)$ as in Definition 3.1 and $\bar{S}_{n}(x)$ is the Appell vector for $g(t)$. This completes the proof.

Note 3.2. Expressing

$$
\bar{S}_{n}(x)=\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \mathcal{W}_{n}\left[e^{x t}\right]_{t=0}
$$

in matrix form as follows:

$$
\left[\begin{array}{c}
s_{0}(x)  \tag{12}\\
s_{1}(x) \\
s_{2}(x) \\
\vdots \\
s_{n}(x)
\end{array}\right]=\left[\begin{array}{ccccc}
s_{00} & 0 & 0 & \cdots & 0 \\
s_{10} & s_{11} & 0 & \cdots & 0 \\
s_{20} & s_{21} & s_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n 0} & s_{n 1} & s_{n 2} & \cdots & s_{n n}
\end{array}\right]\left[\begin{array}{c}
1 \\
x \\
x^{2} \\
\vdots \\
x^{n}
\end{array}\right],
$$

we recognize the matrix $\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0}$ as the coefficient matrix of the Appell polynomials $s_{0}(x), \ldots, s_{n}(x)$.

Note 3.3. Setting $x=0$, we have

$$
\bar{S}_{n}(0)=\mathcal{W}_{n}\left[\frac{1}{g(t)}\right]_{t=0} .
$$

Equivalently

$$
\frac{1}{g(t)}=\sum_{k=0}^{\infty} \frac{s_{k}(0)}{k!} t^{k}
$$

i.e., $\frac{1}{g(t)}$ is the exponential generating function of the number sequence $\left\{s_{k}(0)\right\}$.

Theorem 3.2 (The Appell Identity). Let $\left\{s_{n}(x)\right\}$ be the Appell polynomial sequence for an invertible analytic function $g(t)$. Then

$$
\begin{equation*}
\bar{S}_{n}(y+z)=P_{L}^{z} \bar{S}_{n}(y)=P_{L}[z] \bar{S}_{n}(y) \tag{13}
\end{equation*}
$$

equivalently

$$
\begin{equation*}
s_{n}(y+z)=\sum_{k=0}^{n}\binom{n}{k} s_{k}(y) z^{n-k} \tag{14}
\end{equation*}
$$

Proof. By Theorem 3.1, we have

$$
\bar{S}_{n}(y+z)=\mathcal{W}_{n}\left[\frac{1}{g(t)} e^{(y+z) t}\right]_{t=0}
$$

Using Properties (f) and (c) yields

$$
\begin{aligned}
\bar{S}_{n}(y+z) & =\mathcal{P}_{n}\left[e^{z t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g(t)} e^{y t}\right]_{t=0} \\
& =P_{L}^{z} \bar{S}_{n}(y)=P_{L}[z] \bar{S}_{n}(y)
\end{aligned}
$$

Comparing the last rows of the matrices $\bar{S}_{n}(y+z)$ and $P_{L}[z] \bar{S}_{n}(y)$ leads to Eq. (14).

Let us list a few well-known Apell polynomial sequences and vectors.
Example 3.1. The Appell polynomial sequence for $g(t)=1$ is $\left\{s_{n}(x)\right\}$ $=\left\{x^{n}\right\}$ and the corresponding Appell vector is

$$
\begin{aligned}
\bar{S}_{n}(x) & =\mathcal{W}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0}=\mathcal{W}_{n}\left[e^{x t}\right]_{t=0} \\
& =\left[\begin{array}{lllll}
1 & x & x^{2} & \cdots & x^{n}
\end{array}\right]^{T}
\end{aligned}
$$

Example 3.2. Hermite polynomial $\left\{H_{n}(x)\right\}$ is the Appell polynomial sequence for $g(t)=e^{\frac{t^{2}}{2}}$.

Example 3.3. Bernoulli polynomial sequence $\left\{B_{n}^{(\alpha)}(x)\right\}$ of order $a, a \neq 0$, is the Appell polynomial sequence for $g(t)=\left(\frac{e^{t}-1}{t}\right)^{a}$. When $a=1, \quad B_{n}^{(1)}(x)$ is Bernoulli polynomial and denoted by $B_{n}(x) . B_{n}^{(1)}(0)$ are called Bernoulli numbers and denoted by $b_{n}$.

Example 3.4. Euler polynomial sequence $\left\{E_{n}^{(a)}(x)\right\}$ of order $a, a \neq 0$, is the Appell polynomial sequence for $g(t)=\left(\frac{e^{t}+1}{2}\right)^{a}$. When $a=1$, $E_{n}^{(1)}(x)$ is Euler polynomial and denoted by $E_{n}(x)$.

Noting that the Appell vector $\bar{S}_{n}(x)=\mathcal{W}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0}$ forms the first column of the Pascal functional matrix $\mathcal{P}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0}$, we expand the Appell vector into Pascal functional matrix to take advantage of its nice interaction with Wronskian vector as in Property (f).

Definition 3.4. Let $\left\{s_{n}(x)\right\}$ be the Appell polynomial sequence for $g(t)$. The Pascal matrix of the Appell polynomial sequence for $g(t)$ is defined by

$$
\mathcal{P}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0}=\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0 \\
s_{1}(x) & s_{0}(x) & 0 & \cdots & 0 \\
s_{2}(x) & 2 s_{1}(x) & s_{0}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} s_{n}(x) & \binom{n}{1} s_{n-1}(x) & \binom{n}{2} s_{n-2}(x) & \cdots & \binom{n}{n} s_{0}(x)
\end{array}\right] .
$$

It is relatively simple to generalize the Appell identity.
Theorem 3.3 (Generalized Appell Identity). Let $\left\{s_{n}(x)\right\}$ and $\left\{r_{n}(x)\right\}$ be the Appell polynomial sequences for $g_{s}(t)$ and $g_{r}(t)$, respectively. Then

$$
v_{n}(y+z)=\sum_{k=0}^{n}\binom{n}{k} s_{k}(y) r_{n-k}(z)
$$

where $\left\{v_{n}(x)\right\}$ is the Appell polynomial sequence for $h(t)=g_{s}(t) g_{r}(t)$.
Proof. Let us consider the product of the Pascal functional matrix of the Appell polynomial sequence for $g_{s}(t)$ and the Appell vector for $g_{r}(t)$ and apply Property (f) to get:

$$
\begin{equation*}
\mathcal{P}_{n}\left[\frac{1}{g_{s}(t)} e^{y t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{r}(t)} e^{z t}\right]_{t=0}=\mathcal{W}_{n}\left[\frac{1}{g_{s}(t) g_{r}(t)} e^{(y+z) t}\right]_{t=0} \tag{15}
\end{equation*}
$$

The left side of Eq. (15) equals

$$
\left[\begin{array}{ccccc}
s_{0}(y) & 0 & 0 & \cdots & 0  \tag{16}\\
s_{1}(y) & s_{0}(y) & 0 & \cdots & 0 \\
s_{2}(y) & 2 s_{1}(y) & s_{0}(y) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} s_{n}(y) & \binom{n}{1} s_{n-1}(y) & \binom{n}{2} s_{n-2}(y) & \cdots & \binom{n}{n} s_{0}(y)
\end{array}\right]\left[\begin{array}{c}
r_{0}(z) \\
r_{1}(z) \\
r_{2}(z) \\
\vdots \\
r_{n}(z)
\end{array}\right]
$$

and the right side of Eq. (15) is

$$
\left[\begin{array}{lllll}
v_{0}(y+z) & v_{1}(y+z) & v_{2}(y+z) & \cdots & v_{n}(y+z) \tag{17}
\end{array}\right]^{T}
$$

where $\left\{v_{n}(x)\right\}$ is the Appell polynomial sequence for $g_{s}(t) g_{r}(t)$. Equating the last rows of Eqs. (16) and (17) leads to the theorem.

Note 3.4. The Appell identity is a special case of the generalized Appell identity. By letting $g_{r}(t)=1, r_{n}(x)=x^{n}$ and $v_{n}(x)=s_{n}(x)$.

A good application of the generalized Appell identity is the following new identity.

## Corollary 3.1.

$$
\begin{equation*}
2^{n} B_{n}\left(\frac{y+z}{2}\right)=\sum_{k=0}^{n}\binom{n}{k} B_{k}(y) E_{n-k}(z) \tag{18}
\end{equation*}
$$

where $B_{k}(x)$ and $E_{k}(x)$ are Bernoulli and Euler polynomials, respectively.

Proof. If $g_{s}(t)=\frac{e^{t}-1}{t}$ and $g_{r}(t)=\frac{e^{t}+1}{2}$ in Theorem 3.3, then

$$
h(t)=g_{s}(t) g_{r}(t)=\frac{e^{2 t}-1}{2 t}=g_{s}(2 t) .
$$

Thus $\left\{s_{n}(x)\right\}=\left\{B_{n}(x)\right\},\left\{r_{n}(x)\right\}=\left\{E_{n}(x)\right\}$, and $v_{n}(x)$ is the Appell polynomial sequence for $g_{s}(2 t)$. Here

$$
\bar{V}_{n}(x)=\left.\mathcal{W}_{n}\left[\frac{1}{g_{s}(2 t)} e^{x t}\right]\right|_{t=0}=\left.\operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \mathcal{W}_{n}\left[\frac{1}{g_{s}(t)} e^{\frac{x}{2} t}\right]\right|_{t=0}
$$

i.e., $v_{n}(x)=2^{n} B_{n}\left(\frac{x}{2}\right)$.

In Section 5, we will derive some new and well-known identities using the generalized Appell identity.

## 4. The Connection Constants, Inverse Relations and Duplication Formulas

### 4.1. Connection constants

Since any Appell polynomial sequence $\left\{s_{k}(x)\right\}_{k=0}^{n}$ forms bases for the vector space of polynomials of degree less than or equal to $n$, it is natural to ask how to represent one Appell polynomial sequence $\left\{s_{n}(x)\right\}$ by another Appell polynomial sequence $\left\{r_{n}(x)\right\}$.

Let $\left\{s_{n}(x)\right\}$ and $\left\{r_{n}(x)\right\}$ be the Appell polynomial sequences for $g_{s}(t)$ and $g_{r}(t)$, respectively. Then, using Theorem 3.1, we have

$$
\bar{S}_{n}(x)=\mathcal{W}_{n}\left[\frac{1}{g_{s}(t)} e^{x t}\right]_{t=0}
$$

and

$$
\bar{R}_{n}(x)=\mathcal{W}_{n}\left[\frac{1}{g_{r}(t)} e^{x t}\right]_{t=0}
$$

Employing Property (f), we rewrite

$$
\begin{aligned}
\bar{S}_{n}(x) & =\mathcal{W}_{n}\left[\frac{1}{g_{s}(t)} \frac{g_{r}(t)}{g_{r}(t)} e^{x t}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{g_{r}(t)}{g_{s}(t)}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{r}(t)} e^{x t}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{g_{r}(t)}{g_{s}(t)}\right]_{t=0} \bar{R}_{n}(x)
\end{aligned}
$$

We summarize the above analysis as the following theorem.
Theorem 4.1 (Connection Constants Theorem). Let $\bar{S}_{n}(x)$ and $\bar{R}_{n}(x)$ be the Appell vectors for $g_{s}(t)$ and $g_{r}(t)$, respectively. Then

$$
\bar{S}_{n}(x)=C \bar{R}_{n}(x)
$$

where $C=\mathcal{P}_{n}[h(t)]_{t=0}$ and $h(t)=\frac{g_{r}(t)}{g_{s}(t)}$. Equivalently

$$
s_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} h^{(n-k)}(0) r_{k}(x)
$$

The following new identity is an immediate consequence of Theorem 4.1.

Corollary 4.1. For Bernoulli polynomial $B_{n}(x)$ and Euler polynomial $E_{n}(x)$, we have

$$
B_{n}(x)=\frac{n E_{n-1}(x)}{2}+\sum_{k=0}^{n}\binom{n}{k} b_{n-k} E_{k}(x)
$$

where $b_{k}$ is the kth Bernoulli number.
Proof. Let $g_{s}(t)=\frac{e^{t}-1}{t}$ and $g_{r}(t)=\frac{e^{t}+1}{2}$ in Theorem 4.1. Thus

$$
h(t)=\frac{g_{r}(t)}{g_{s}(t)}=\frac{t\left(e^{t}+1\right)}{2\left(e^{t}-1\right)}=\frac{t}{2}+\frac{t}{e^{t}-1}
$$

It is easy to obtain

$$
h^{(k)}(0)=\left\{\begin{array}{ll}
1 / 2+b_{k} & \text { if } k=1, \\
b_{k} & \text { otherwise },
\end{array} \text { for } k=0,1,2, \ldots, n\right.
$$

Therefore, we have

$$
B_{n}(x)=\frac{n E_{n-1}(x)}{2}+\sum_{k=0}^{n}\binom{n}{n-k} b_{n-k} E_{k}(x) .
$$

This completes the proof.
Note 4.1. [7] presents $E_{n}(x)$ in terms of $B_{n}(x)$.

### 4.2. Inverse relations

The most popular inverse relation is the binomial inverse relation, which is

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n}\binom{n}{k} x_{k}, \\
x_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} y_{k} .
\end{array}\right.
$$

In general, an inverse relation is

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n} a_{n, k} x_{k}, \\
x_{n}=\sum_{k=0}^{n} b_{n, k} y_{k}
\end{array}\right.
$$

If we write the inverse relation in a matrix form, we have

$$
\left\{\begin{array}{l}
\bar{x}_{n}=C_{1} \bar{y}_{n},  \tag{19}\\
\bar{y}_{n}=C_{2} \bar{x}_{n},
\end{array}\right.
$$

where $C_{1}$ and $C_{2}$ are $(n+1) \times(n+1)$ matrices and $\bar{x}_{n}=\left[x_{0}, x_{1}, \ldots, x_{n}\right]^{T}$ and $\bar{y}_{n}=\left[y_{0}, y_{1}, \ldots, y_{n}\right]^{T}$. It is equivalent to a single orthogonality relation

$$
\begin{equation*}
C_{1} C_{2}=I_{n+1} . \tag{20}
\end{equation*}
$$

By Property (e) we find a new inverse relation.

Note 4.2. $\bar{x}_{n}=C_{1} \bar{y}_{n}$ and $\bar{y}_{n}=C_{2} \bar{x}_{n}$ are called inverse pairs.

Theorem 4.2 (New Inverse Relation). For an invertible analytic function $f(t)$,

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(0) x_{k}  \tag{21}\\
x_{n}=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{f}\right)^{(n-k)}(0) y_{k}
\end{array}\right.
$$

is an inverse relation.

Proof. Note that $C_{2}=\mathcal{P}_{n}[f(t)]_{t=0}$ and $C_{1}=\mathcal{P}_{n}[1 / f(t)]_{t=0}$ in Eq. (21). Employing Property (e) yields $C_{1} C_{2}=I_{n+1}$.

There are many well-known inverse relations, which are special cases of Theorem 4.2. We illustrate a few of them.

Example 4.1 (Binomial). Choosing $f(t)=e^{t}$ in Theorem 4.2, we obtain the binomial inverse relation

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n}\binom{n}{k} x_{k} \\
x_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} y_{k}
\end{array}\right.
$$

Similarly,

Example 4.2 (Generalized Binomial). Choosing $f(t)=e^{a t}$ in Theorem 4.2, we obtain the generalized binomial inverse relation

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n}\binom{n}{k} a^{n-k} x_{k} \\
x_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} a^{n-k} y_{k}
\end{array}\right.
$$

Example 4.3 (Ox-Plowing or Boustrophedon Transformation). Choosing

$$
f(x)=\sec x+\tan x=\sum_{n=0}^{\infty} T_{n} \frac{x^{n}}{n!},
$$

where $T_{n}$ is the $n$th tangent number, we obtain the following inverse relation known as Ox-Plowing Transformation

$$
\left\{\begin{array}{l}
b_{n}=\sum_{k=0}^{n}\binom{n}{k} T_{n-k} a_{k} \\
a_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} T_{n-k} b_{k}
\end{array}\right.
$$

We note that $\frac{1}{f(x)}=f(-x)$ and, by Theorem 4.2, the inverse relation holds.
Example 4.4 (Bernoulli Inverse Relation). Choosing

$$
f(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{(k+1)!}=\frac{e^{t}-1}{t}
$$

in Theorem 4.2, we have

$$
\left\{\begin{array}{l}
y_{n}=\sum_{k=0}^{n}\binom{n}{k} \frac{1}{k+1} x_{n-k}, \\
x_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{k} y_{n-k},
\end{array}\right.
$$

where $b_{k}$ is the $k$ th Bernoulli number. Note that $\frac{1}{f(t)}=\frac{t}{e^{t}-1}$ is the exponential generating function of Bernoulli numbers.

Theorem 4.3 (New Inverse Relation related to Appell Sequences).

$$
\left\{\begin{array}{l}
y_{n}(\omega)=\sum_{k=0}^{n}\binom{n}{k} s_{n-k}(\omega) x_{k}(\omega),  \tag{22}\\
x_{n}(\omega)=\sum_{k=0}^{n}\binom{n}{k} r_{n-k}(-\omega) y_{k}(\omega),
\end{array}\right.
$$

is an inverse relation if $\left\{s_{n}(\omega)\right\}$ is the Appell polynomial sequence for $\frac{1}{g(t)}$ and $\left\{r_{n}(\omega)\right\}$ is the Appell polynomial sequence for $g(t)$.

Proof. If we write the relation of Eq. (22) in a matrix form, we have

$$
\left\{\begin{array}{l}
\bar{x}_{n}(\omega)=C_{1} \bar{y}_{n}(\omega),  \tag{23}\\
\bar{y}_{n}(\omega)=C_{2} \bar{x}_{n}(\omega),
\end{array}\right.
$$

where $C_{2}=\mathcal{P}_{n}\left[\frac{e^{\omega t}}{g(t)}\right]_{t=0}$ and $C_{1}=\mathcal{P}_{n}\left[g(t) e^{-\omega t}\right]_{t=0}$. By Property (e), it is easy to see $C_{1} C_{2}=I_{n+1}$.

Example 4.5 (Hermite Inverse Relation).

$$
\left\{\begin{array}{l}
y_{n}(\omega)=\sum_{k=0}^{n}\binom{n}{k} H_{n-k}(\omega) x_{k}(\omega), \\
x_{n}(\omega)=\sum_{k=0}^{n}\binom{n}{k} r_{n-k}(-\omega) y_{k}(\omega)
\end{array}\right.
$$

where $\left\{H_{k}(\omega)\right\}$ is Hermite polynomial sequence, which has its exponential generating function

$$
e^{\omega t-t^{2} / 2}=\sum_{k=0}^{\infty} \frac{H_{k}(\omega) t^{k}}{k!}
$$

and $\left\{r_{k}(\omega)\right\}$ is an Appell sequence with the exponential generating function

$$
e^{\omega t+t^{2} / 2}=\sum_{k=0}^{\infty} \frac{r_{k}(\omega) t^{k}}{k!}
$$

Theorem 4.4 (Inverse Relations between any two Appell sequences). Let $\bar{S}_{n}(x)$ and $\bar{R}_{n}(x)$ be the Appell vectors for $g_{s}(t)$ and $g_{r}(t)$, respectively. Then the following inverse relation holds:

$$
\bar{S}_{n}(x)=C_{1} \bar{R}_{n}(x)
$$

and

$$
\bar{R}_{n}(x)=C_{2} \bar{S}_{n}(x)
$$

where $C_{1}=\mathcal{P}_{n}[h(t)]_{t=0}, h(t)=\frac{g_{r}(t)}{g_{s}(t)}$, and $C_{2}=C_{1}^{-1}=\mathcal{P}_{n}[1 / h(t)]_{t=0}$.

Proof. The proof is an immediate consequence of the Connection Constants Theorem.

Corollary 4.2 (Inverse pairs of Bernoulli polynomials and $x^{n}$ ).

$$
B_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} x^{k}, \quad x^{n}=\sum_{k=0}^{n}\binom{n}{k} \frac{1}{n-k+1} B_{k}(x)
$$

where $b_{k}$ is the kth Bernoulli number.
Proof. Let $g_{r}(t)=1$ and $g_{s}(t)=\frac{e^{t}-1}{t}$. Then $h(t)=\frac{1}{g_{s}(t)}, r_{n}(x)=x^{n}$,
and $\bar{S}_{n}(x)=\bar{B}_{n}(x)$. Hence

$$
\bar{B}_{n}(x)=\mathcal{P}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0} \bar{R}_{n}(x)
$$

$$
=\left[\begin{array}{ccccc}
B_{0}(0) & 0 & 0 & \cdots & 0 \\
B_{1}(0) & B_{0}(0) & 0 & \cdots & 0 \\
B_{2}(0) & \binom{2}{1} B_{1}(0) & B_{0}(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{n}(0) & \binom{n}{1} B_{n-1}(0) & \binom{n}{2} B_{n-2}(0) & \cdots & B_{0}(0)
\end{array}\right]\left[\begin{array}{c}
1 \\
x \\
x^{2} \\
\vdots \\
x^{n}
\end{array}\right] .
$$

Equating the last rows of the first and the last matrices yields

$$
B_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} x^{k}
$$

The inverse of

$$
\mathcal{P}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0}=\mathcal{P}_{n}\left[\frac{e^{t}-1}{t}\right]_{t=0}
$$

Since

$$
\left.\left(\frac{d}{d t}\right)^{(k)} \frac{e^{t}-1}{t}\right|_{t=0}=\frac{1}{k+1}
$$

we have

$$
x^{n}=\sum_{k=0}^{n}\binom{n}{k} \frac{1}{n-k+1} B_{k}(x)
$$

### 4.3. Duplication formulas

A duplication formula is a formula of the form

$$
s_{n}(a x)=\sum_{k=0}^{n} d_{n, k} s_{k}(x)
$$

Theorem 4.5 (Duplication Formula). Let $\left\{s_{n}(x)\right\}$ be an Appell polynomial sequence for any invertible analytic function $g(t)$. Then

$$
\begin{equation*}
\bar{S}_{n}(a x)=D \bar{S}_{n}(x) \tag{24}
\end{equation*}
$$

where

$$
D=\mathcal{P}_{n}\left[\frac{g(a t)}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right]
$$

Proof. Using Theorem 3.1, and applying Properties (f) and (g) to Eq. (24), we have

$$
\begin{align*}
\bar{S}_{n}(a x) & =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \mathcal{W}_{n}\left[e^{a x t}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{W}_{n}\left[e^{x t}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{P}_{n}[g(t)]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g(t)} e^{x t}\right]_{t=0} \\
& =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{P}_{n}[g(t)]_{t=0} \bar{S}_{n}(x) \tag{25}
\end{align*}
$$

Applying properties (h) and (e) to Eq. (25) yields

$$
\begin{align*}
\bar{S}_{n}(a x) & =\mathcal{P}_{n}\left[\frac{1}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right] \mathcal{P}_{n}[g(t)]_{t=0} \bar{S}_{n}(x) \\
& =\mathcal{P}_{n}\left[\frac{g(a t)}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, a, \ldots, a^{n}\right] \bar{S}_{n}(x) \\
& =D \bar{S}_{n}(x) \tag{26}
\end{align*}
$$

This completes the proof.

Raabe found the following interesting multiplication identity for Bernoulli polynomials in [6]

$$
B_{n}(m x)=m^{n-1} \sum_{k=0}^{m-1} B_{n}\left(x+\frac{k}{m}\right) .
$$

Here, we use Theorem 4.5 to develop a new identity, which allows us to represent $B_{n}(m x)$ in terms of $B_{k}(x), k=0,1, \ldots, n$.

Corollary 4.3. For any positive integer $m \geq 2$,

$$
\begin{equation*}
B_{n}(m x)=m^{n} B_{n}(x)+m^{n-1} \sum_{k=1}^{n}\binom{n}{k} \Omega_{k} m^{-k} B_{n-k}(x), \tag{27}
\end{equation*}
$$

where $\Omega_{k}=1^{k}+2^{k}+\cdots+(m-1)^{k}$.
Proof. Let $g(t)=\frac{e^{t}-1}{t}$ and $a=m$ in Theorem 4.5. Thus $S_{n}(x)=B_{n}(x)$.
Clearly,

$$
\frac{g(a t)}{g(t)}=m^{-1} \sum_{j=0}^{m-1} e^{j t}
$$

Since

$$
\left.\begin{array}{rl}
\left(\frac{d}{d t}\right)^{(k)}\left(\sum_{j=0}^{m-1} e^{j t}\right)_{t=0}=\left\{\begin{array}{l}
m \\
\Omega_{k}=1^{k}+2^{k}+\cdots+(m-1)^{k} \\
\text { if } k>0,
\end{array}\right. \\
\bar{B}_{n}(m x) & =\mathcal{P}_{n}\left[\frac{g(m t)}{g(t)}\right]_{t=0} \operatorname{diag}\left[1, m, \ldots, m^{n}\right] \bar{B}_{n}(x)
\end{array}\right] \begin{gathered}
\text { in }\left[\begin{array}{ccccc}
m & 0 & 0 & \cdots & 0 \\
\Omega_{1} & m & 0 & \cdots & 0 \\
\Omega_{2} & \binom{2}{1} \Omega_{1} & m & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Omega_{n} & \binom{n}{1} \Omega_{n-1} & \binom{n}{1} \Omega_{n-2} & \cdots & m
\end{array}\right]\left[\begin{array}{c}
B_{0}(x) \\
m B_{1}(x) \\
m^{2} B_{2}(x) \\
\vdots \\
m^{n} B_{n}(x)
\end{array}\right] .
\end{gathered}
$$

Equating the last rows of both sides of Eq. (29) yields

$$
\begin{align*}
B_{n}(m x) & =m^{n} B_{n}(x)+m^{-1} \sum_{k=1}^{n}\binom{n}{n-k} \Omega_{k} m^{n-k} B_{n-k}(x) \\
& =m^{n} B_{n}(x)+m^{n-1} \sum_{k=1}^{n}\binom{n}{k} \Omega_{k} m^{-k} B_{n-k}(x) . \tag{30}
\end{align*}
$$

This completes the proof.

## 5. Combinatorial Identities

The first set of identities are immediate consequences of the Generalized Appell Identity, Theorem 3.3. The first part of the following identities was presented in [7] and [9].

Identity 5.1. Let $B_{n}^{(a)}(x)$ and $E_{n}^{(a)}(x)$ be Bernoulli and Euler polynomials, respectively, of order $a$. Then

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k} B_{k}^{(a)}(y) B_{n-k}^{(b)}(z)=B_{n}^{(a+b)}(y+z) \\
& \sum_{k=0}^{n}\binom{n}{k} E_{k}^{(a)}(y) E_{n-k}^{(b)}(z)=E_{n}^{(a+b)}(y+z)
\end{aligned}
$$

Proof. The identity for Bernoulli polynomial is verified by noting that the product of the generating functions for polynomials of order $a$, $g_{s}(t)=\left(\frac{e^{t}-1}{t}\right)^{a}$ and order $b, \quad g_{r}(t)=\left(\frac{e^{t}-1}{t}\right)^{b}$ is the generating function for polynomial of order $a+b, g(t)=\left(\frac{e^{t}-1}{t}\right)^{a+b}$.

The same is true of the identity for the Euler polynomials.
Identity 5.2. Let $\left\{s_{n}(x)\right\}$ and $\left\{r_{n}(x)\right\}$ be the Appell polynomial sequences for any invertible analytic functions $g_{s}(t)$ and $g_{r}(t)$, respectively. Then

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k} s_{k}(y) r_{n-k}(-y)=\sum_{k=0}^{n}\binom{n}{k} s_{k} r_{n-k}, \\
& \sum_{k=0}^{n}\binom{n}{k} s_{k}(y) r_{n-k}(z)=\sum_{k=0}^{n}\binom{n}{k} s_{k}(y+z) r_{n-k},
\end{aligned}
$$

where $s_{k}=s_{k}(0)$ and $r_{k}=r_{k}(0)$.
Proof. By Generalized Appell Identity Theorem, we have

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k} s_{k}(y) r_{n-k}(-y) & =v_{n}(0) \\
& =\sum_{k=0}^{n}\binom{n}{k} s_{k}(0) r_{n-k}(0)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k} s_{k}(y) r_{n-k}(z) & =v_{n}(y+z) \\
& =\sum_{k=0}^{n}\binom{n}{k} s_{k}(y+z) r_{n-k}(0),
\end{aligned}
$$

where $\left\{v_{n}(x)\right\}$ is the Appell polynomial sequence for $g_{s}(t) g_{r}(t)$.
In particular, we have

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k} B_{k}(y) B_{n-k}(-y)=\sum_{k=0}^{n}\binom{n}{k} b_{k} b_{n-k}, \\
& \sum_{k=0}^{n}\binom{n}{k} B_{k}(y) B_{n-k}(z)=\sum_{k=0}^{n}\binom{n}{k} B_{k}(y+z) b_{n-k}, \\
& \sum_{k=0}^{n}\binom{n}{k} E_{k}(y) B_{n-k}(z)=\sum_{k=0}^{n}\binom{n}{k} E_{k}(y+z) b_{n-k}, \\
& \sum_{k=0}^{n}\binom{n}{k} B_{k}(y) H_{n-k}(-y)=\sum_{k=0}^{n}\binom{n}{k} \frac{b_{k} h_{n-k}}{\frac{n-k}{2}},
\end{aligned}
$$

where $B_{k}(x), E_{k}(x)$, and $H_{k}(x)$ are Bernoulli, Euler, and Hermite polynomials, respectively. $b_{k}$ and $h_{k}$ are the $k$ th Bernoulli and Hermite numbers, respectively. The $k$ th Hermite number $h_{k}$ equals $2^{n / 2} H_{k}(0)$.

Identity 5.3. Let $\left\{s_{n}(x)\right\}$ and $\left\{r_{n}(x)\right\}$ be the Appell sequences for $g_{s}(t)$ and $g_{r}(t)$, respectively. Then

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} s_{k}(x) r_{n-k}(x)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} s_{k} r_{n-k}
$$

where $s_{k}=s_{k}(0)$ and $r_{n-k}=r_{n-k}(0)$.
Proof. By Property (g), we know

$$
\begin{align*}
& \mathcal{P}_{n}\left[\frac{1}{g_{s}(t)} e^{x t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{r}(-t)} e^{-x t}\right]_{t=0} \\
= & {\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0 \\
s_{1}(x) & s_{0}(x) & 0 & \cdots & 0 \\
s_{2}(x) & 2 s_{1}(x) & s_{0}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} s_{n}(x) & \binom{n}{1} s_{n-1}(x) & \binom{n}{2} s_{n-2}(x) & \cdots & \binom{n}{n} s_{0}(x)
\end{array}\right]\left[\begin{array}{c}
r_{0}(x) \\
-r_{1}(x) \\
r_{2}(x) \\
\vdots \\
(-1)^{n} r_{n}(x)
\end{array}\right] . } \tag{31}
\end{align*}
$$

On the other hand, by Property (f), we have

$$
\begin{align*}
& \mathcal{P}_{n}\left[\frac{1}{g_{s}(t)} e^{x t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{r}(-t)} e^{-x t}\right]_{t=0} \\
= & \mathcal{P}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{r}(-t)}\right]_{t=0} \\
= & {\left[\begin{array}{ccccc}
s_{0}(0) & 0 & 0 & \cdots & 0 \\
s_{1}(0) & s_{0}(0) & 0 & \cdots & 0 \\
s_{2}(0) & 2 s_{1}(0) & s_{0}(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} s_{n}(0) & \binom{n}{1} s_{n-1}(0) & \binom{n}{2} s_{n-2}(0) & \cdots & \binom{n}{n} s_{0}(0)
\end{array}\right]\left[\begin{array}{c}
r_{0}(0) \\
-r_{1}(0) \\
r_{2}(0) \\
\vdots \\
(-1)^{n} r_{n}(0)
\end{array}\right] . } \tag{32}
\end{align*}
$$

Equating the last rows of Eqs. (31) and (32) produces the identity.

The followings are several special cases of Identity 5.3:

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} B_{k}(x) B_{n-k}(x)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} b_{k} b_{n-k},
$$

and

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} B_{k}(x) H_{n-k}(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k} \frac{b_{k} h_{n-k}}{2^{\frac{n-k}{2}}} .
$$

Identity 5.4 (Binomial Convolution). For any $a \neq 0$,

$$
\sum_{k=0}^{n}\binom{n}{k} B_{k}^{(a)}(y) E_{n-k}^{(a)}(z)=2^{n} B_{n}^{(a)}\left(\frac{y+z}{2}\right)
$$

Proof. Again, referring to the proof of Theorem 3.3, Generalized Appell Identity Theorem, with $g_{s}(t)=\left(\frac{e^{t}-1}{t}\right)^{(a)}$ and $g_{r}(t)=\left(\frac{e^{t}+1}{2}\right)^{(a)}$, we note that their product $h(t)=g_{s}(t) g_{r}(t)=g_{s}(2 t)$. Let $\left\{v_{n}(x)\right\}$ be an Appell sequence for $h(t)$. Then

$$
\begin{equation*}
\bar{V}_{n}(x)=\mathcal{W}_{n}\left[\frac{1}{h(t)} e^{x t}\right]_{t=0}=\mathcal{P}_{n}\left[e^{x t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{h(t)}\right]_{t=0} \tag{33}
\end{equation*}
$$

Employing Properties (g) and (h) in Eq. (33) yields

$$
\begin{aligned}
\bar{V}_{n}(x)= & \mathcal{P}_{n}\left[e^{x t}\right]_{t=0} \operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \mathcal{W}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0} \\
= & \operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \operatorname{diag}\left[1,2^{-1}, \ldots, 2^{-n}\right] \mathcal{P}_{n}\left[e^{x t}\right]_{t=0} \\
& \times \operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \mathcal{W}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0} \\
= & \operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \mathcal{P}_{n}\left[e^{\frac{x}{2} t}\right]_{t=0} \mathcal{W}_{n}\left[\frac{1}{g_{s}(t)}\right]_{t=0}
\end{aligned}
$$

$$
\begin{align*}
& =\operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \mathcal{W}_{n}\left[\frac{1}{g_{s}(t)} e^{\frac{x}{2} t}\right]_{t=0} \\
& =\operatorname{diag}\left[1,2, \ldots, 2^{n}\right] \bar{B}_{n}^{(a)}\left(\frac{x}{2}\right) \tag{34}
\end{align*}
$$

From Theorem 3.3, we have

$$
\begin{equation*}
v_{n}(y+z)=\sum_{k=0}^{n}\binom{n}{k} B_{k}^{(a)}(y) E_{n-k}^{(a)}(z) \tag{35}
\end{equation*}
$$

Combining Eqs. (34) and (35) leads

$$
\sum_{k=0}^{n}\binom{n}{k} B_{k}^{(a)}(y) E_{n-k}^{(a)}(z)=2^{n} B_{n}^{(a)}\left(\frac{y+z}{2}\right)
$$

This completes the proof.
Identity 5.5. Let $\left\{B_{n}(x)\right\}$ be Bernoulli polynomial sequence. Then

$$
\begin{equation*}
(n+1) x^{n}=\sum_{k=0}^{n}\binom{n+1}{k+1} B_{n-k}(x) \tag{36}
\end{equation*}
$$

Proof. Applying the Connection Constants Theorem to $g_{s}(t)=1$ and $g_{r}(t)=\frac{e^{t}-1}{t}$, we have $s_{n}(x)=x^{n}$ and $r_{n}(x)=B_{n}(x)$. Hence by the Connection Constants Theorem,

$$
\bar{S}_{n}(x)=\left[\begin{array}{lllll}
1 & x & x^{2} & \cdots & x^{n} \tag{37}
\end{array}\right]^{T}=C \bar{B}_{n}(x)
$$

where $C=\mathcal{P}_{n}\left[\frac{e^{t}-1}{t}\right]_{t=0}$. Since

$$
\left.\left(\frac{e^{t}-1}{t}\right)^{(k)}\right|_{t=0}=\frac{1}{k+1}
$$

and

$$
\binom{n}{k} \frac{1}{k+1}=\binom{n+1}{k+1} \frac{1}{n+1},
$$

the last row of $C \bar{B}_{n}(x)$ becomes

$$
\sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1} B_{n-k}(x) .
$$

Equating the last rows of both sides of Eq. (37) leads to the identity in Eq. (36).

## Identity 5.6.

$$
\begin{aligned}
& 2^{n}\left(B_{n}\left(\frac{x+1}{2}\right)-B_{n}\left(\frac{x}{2}\right)\right)=n E_{n-1}(x), \\
& E_{n}^{(a)}(x+1)+E_{n}^{(a)}(x)=2 E_{n}^{(a-1)}(x) .
\end{aligned}
$$

For $a=1$,

$$
E_{n}(x+1)+E_{n}(x)=2 x^{n} .
$$

Proof. By Theorem 3.1, we have

$$
\begin{align*}
\bar{B}_{n}\left(\frac{x+1}{2}\right)-\bar{B}_{n}\left(\frac{x}{2}\right) & =\mathcal{W}_{n}\left[\frac{t}{e^{t}-1} e^{\frac{x+1}{2} t}\right]_{t=0}-\mathcal{W}_{n}\left[\frac{t}{e^{t}-1} e^{\frac{x}{2} t}\right]_{t=0} \\
& =\mathcal{W}_{n}\left[\frac{t}{e^{t}-1}\left(e^{\frac{x+1}{2} t}-e^{\frac{x}{2} t}\right)\right]_{t=0} \\
& =\operatorname{diag}\left[1,2^{-1}, \ldots, 2^{-n}\right] \mathcal{W}_{n}\left[\frac{2 t}{e^{2 t}-1}\left(e^{(x+1) t}-e^{x t}\right)\right]_{t=0} \\
& =\operatorname{diag}\left[1,2^{-1}, \ldots, 2^{-n}\right] \mathcal{P}_{n}[t]_{t=0} \mathcal{W}_{n}\left[\frac{2}{e^{t}+1} e^{x t}\right]_{t=0} \\
& =\operatorname{diag}\left[1,2^{-1}, \ldots, 2^{-n}\right] \mathcal{P}_{n}[t]_{t=0} \bar{E}_{n}(x) \tag{38}
\end{align*}
$$

Comparison of the last rows in Eq. (38) gives

$$
B_{n}\left(\frac{x+1}{2}\right)-B_{n}\left(\frac{x}{2}\right)=2^{-n} n E_{n-1}(x),
$$

and hence the first identity. The second identity can be derived much the same way as the first one by combining the Wronskian vectors and simplifying it. For a special case of the second identity when $a=1$, we note that $E_{n}^{(0)}(x)=x^{n}$.

## 6. Differential Equations and Recurrence Relations for Appell Polynomials

In the recent paper [4], He and Ricci developed the differential equations and recurrence relations for Appell polynomials via differential operator factorization method. Here we will redevelop those results by the techniques established in this paper.

Theorem 6.1. Let $\left\{s_{n}(x)\right\}$ be the Appell polynomial sequence for $g(t)$. Then the Appell polynomials $s_{n}(x)$ satisfy the differential equation:

$$
\begin{equation*}
\frac{\alpha_{n-1}}{(n-1)!} y^{(n)}+\frac{\alpha_{n-2}}{(n-2)!} y^{(n-1)}+\cdots+\frac{\alpha_{1}}{1!} y^{\prime \prime}+\left(x+\alpha_{0}\right) y^{\prime}-n y=0, \tag{39}
\end{equation*}
$$

where $\alpha_{k}=\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}$.

## Proof.

$$
\begin{gather*}
\mathcal{W}_{n}\left[\left(\frac{1}{g(t)} e^{x t}\right)^{\prime}\right]_{t=0}=\left[\begin{array}{c}
s_{1}(x) \\
s_{2}(x) \\
\vdots \\
s_{n+1}(x)
\end{array}\right]  \tag{40}\\
=\mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(t)}{g(t)}\right) \frac{e^{x t}}{g(t)}\right]_{t=0}=\mathcal{P}_{n}\left[\frac{e^{x t}}{g(t)}\right]_{t=0} \mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(t)}{g(t)}\right)\right]_{t=0}
\end{gather*}
$$

$$
=\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0  \tag{41}\\
s_{1}(x) & s_{0}(x) & 0 & \cdots & 0 \\
s_{2}(x) & 2 s_{1}(x) & s_{0}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} s_{n}(x) & \binom{n}{1} s_{n-1}(x) & \binom{n}{2} s_{n-2}(x) & \cdots & \binom{n}{n} s_{0}(x)
\end{array}\right]\left[\begin{array}{c}
x+\alpha_{0} \\
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right] .
$$

Comparing the last rows of Eqs. (40) and (41), we get

$$
\begin{equation*}
s_{n+1}(x)=\left(x+\alpha_{0}\right) s_{n}(x)+\sum_{k=1}^{n}\binom{n}{k} \alpha_{k} s_{n-k}(x) . \tag{42}
\end{equation*}
$$

Taking derivative of both sides of Eq. (42) with respect to $x$ and noting that

$$
s_{n+1}^{\prime}(x)=(n+1) s_{n}(x), \quad s_{n}^{(k)}(x)=n(n-1)(n-2) \cdots(n-k+1) s_{n-k}(x)
$$

and $s_{n}^{(n+1)}(x)=0$, we get

$$
\begin{equation*}
(n+1) s_{n}(x)=s_{n}(x)+\left(x+\alpha_{0}\right) s_{n}^{\prime}(x)+\sum_{k=1}^{n} \alpha_{k} \frac{s_{n}^{(k+1)}(x)}{k!} . \tag{43}
\end{equation*}
$$

Simplifying Eq. (43) yields

$$
\begin{align*}
& \frac{\alpha_{n-1}}{(n-1)!} s_{n}^{(n)}(x)+\frac{\alpha_{n-2}}{(n-2)!} s_{n}^{(n-1)}(x) \\
& +\cdots+\frac{\alpha_{1}}{1!} s^{\prime \prime}(x)+\left(x+\alpha_{0}\right) s_{n}^{\prime}(x)-n s_{n}(x)=0 . \tag{44}
\end{align*}
$$

This completes the proof.
When $g(t)=\frac{e^{t}-1}{t}$ in Theorem 6.1, we obtain the differential equation for Bernoulli polynomials $B_{n}(x)$. Note that a similar equation has been presented in Theorem 2.3 in [4], which is incorrect.

Corollary 6.1. The Bernoulli polynomials $B_{n}(x)$ satisfy the following differential equation

$$
\begin{equation*}
\frac{b_{n}}{n!} y^{(n)}+\frac{b_{n-1}}{(n-1)!} y^{(n-1)}+\cdots+\frac{b_{2}}{2!} y^{\prime \prime}+\left(b_{1}-x\right) y^{\prime}+n y=0 \tag{45}
\end{equation*}
$$

where $b_{k}$ is the kth Bernoulli number.
Proof. To show the corollary, we need to calculate the coefficient $\alpha_{k}$ in Eq. (39). Given $g(t)=\frac{e^{t}-1}{t}$ and $\alpha_{k}=\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}$, we have

$$
\begin{align*}
\alpha_{k} & =\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}=\left.\left(\left(\frac{1}{g(t)}\right)^{\prime} g(t)\right)^{(k)}\right|_{t=0} \\
& =\left.\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{g(t)}\right)^{(j+1)}(g(t))^{(k-j)}\right|_{t=0} \tag{46}
\end{align*}
$$

Noting that

$$
\left.\left(\frac{1}{g(t)}\right)^{(j+1)}\right|_{t=0}=b_{j+1}
$$

and

$$
\left.(g(t))^{(k-j)}\right|_{t=0}=\frac{1}{k-j+1}
$$

we have

$$
\begin{equation*}
\alpha_{k}=\sum_{j=0}^{k}\binom{k}{j} \frac{1}{k-j+1} b_{j+1}=\frac{1}{k+1} \sum_{j=0}^{k}\binom{k+1}{j} b_{j+1} . \tag{47}
\end{equation*}
$$

Making an index change $l=j+1$ in Eq. (47) and applying

$$
\binom{k+1}{l}+\binom{k+1}{l-1}=\binom{k+2}{l}
$$

to the resulting equation yields

$$
\begin{equation*}
\alpha_{k}=\frac{1}{k+1} \sum_{l=1}^{k+1}\binom{k+1}{l-1} b_{l}=\frac{1}{k+1} \sum_{l=1}^{k+1}\left(\binom{k+2}{l}-\binom{k+1}{l}\right) b_{l} . \tag{48}
\end{equation*}
$$

Using the well-known identity

$$
\sum_{j=0}^{m}\binom{m+1}{j} b_{j}=0
$$

in Eq. (48) leads to $\alpha_{k}=-\frac{1}{k+1} b_{k+1}$.
Similarly, we develop the differential equation for Euler polynomials $E_{n}(x)$. Note that a similar equation has been presented in Theorem 2.4 in [4], which is incorrect.

Corollary 6.2. The Euler polynomials $E_{n}(x)$ satisfy the following differential equation

$$
\begin{equation*}
\frac{e_{n-1}}{(n-1)!} y^{(n)}+\frac{e_{n-2}}{(n-2)!} y^{(n-1)}+\cdots+\frac{e_{1}}{1!} y^{\prime \prime}+\left(x+e_{0}\right) y^{\prime}-n y=0, \tag{49}
\end{equation*}
$$

where

$$
e_{k}=-\frac{1}{2^{k+1}} \sum_{l=0}^{k}\binom{k}{l} E_{l}
$$

for $k=0,1,2, \ldots, n-1$, and $E_{l}$ is the lth term of Euler number sequence.
Proof. To show the corollary, we need to calculate the coefficient $\alpha_{k}$ in Eq. (39). Given $g(t)=\frac{e^{t}+1}{2}$ and $\alpha_{k}=\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}$, we have

$$
\begin{align*}
\alpha_{k} & =\left.\left(-\frac{e^{t}}{e^{t}+1}\right)^{(k)}\right|_{t=0} \\
& =-\left.\frac{1}{2}\left(\left(\frac{2 e^{\frac{1}{2} t}}{e^{t}+1}\right)^{\frac{1}{2} t}\right)^{(k)}\right|_{t=0} \\
& =-\left.\frac{1}{2} \sum_{j=0}^{k}\binom{k}{j}\left(\frac{2 e^{\frac{1}{2} t}}{e^{t}+1}\right)^{(j)}\left(e^{\frac{1}{2} t}\right)^{(k-j)}\right|_{t=0} . \tag{50}
\end{align*}
$$

Noting that

$$
\left.\left(\frac{2 e^{\frac{1}{2} t}}{e^{t}+1}\right)^{(j)}\right|_{t=0}=E_{j}\left(\frac{1}{2}\right)
$$

and $E_{j}\left(\frac{1}{2}\right)=2^{-j} E_{j}$, we have

$$
\begin{align*}
\alpha_{k} & =-\frac{1}{2} \sum_{j=0}^{k}\binom{k}{j} 2^{-j} E_{j}\left(\frac{1}{2}\right)^{k-j} \\
& =-\frac{1}{2^{k+1}} \sum_{j=0}^{k}\binom{k}{j} E_{j} \tag{51}
\end{align*}
$$

This completes the proof.
Remark. Equation (49) can be written as

$$
\begin{aligned}
& \frac{E_{n-1}(1)}{(n-1)!} y^{(n)}+\frac{E_{n-2}(1)}{(n-2)!} y^{(n-1)} \\
& +\cdots+\frac{E_{1}(1)}{1!} y^{\prime \prime}+\left(-2 x+E_{0}(1)\right) y^{\prime}+2 n y=0
\end{aligned}
$$

where $E_{k}(1)$ is the value of Euler polynomial $E_{k}(x)$ at $x=1$.
For the differential equation for Hermite polynomials $H_{n}^{(v)}(x)$ of variance $v$, we set $g(t)=e^{v t^{2} / 2}$ in Theorem 6.1.

Corollary 6.3. The Hermite polynomials $H_{n}^{(v)}(x)$ of variance v satisfy the following differential equation

$$
\begin{equation*}
v y^{\prime \prime}-x y^{\prime}+n y=0 \tag{52}
\end{equation*}
$$

Proof. Since the coefficient $\alpha_{k}$ in Eq. (39) is $\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}$ and $g(t)=e^{v t^{2} / 2}$, we have

$$
\begin{align*}
\alpha_{k} & =\left.\left(-\frac{v t e^{v t^{2} / 2}}{e^{v t^{2} / 2}}\right)^{(k)}\right|_{t=0}=-\left.(v t)^{(k)}\right|_{t=0} \\
& = \begin{cases}-v & \text { if } k=1, \\
0 & \text { otherwise } .\end{cases} \tag{53}
\end{align*}
$$

This completes the proof.
In the process of proving Theorem 6.1, we obtained the recurrence relations for Appell polynomial sequences.

Theorem 6.2. Let $\left\{s_{n}(x)\right\}$ be the Appell polynomial sequence for $g(t)$. Then, for $n \geq 1$, the Appell polynomials $s_{n}(x)$ have the following recurrence relation:

$$
\begin{equation*}
s_{n}(x)=x s_{n-1}(x)+\sum_{j=0}^{n-1}\binom{n-1}{j} \alpha_{n-j-1} s_{j}(x) \tag{54}
\end{equation*}
$$

where $\alpha_{k}=\left.\left(-\frac{g^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}$.
Proof. Reindexing and rearranging Eq. (42) yields the desired result.

Choosing $g(t)=\frac{e^{t}-1}{t}$ in Theorem 6.2, we regain the recurrence relation, which is the Theorem 2.3 in [4], for Bernoulli polynomial sequence $\left\{B_{n}(x)\right\}$ :

Corollary 6.4. For $n \geq 1$,

$$
\begin{equation*}
B_{n}(x)=x B_{n-1}(x)-\frac{1}{n} \sum_{j=0}^{n-1}\binom{n}{j} b_{n-j} B_{j}(x) \tag{55}
\end{equation*}
$$

where $b_{k}$ is the Bernoulli number.

Similarly, choosing $g(t)=\frac{e^{t}+1}{2}$ in Theorem 6.2, we obtain the recurrence relation for Euler polynomial sequence $\left\{E_{n}(x)\right\}$ :

Corollary 6.5. For $n \geq 1$,

$$
\begin{equation*}
E_{n}(x)=x E_{n-1}(x)+\sum_{j=0}^{n-1}\binom{n-1}{j} e_{n-1-j} E_{j}(x) \tag{56}
\end{equation*}
$$

where

$$
e_{k}=-\frac{1}{2^{k+1}} \sum_{l=0}^{k}\binom{k}{l} E_{l}
$$

for $k=0,1,2, \ldots, n-1$, and $E_{l}$ is the lth term of Euler number sequence.
Remark. The recurrence relation for Euler polynomial sequence $\left\{E_{n}(x)\right\}$ presented in Theorem 2.4 in [4] is incorrect because of the wrong $e_{k}$.

Finally, choosing $g(t)=e^{v t^{2} / 2}$ in Theorem 6.2, we obtain the recurrence relation for Hermite polynomials $H_{n}^{(v)}(x)$ of variance v:

Corollary 6.6. For $n \geq 1$,

$$
\begin{equation*}
H_{n}^{(v)}(x)=x H_{n-1}^{(v)}(x)-v(n-1) H_{n-2}^{(v)}(x) \tag{57}
\end{equation*}
$$

## 7. Conclusion

In this paper, we demonstrated that the linear algebra approach in studying the Appell polynomials is very useful in deriving new and existing properties of Appell polynomial sequences. This novel technique will shed some light upon the broader category of polynomial sequences, i.e., Sheffer sequences discussed in [7]. In future work, we shall pursue this direction.

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