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VARIETY OF PERFECT GROUPS AND ISOLOGIC GROUPS

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Abstract

Let ϑ be a variety of groups defined by the set of laws V. A group G is called V-perfect if $G = V(G)$, where $V(G)$ is the verbal subgroup of G. Let G be a finite group and G^* be a V-covering group of G and let H be of order G^* such that H is V-isologic to G^*. It is of interest to know whether H is a V-covering group of G. The answer is negative in general. In this paper, we give some conditions in the affirmative case.

1. Notation and Necessary Results

Let F_∞ be a free group freely generated by a countable set $\{x_1, x_2, ...\}$. Let ϑ be a variety of groups defined by a subset V of F_∞. It will be assumed that the reader is familiar with the notions of the verbal subgroup, $V(G)$, and the marginal subgroup, $V^*(G)$, associated with the variety of groups and a given group G. See also [7] for more information on the variety of groups.

Throughout the paper we always assume that ϑ is the variety of groups defined by the set of laws V.

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Let G be any group with a normal subgroup N. Then we define $[NV^*G]$ to be the subgroup of G generated by the following set:

$$\{v(g_1, ..., g_i n, ..., g_r)(v(g_1, ..., g_r))^{-1} | 1 \leq i \leq r; v \in V; g_1, ..., g_r \in G; n \in N\}.$$

It is easily checked that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that $N/T \subseteq V^*(G/T)$.

The following lemma gives basic properties of the verbal and the marginal subgroups of a group G with respect to the variety ϑ, which is useful in our investigation, see [2].

Lemma 1.1. *Let* ϑ *be a variety of groups, and N be a normal subgroup of a group G. Then the following statements are true*:

(i) $V(V^*(G)) = 1$, $V^*(G/V(G)) = G/V(G)$;

(ii) $V(G) = 1 \Leftrightarrow V^*(G) = G \Leftrightarrow G \in \vartheta$;

(iii) $[NV^*G] = 1 \Leftrightarrow N \subseteq V^*(G)$;

(iv) $V(G/N) = V(G)N/N$, $V^*(G)N/N \subseteq V^*(G/T)$;

(v) $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$, $V(G) = [GV^*G]$;

(vi) *If* $N \cap V(G) = 1$, *then* $N \subseteq V^*(G)$ *and* $V^*(G/N) = V^*(G)/N$.

Get ϑ be a variety of groups defined by the set of laws V and let G be a group with a free presentation

$$1 \to R \to F \to G \to 1,$$

where F is a free group. Then the Bear-invariant of G with respect to the variety ϑ, denoted by $VM(G)$, is defined to be $R \cap V(F)/[RV^*F]$.

We also denote the factor group $V(F)/[RV^*F]$ by $VP(G)$. Of course, if G is in ϑ, then $VM(G) = VP(G)$.

It is easily seen that the Bear-invariant of the group G with respect to the variety ϑ is always abelian and independent of the choice of the free presentation of G, see [5] or [6].

In particular, if ϑ is the variety of abelian group, then the Bear-invariant of the group G with respect to the variety ϑ will be $(R \cap F')/[R, F]$, which by Schur [8] is isomorphic to the multiplicator of G, and if ϑ is the variety of nilpotent groups of class at most c, N_c, say, then the Bear-invariant of G with respect to the variety N_c is (see [6]) $(R \cap F_{c+1})/[R,_c F]$ where F_{c+1} is the $(c+1)$-th term of the lower central series of F, and $[R,_c F] = [R, F, \ldots, F]$.

Equating $V(G)$ with $V(F)/R \cap V(F)$ gives rise to a natural central extension of G, $1 \to VM(G) \to VP(G) \to V(G) \to 1$.

Definition 1.2. Let ϑ be a variety of groups defined by the set of laws V and let G and H be two groups. Then (α, β) is said to be a *V-isologism between G and H*, if

$$\alpha : G/V^*(G) \to H/V^*(H), \quad \beta : V(G) \to V(H)$$

are isomorphisms such that for all $v(x_1, \ldots, x_r) \in V$ and all $g_1, \ldots, g_r \in G$, we have $\beta(v(g_1, \ldots, g_r)) = v(h_1, \ldots, h_r)$ whenever $h_i \in \alpha(g_i V^*(G))$; $1 \leq i \leq r$.

In this case, we write $G \sim H$, and say that G is V-isologic to H.

In particular, if ϑ is the variety of abelian groups we obtain the notion of isoclinism due to Hall [1].

The Frattini subgroup, $\Phi(G)$, of a group G is defined to be the intersection of all the maximal subgroups of G.

The following theorem in [2] gives the connection among the verbal, the marginal and the Frattini subgroups of a group G.

Theorem 1.3 [2]**.** *Let ϑ be a variety of groups. Then for a group G,*

$$V^*(G) \cap V(G) \subseteq \Phi(G).$$

A group G is said to be *V-perfect with respect to the variety ϑ*, if $G = V(G)$. The following result is needed later on.

Theorem 1.4. *Let ϑ be a variety of groups and G be a finite group. If H is a V-perfect subgroup of G such that $H \sim G$, then $G = V(G)V^*(G)$.*

Proof. Since $G/V^*(G)$ is finite and $H \sim G$, by [2, Lemma 4.4], $G = HV^*(G)$ hence $G = V(G)V^*(G)$.

We need the following theorem in [2].

Theorem 1.5. *Let ϑ be a variety of groups and G_1 and G_2 be two arbitrary groups. Then $G_1 \sim G_2$ if and only if there exists a group G containing normal subgroups N_1 and N_2, such that $G_1 \cong G/N_1$ and $G_2 \cong G/N_2$ and $G_1 \sim G \sim G_2$.*

Definition 1.6. A *V-stem cover* of G is an exact sequence $1 \to A \to G^* \to G \to 1$ such that (i) $A \subseteq V^*(G^*) \cap V(G^*)$, (ii) $A \cong VM(G)$. In this case, G^* is said to be a *V-covering group* of G.

The following lemma can be proved easily.

Lemma 1.7. *Let G be a V-perfect group. Then every V-covering group of G is also V-perfect.*

The following theorem is a vast generalization of [4, Theorem 3.19.2].

Theorem 1.8. *Let G be a V-perfect group. Then $VP(G)$ is a V-covering group of G.*

Proof. Let G be a V-perfect group with a free presentation $F/R \cong G$ which gives rise to the following natural exact sequence:

$$1 \to \frac{R}{[RV^*F]} \to \frac{F}{[RV^*F]} \xrightarrow{\pi} G \to 1$$

and

$$\pi\left(\frac{V(F)}{[RV^*F]}\right) = G.$$

So we may restrict π and obtain the following exact sequence:

$$1 \to \frac{R \cap V(F)}{[RV^*F]} \to \frac{V(F)}{[RV^*F]} \to G \to 1.$$

Now, to prove that $VP(G)$ is a V-covering group of G, we must show that $VM(G) \subseteq V(VP(G))$. But this is enough to show that $VP(G)$ is V-perfect.

By the assumption $G = V(G)$, which implies that $F = V(F)R$. Now, by [2, Theorem 2.4], we have $V(F) = V(V(F))[RV^*F]$, which completes the proof.

Lemma 1.7 and the above theorem give the following:

Corollary 1.9. *If G be a V-perfect group, then $VP(G)$ is also V-perfect.*

2. *V*-perfect and *V*-isologic Groups

Let ϑ be a variety of groups defined by the set of laws V. In this section, we deal with the following question:

Are there finite groups G with a V-covering G^* such that all V-isologic groups H, say, to G^* and of the same order are also V-covering of G? This is false in general, see [3]. However, we give a positive answer when G is a V-perfect group.

Theorem 2.1. *Let ϑ be a variety of groups and G be a finite group with a free presentation $1 \to R \to F \to G \to 1$. Then every V-covering group of G is a homomorphic image of $F/[RV^*F]$.*

Proof. Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be the free presentation of the finite group G, where F is a free group on the set X and $R = Ker\pi$. Given a covering group G^* of G, choose an exact sequence $1 \to A \to G^* \xrightarrow{\varphi} G \to 1$ with $A \subseteq V(G^*) \cap V^*(G^*)$ and $A \cong VM(G)$. For every $x \in X$ there exists l_x in G^* such that $\varphi(l_x) = \pi(x)$. So $G^* = \langle A, l_x;\ x \in X\rangle$ and hence $G^* = \langle l_x;\ x \in X\rangle$ by the virtue of Theorem 1.3. Consider the homomorphism $\psi : F \to G^*$ defined by $\psi(x) = l_x,\ x \in X$. Then ψ is surjective and $\pi = \varphi \circ \psi$. Clearly $1 = \pi(R) = \varphi(\psi(R))$, and we have $\psi(R) \subseteq A$.

So

$$\psi([RV^*F]) = [\psi(R)V^*\psi(F)] \subseteq [AV^*G^*] \subseteq [V^*(G^*)V^*G^*] = 1.$$

It follows that ψ induces an epimorphism

$$\psi_1 : \frac{F}{[RV^*F]} \to G^*.$$

Now, the above theorem gives the following important result.

Theorem 2.2. *All the V-covering groups of a finite group G are mutually V-isologic.*

Proof. Let G be a finite group with a free presentation

$$1 \to R \to F \to G \to 1.$$

Then $VM(G) = R \cap V(F)/[RV^*F]$ is the Bear-invariant of G, $F_1 = F/[RV^*F]$ and $R_1 = R/[RV^*F]$. Now, if G^* is any V-covering group of G, then by Theorem 2.1, there is an epimorphism $\varphi : F_1 \to G^*$, such that $Ker\varphi$ is a complement for $VM(G)$ in R_1. Hence

$$Ker\varphi \cap V(F_1) = Ker\varphi \cap V(F_1) \cap R_1 = Ker\varphi \cap VM(G) = 1,$$

which implies that $\beta = \varphi|_{V(F_1)} : V(F_1) \to V(G^*)$ and $\alpha : F_1/V^*(F_1) \to G^*/V^*(G^*)$ are both isomorphisms, and $\alpha(f_1V^*(F_1)) = \varphi(f_1)V^*(G^*)$. Hence $G^* \sim F_1$.

Now, having the above information in hand we shall have the following straightforward result.

Lemma 2.3. *Let G be a finite V-perfect group and H be any other finite group V-isologic to G. Then H is isomorphic to the direct product of G and $V^*(H)$ amalgamating $V(H) \cap V^*(H)$ and $V^*(G)$.*

Now, we are ready to give an affirmative answer to the question raised in the beginning of this section.

Theorem 2.4. *Let G be a finite V-perfect group and G^* be a V-covering group of G. If H is a V-isologic group to G^* and of the same order, then $H \cong G^*$.*

Proof. Since G is V-perfect, it follows that G^* is also V-perfect. By the above lemma, $H \cong \frac{G_1 \times A}{N}$, where $|N| = |V^*(G^*)|$ and $G_1 \cong G^*$.

But $|H| = |G_1|$ implies that $H \cong G_1 \cong G^*$.

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