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FINITENESS OF THE CLASS GROUP OF A NUMBER FIELD VIA LATTICE PACKINGS

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Abstract

An alternative proof of the finiteness of the class group of a number field of degree n is presented. It is based solely on the fact that the center density of an n-dimensional lattice packing is bounded away from infinity.

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1. Introduction

Let K be an algebraic number field of degree n with ring of algebraic integers $\mathfrak{O}_K$. The well-known proof of the theorem on the finiteness of the class group $\mathfrak{C}_K$ of K [2, pp. 155-156] involves Minkowski's criterion for a convex set to contain a point of a lattice [2, Theorem 2.50, p. 97] and the existence of an integral $\mathfrak{O}_K$-ideal whose norm does not exceed M_K, the Minkowski bound of K [2, Theorem 2.56, p. 100].

The purpose of this paper is to present an alternative proof for that classical theorem in number theory by means of elementary notions and results from sphere packings [1]. One consequence of the new proof is a lower bound on the center density of lattice packings associated with integral $\mathfrak{O}_K$-ideals. In this section, we review the necessary facts about number fields [2] and sphere packings [1], and establish the notation. In Section 2, we prove the main result.

Let $\sigma_1, \ldots, \sigma_n$ be the embeddings of K into $\mathbb{C}$, the field of complex numbers. As usual, $\sigma_1, \ldots, \sigma_r$ are real and $\sigma_{r+1}, \ldots, \sigma_n$ are complex embeddings, with σ_{j+s} being the complex conjugate of σ_j for $r+1 \leq j \leq r+s$. Hence, $n = r + 2s$. The canonical embedding $\sigma_K : K \to \mathbb{R}^n$ is the injective ring homomorphism defined by

$$\sigma_K(x) = (\sigma_1(x), \ldots, \sigma_r(x), \Re\sigma_{r+1}(x), \Im\sigma_{r+1}(x), \ldots, \Re\sigma_{r+s}(x), \Im\sigma_{r+s}(x))$$

for all $x \in K$ and where $\Re z$ and $\Im z$ are the real and imaginary parts of the complex number z, respectively. It follows that

$$|\sigma_K(x)|^2 = \sigma_1(x)^2 + \cdots + \sigma_r(x)^2 + |\sigma_{r+1}(x)|^2 + \cdots + |\sigma_{r+s}(x)|^2.$$

Since $\sigma_{r+s+j} = \overline{\sigma_{r+j}}$ for $j = 1, \ldots, s$, we have

$$2|\sigma_K(x)|^2 = 2(\sigma_1(x)^2 + \cdots + \sigma_r(x)^2) + |\sigma_{r+1}(x)|^2 + \cdots + |\sigma_{r+2s}(x)|^2.$$

Thus

$$2|\sigma_K(x)|^2 \geq \sum_{i=1}^{n} |\sigma_i(x)|^2 \geq n \sqrt[n]{\left|\prod_{i=1}^{n} \sigma_i(x)\right|^2} = n\sqrt[n]{|N_K(x)|^2}, \tag{1}$$

where the second inequality follows from the Arithmetic-Geometric Mean Inequality and $N_K(x)$ denotes the relative norm of x in $K/\mathbb{Q}$. From (1), it follows that

$$|\sigma_K(x)| \geq \frac{\sqrt{2n}}{2} \sqrt[n]{|N_K(x)|}. \tag{2}$$

Let $\mathfrak{a}$ be an integral $\mathfrak{O}_K$-ideal of norm $N(\mathfrak{a}) = |\mathfrak{O}_K/\mathfrak{a}|$ and let $\mathfrak{a}^* = \mathfrak{a}\setminus\{0\}$. The set $\Lambda(\mathfrak{a}) = \{\sigma_K(x) | x \in \mathfrak{a}\}$ is an n-dimensional lattice in Euclidean space $\mathbb{R}^n$, see [1, p. 225]. We shall refer to $\Lambda(\mathfrak{a})$ as the lattice associated to $\mathfrak{a}$. Denote the discriminant of K by Δ_K. The center density [1, p. 13] of $\Lambda(\mathfrak{a})$ is $\delta(\Lambda(\mathfrak{a})) = \rho^n/v(\Lambda(\mathfrak{a}))$, where

$$\rho = \frac{1}{2}\min\{\| \sigma_K(x) \| \,|\, x \in \mathfrak{a}^*\}$$

is, by definition, the packing radius of $\Lambda(\mathfrak{a})$ [1, p. 10] and

$$v(\sigma_K(\mathfrak{a})) = 2^{-s}\sqrt{|\Delta_K|}\,N(\mathfrak{a})$$

is the volume of $\Lambda(\mathfrak{a})$, see [1, Theorem 9, p. 226]. In passing, we observe that the center density of any n-dimensional lattice cannot be greater than $1/V_n$, where V_n [1, p. 9] denotes the volume of a sphere of radius 1 in $\mathbb{R}^n$. Thus

$$\delta(\Lambda(\mathfrak{a})) = \frac{2^s\rho^n}{\sqrt{|\Delta_K|}N(\mathfrak{a})} = \frac{2^s\left(\frac{1}{2}\min\limits_{x\in\mathfrak{a}^*}\| \sigma_K(x) \|\right)^n}{\sqrt{|\Delta_K|}\,N(\mathfrak{a})},$$

that is,

$$\delta(\Lambda(\mathfrak{a})) = \frac{1}{2^{r+s}\cdot\sqrt{|\Delta_K|}} \cdot \frac{\min\limits_{x\in\mathfrak{a}^*}\| \sigma_K(x) \|^n}{N(\mathfrak{a})}. \tag{3}$$

From (2) and (3), we obtain

$$\delta(\Lambda(\mathfrak{a})) \geq \frac{1}{2^{r+s}\cdot\sqrt{|\Delta_K|}} \cdot \frac{\left(\frac{\sqrt{2n}}{2}\cdot\min\limits_{x\in\mathfrak{a}^*}\sqrt[n]{|N_K(x)|}\right)^n}{N(\mathfrak{a})},$$

whence

$$\delta(\Lambda(\mathfrak{a})) \geq \frac{n^{n/2}}{2^{\frac{3}{2}r+2s} \cdot \sqrt{|\Delta_K|}} \cdot \frac{\min_{x\in\mathfrak{a}^*} |N_K(x)|}{N(\mathfrak{a})}. \tag{4}$$

Let $c_{\mathfrak{a}} \in \mathfrak{C}_K$ be the ideal class containing $\mathfrak{a}$ and $\widetilde{c_{\mathfrak{a}}}$ be the set of integral $\mathfrak{O}_K$-ideals contained in $c_{\mathfrak{a}}$. For $x \in \mathfrak{a}$, we can write $\langle x \rangle = \mathfrak{a} \cdot \mathfrak{b}_x$, where $\mathfrak{b}_x$ is an integral $\mathfrak{O}_K$-ideal. Thus, $|N_K(x)| = N(\langle x \rangle) = N(\mathfrak{a}) \cdot N(\mathfrak{b}_x)$. Moreover, the mapping $\varphi : \mathfrak{a} \to \widetilde{c_{\mathfrak{a}}^{-1}}$ given by $\varphi(x) = \langle x \rangle \cdot \mathfrak{a}^{-1}$ is bijective. It follows that

$$\frac{\min_{x\in\mathfrak{a}^*} |N_K(x)|}{N(\mathfrak{a})} = \min_{\mathfrak{b}\in\widetilde{c_{\mathfrak{a}}^{-1}}} \frac{N(\mathfrak{a}) \cdot N(\mathfrak{b})}{N(\mathfrak{a})} = \min_{\mathfrak{b}\in\widetilde{c_{\mathfrak{a}}^{-1}}} N(\mathfrak{b}). \tag{5}$$

Together, (4) and (5) imply that

$$\delta(\Lambda(\mathfrak{a})) \geq \frac{n^{n/2}}{2^{\frac{3}{2}r+2s} \cdot \sqrt{|\Delta_K|}} \cdot \min_{\mathfrak{b}\in\widetilde{c_{\mathfrak{a}}^{-1}}} N(\mathfrak{b}). \tag{6}$$

2. Main Result

With the notation from Section 1, we are now ready to prove the following.

Theorem 1 [2, Theorem 3.60, p. 155]**.** *The ideal class group $\mathfrak{C}_K$ is finite.*

Proof. By way of contradiction, suppose that $\mathfrak{C}_K$ has infinitely many ideal classes. Let i be a positive integer and define

$$\mathscr{C}_i = \{c \in \mathfrak{C}_K \mid \min_{\mathfrak{b}\in\widetilde{c}} N(\mathfrak{b}) = i\}.$$

From the elementary result that only finitely many integral $\mathfrak{O}_K$-ideals have norm equal to i, it follows that $\mathscr{C}_i$ is a finite set. Moreover, we can write $\mathfrak{C}_K = \bigcup_{i=1}^{\infty} \mathscr{C}_i$, where the $\mathscr{C}_i$ are disjoint members of $\mathfrak{C}_K$. By hypothesis, $\mathfrak{C}_K$ has infinitely many elements. Then for all $M \in \mathbb{N}$, there exists $i > M$ such that $\mathscr{C}_i \neq \varnothing$, that is, there exists $c \in \mathfrak{C}_K$ such that $\min\limits_{\mathfrak{b} \in \widetilde{c}} N(\mathfrak{b}) > M$. Choose $\mathfrak{c} \in \widetilde{c^{-1}}$. From (6), we have

$$\delta(\Lambda(\mathfrak{c})) \geq \frac{n^{n/2}}{2^{\frac{3n}{2}-s} \cdot \sqrt{|\Delta_K|}} \cdot M.$$

Hence, the center density of lattices associated to integral $\mathfrak{O}_K$ -ideals can be made arbitrarily large, a contradiction. Therefore, $\mathfrak{C}_K$ has finitely many elements.

References

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[2] R. A. Mollin, Algebraic Number Theory, Chapman and Hall/CRC, New York, 1999.