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CONVEXITY OF MORSE STRATIFICATIONS AND GRADIENT SPINES OF 3-MANIFOLDS

(In memory of Jerry Levine, friend and mentor)

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Abstract

We notice that a generic nonsingular gradient field $v = \nabla f$ on a compact 3-fold X with boundary canonically generates a simple spine $K(f, v)$ of X. We study the transformations of $K(f, v)$ that are induced by deformations of the data (f, v). We link the Matveev complexity $c(X)$ of X with counting the *double-tangent* trajectories of the v-flow, i.e., the trajectories that are tangent to the boundary ∂X at a pair of distinct points. Let $gc(X)$ be the minimum number of such trajectories, minimum being taken over all nonsingular v's. We call $gc(X)$ the *gradient complexity* of X. Next, we prove that there are only finitely many X of bounded gradient complexity, provided that X is irreducible and has no essential annuli. In particular, there exists only finitely many hyperbolic manifolds X with bounded $gc(X)$. For such X, their normalized hyperbolic volume gives a lower bound of $gc(X)$. If an orientable X with $\partial X = \coprod S^2$ admits a non-singular gradient flow with one double-tangent trajectory at most, then X is a connected sum of

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several 3-balls and products $S^2 \times S^1$. All these and many other results of the paper rely on a careful study of the stratified geometry of ∂X relative to the v-flow. It is characterized by failure of ∂X to be *convex* with respect to a generic flow v. It turns out that convexity or its lack have profound influence on the topology of X. This interplay between intrinsic concavity of ∂X with respect to any gradient-like flow and the complexity $gc(X)$ is in the focus of the paper.

1. Introduction

Classical Morse theory links singularities of Morse functions with the topology of closed manifolds. Specifically, singularities of Morse functions $f : X \to \mathbb{R}$ cause interruptions of the f-gradient flow, and the homology or even the topological type of a manifold X can be expressed in terms of such interruptions (see [4]). These terms include descending and ascending disks, attaching maps, and spaces of flow trajectories which connect the singularities.

On manifolds with boundary, an additional source of the flow interruption occurs: it comes from a special geometry of the boundary ∂X, or rather from the failure of ∂X to be *convex* with respect to the flow (see Definition 4.1). In fact, on manifolds with boundary, one can *trade* the f-singularities in the interior of X for these boundary effects. In our approach, the boundary effects take the central stage, while the singularities themselves remain in the background. In the paper, we apply this philosophy to 3-manifolds. Many of our results allow for straightforward multidimensional generalizations, the other are specifically three-dimensional.

Some of our theorems are in the spirit of the pioneering work of Ishii on, so-called, *flow-spines* [18, 19] (see also a recent paper by Koda [23] and an excellent monograph "Branched Standard Spines of 3-manifolds" by Benedetti and Petronio [2], followed by [3]). In an earlier version of this paper [21], we managed to overlook all this line of research ...

As we will introduce the relevant constructions, we will describe some technical differences between the flow-spines of [18], the branched spines of [9], [2], on one hand, and the gradient spines on the other. For now, it is sufficient to say that the branched spines break the symmetry of generic spines, and the gradient spines break it further (in a way similar to the symmetry breaking in the category of oriented branched spines). We stress that any generic gradient flow defines its gradient spine in a *canonical* way (in contrast with [18], we use the boundary itself to construct a surrogate of Ishii's "normal pair") which makes the connection between the space of flow trajectories and the spine into an instant one. Also, we deal with *generic* gradient-like non-singular fields v; they are not necessarily *concave* with respect to the boundary ∂X as the fields in [2], [3]. For a concave traversing fields, the affiliated spine K (see [2, 3]) is not uniquely determined by the flow, but, after choosing an embedding $K \subset X$, the collapse $p : X \to K$ is; in contrast, the gradient spines are uniquely determined by the flow, but the collapse p is not.

However, the most important difference between this paper and the above results on spines in 3-folds does not reside in the special nature of gradient spines (distinctions that are intrinsic to the theory and, perhaps, of little importance to non-experts), but rather in the ultimate goals of our program. We do not seek here to develop a version of combinatorial calculus (say, as in [2], [3], or [23]) for the gradient spines (although this would be a useful project, if feasible), but rather to use the existing combinatorial machinery to describe the behavior of gradient or traversing flows on a given 3-fold.

In 3-dimensional topology, there is no lack of combinatorics-inspired invariants... The geometrically meaningful and transparent invariants are in a somewhat short supply. We suggest that counting some special (so-called, double-tangent) trajectories of flows which interact in a particular way with the boundary of 3-folds is an example of such geometrically interesting invariant. There is some similarity between such count on manifolds with boundary and the classical count of closed isolated trajectories of generic flows on closed manifolds.

Partially, our motivation comes from the desire to understand better the interplay between the intrinsic concavity of ∂X with respect to generic gradient flows and the topology of the underlining 3-fold X. We conjecture that there exists a numerical topological invariant that measures the failure of convexity with respect to any nonsingular gradient flow—"*some manifolds intrinsically are just more concave than others...*". In a sense, the gradient complexity $gc(X)$, introduced in this paper, can serve as a crude measure of intrinsic concavity of X. In fact, a 3-manifold X with a connected boundary which admits a convex gradient-like field v is a handlebody; so a random manifold does not admit convex nonsingular gradient flows. For instance, $H_2(X;\, \mathbb{Z}) \neq 0$ constitutes an obstruction to the convexity for any nonsingular gradient flow. At the same time, any manifold with boundary admits a *strictly concave* traversing (but not necessarily gradient!) flow [2].

The combinatorial complexity theory of Matveev [25] helps us to uncover the behavior of generic nonsingular gradient flows on 3-folds in connection to their boundaries. Before describing these results in the full generality, let us give to the reader their taste. For example, we prove that on a manifold X, obtained from the Poincaré homology sphere by removing an open disk, any nonsingular gradient flow has at least *five* double-tangent trajectories; moreover, X admits a gradient flow with not more than $6 \cdot 5 = 30$ such trajectories. Another example is provided by the remarkable hyperbolic manifold M_1 that has the minimal (among hyperbolic manifolds) volume $V \approx 0.94272$. By removing an open disk from M_1 we get a manifold X on which any nonsingular gradient flow has at least *nine* trajectories, each one tangent to the sphere ∂X at a pair of distinct points; moreover, X admits a gradient flow with not more than $6 \cdot 9 = 54$ double-tangent trajectories.

A generic vector field v on X gives rise to a natural stratification

$$X \supset \partial_1^+ X \supset \partial_2^+ X \supset \partial_3^+ X \tag{1.1}$$

by compact submanifolds, where $\dim(\partial_j^+ X) = 3 - j$. Here $\partial_1^+ X$ is the part of the boundary $\partial_1 X := \partial X$, where v points inside X. $\partial_2 X$ is a 1-dimensional locus, where v is *tangent* to the boundary ∂X. Its portion $\partial_2^+ X \subset \partial X$ consists of points, where v points inside $\partial_1^+ X$. Similarly, $\partial_3 X$ is a finite locus, where v is tangent to $\partial_2 X$. Finally, $\partial_3^+ X \subset \partial_3 X$ consists of points, where v points inside $\partial_2^+ X$.

In his groundbreaking 1929 paper [28], Morse discovered a beautiful connection between this stratification and the index of the field v.[1]

Now, let us describe the content of our paper section by section.

Section 2. This section starts with a sketch of main results from [28] (see Theorem 2.1 and Corollary 2.1). It also contains one remark about the role that stratification (1.1) plays in the Gauss-Bonnet Theorem (see Theorem 2.2 and [13] for an interesting general discussion). More importantly, we notice that, at any point $x \in \partial X$, the v-flow defines a projection of the boundary ∂X into a germ of the constant level surface $f^{-1}(f(x))$. At a generic point $x \in \partial_2 X$ this projection is a *fold*, while at $x \in \partial_3 X$ it is a *cusp*. Throughout the paper, these folds and cusps provide us with crucial measuring devices for probing the topology of X. A significant portion of the paper is preoccupied with role of the cusps and attempts to eliminate them.

Section 3. As in [28], the stratification $\{\partial_j^+ X\}_j$ is in the focus of our investigation. Here we prove that the surface $\partial_1^+ X$ can be subjected to 1-surgery via a deformation of the gradient-like field v. This allows one to change the topology of the stratum $\partial_1^+ X$ almost at will (see Lemma 3.1 and Corollary 3.1).

Section 4. For given nonsingular Morse data (f, v), we introduce the notion of *s-convexity*, $s = 2$ or 3. The 2-convexity of v is defined as the property $\partial_2^+ X = \varnothing$. It puts a severe restrictions on the topology of X (see Theorem 4.2 and Corollary 4.5). In contract, the 3-convexity, $\partial_3^+ X = \varnothing$, by itself has no topological significance: one can always deform (f, v) to eliminate $\partial_3^+ X$ together with all other cusps (Theorem 9.5).[2] However, when we fix the topology of $\partial_1^+ X$, some combinations of cusps from $\partial_3 X$ acquire topological invariance (Corollary 9.2).

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[1]Actually, the results of [28] apply to compact manifolds X of any dimension.

[2]Note that Theorem 4.1.9 in [2] implies $\partial_3^+ X = \varnothing$ for all, so-called, *traversing flows*.

Although convexity or its lack are defined in terms of the gradient-like fields, we can arrange for the 2-convexity if we know that singularities of $f|_{\partial X}$ admit a particular *ordering* induced by f (similar to the self-indexing property). Specifically, the singularities of $f|_{\partial X}$ can be divided into two groups: the positive Σ_1^+ where the gradient $v = \nabla f$ is directed inwards X, and the negative Σ_1^- where v is directed outwards (see Figure 1). Theorem 4.1 claims that when $f(\Sigma_1^-)$ is *above* $f(\Sigma_1^+)$, then one can deform the riemannian metric on X so that the convexity of the gradient flow will be guaranteed. Hence, on 3-folds X that are not handlebodies, it is impossible to find a nonsingular function f with the property $f(\Sigma_1^-) > f(\Sigma_1^+)$. In addition, Theorem 4.2 describes an interplay between the dynamics of the flow v through the "bulk" X and of the v-induced flow v_1 in $\partial_1 X$, on the one hand, and the convexity phenomenon, on the other.

In Corollary 4.5, we prove that an acyclic X is a 3-disk if and only if one of the two properties are satisfied: (1) X admits nonsingular 2-convex Morse data (f, v), (2) X admits nonsingular 3-convex Morse data (f, v) with a connected $\partial_1^+ X$.

Section 5 is devoted to properties of *gradient spines*, a construction central to our investigations. In spirit, but not technically, it represents a special class of flow spines [18]. The difference between the two classes reflects the difference between the nonsingular vector and gradient fields on a given manifold.

Recall that a *spine* $K \subset X$ is a compact cellular two-dimensional subcomplex K of the 3-fold X, such that $X \setminus K$ is homeomorphic to the product

$$[\partial X \setminus (\partial X \cap K)] \times [0, 1),$$

so that X is collapsible onto K.

The relation between general spines and ambient 3-folds is subtle: a manifold X has many non-homeomorphic spines K, and there are topologically distinct X that share the same spine. In order to make the reconstruction of X from K possible, K has to be rather special (cf. [25], [2]).

In fact, a generic nonsingular gradient-like field v canonically gives rise to a spine that we call *gradient* (see Figures 10, 12). A gradient spine K is a union of $\partial_1^+ X$ with the descending v-trajectories through $\partial_2^+ X$. Like branched spines (Definition 6.5), gradient spines inherit *orientations* from the boundary ∂X and have a *preferred side* in the ambient X. Exactly these properties of a gradient spine K allow for its resolution into a surface S homeomorphic to $\partial_1^+ X$ and, eventually, for a reconstruction of X from K (Theorem 6.1). By modifying the field v, we can arrange for $\partial_1^+ X$, and thus for S, to be homeomorphic to a disk D^2. As a result, we get our Origami Theorem 5.2: any 3-manifold X with a connected boundary has a gradient spine K obtained from a disk D^2 by identifying certain arcs in ∂D^2 with the appropriate arcs in the interior of D^2 (see Figures 14, 15). This statement is similar in flavor to Theorem 1.2 from [18], where disk-shaped "sections" of generic (non-gradient) flows on closed manifolds are employed for the same goal.

Section 6. This section deals with combinatorial structures that generalize the notion of gradient spine K (see Definitions 6.1-6.4). We start with a 2-complex K whose local geometry is modeled after gradient spines (see Figure 20). Adding a system of, so-called, *TN-markers* to K along its singularity set $s_\dagger(K)$ produces an object which captures the topology of the ambient X and admits a canonic resolution into an oriented surface. Such a polyhedron K with markers is called an *abstract gradient spine*. Unlike generic 2-complexes, each abstract gradient spine K is a spine of some manifold (cf. [3], [25]) and determines it (Theorem 6.1).[3]

The notion of a $\vec{Y}$-*spine* (see Definition 6.6) is still another generalization of gradient spines. It is a very close relative of branched

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[3]In a sense, the category of abstract gradient spines is equivalent to the category of compact 3-manifolds with non-empty boundary.

spines. In fact, for an oriented X, the notions of an oriented branched spine $K \subset X$ and of $\vec{Y}$-spine are equivalent, (Lemma 6.3). Unlike abstract gradient or branched spines, the $\vec{Y}$-spines K are defined *extrinsically*, that is, in terms of an embedding in X of the vicinity of the singular set $s(K) \subset K$. By Lemma 6.2, any gradient spine is a $\vec{Y}$-spine. Moreover, by pivotable Theorem 8.1, $\vec{Y}$-spines admit a "nice" approximation by the gradient spines of the same complexity.

Section 7. We apply ideas and results of [25], which revolve around Matveev's notion of combinatorial complexity of simple 2-complexes and compact 3-folds, to the gradient and $\vec{Y}$-spines. We introduce the *gradient complexity* $gc(X)$ of a 3-fold X with boundary as the minimal number of *double-tangent* trajectories that a nonsingular gradient-like field on X can have. In general, $gc(X) \geq c(X)$, where $c(X)$, the Matveev combinatorial complexity, is defined to be the minimal number of special isolated singularities[4] that a simple spine $K \subset X$ can have. One can restrict the scope of this definition only to $\vec{Y}$-spines (equivalently, to oriented branched spines) in order to get the notion of $\vec{Y}$-complexity $c_{\vec{Y}}(X)$. We prove that $gc(X) \geq c_{\vec{Y}}(X) \geq c(X)$. In fact, Theorem 8.1 claims that $gc(X) = c_{\vec{Y}}(X)$.

The inequality $gc(X) \geq c(X)$ helps us to restate many results from [25] in the language of double-tangent trajectories. For instance, by Theorem 7.3, for any natural c, there is no more than finitely many boundary irreducible with no essential annuli 3-folds X that admit nonsingular gradient-like flows with c double-tangent trajectories. The number $N(c)$ of such 3-folds has a crude upper bound $\Gamma_4(c) \cdot 12^c$, where $\Gamma_4(c)$ stands for the number of topological types of regular four-valent graphs with c vertices at most. In particular, there is no more than $\Gamma_4(c) \cdot 12^c$ hyperbolic manifolds with c double-tangent trajectories.

[4]called, butterflies in [25] and Q-singularities in this paper.

Let X be obtained from a closed hyperbolic 3-fold Y by removing a number of open 3-balls. By Theorem 7.5, any non-singular gradient-like flow (as well as any convex traversing flow) on X has at least $V(Y)/V_0$ double-tangent trajectories. Here $V(Y)$ stands for the hyperbolic volume of Y and V_0 for the volume of the perfect ideal tetrahedron.

Fortunately, all orientable irreducible and closed 3-folds of combinatorial complexity at most six (there are 74 members in this family) have been classified and their special minimal spines have been listed [25][5]. Some partial results are available for the 1155 closed irreducible manifolds of complexity at most nine. This has been accomplished by an algorithmic computation coupled with "hands on" analysis of spines that look different, but share the same values of the Turaev-Viro invariants [34]. The bottom line is that all X with $c(X) \leq 6$ are distinguished by their Turaev-Viro invariants! Thus, for each manifold Y on the Matveev list and any generic nonsingular gradient flow on $X = Y \setminus D^3$, we get a lower bound on the number of double-tangent trajectories.

In Theorem 7.4, we notice that $gc(X)$ is semi-additive under the connected sum operation and can increase only as a result of 2-surgery on X.

Section 7 contains a few more results about upper and lower estimates of $gc(X)$ for manifolds obtained from closed manifolds Y by removing a number of 3-balls. Theorem 7.6 provides a lower bound for $gc(X)$ in terms of the presentational complexity of the fundamental group $\pi_1(X)$. At the same time, any self-indexing Morse function h on Y gives rise to an upper estimate of $gc(X)$ (given in terms of the attaching maps for the unstable 2-disks of index two h-critical points).

Section 8. Here, we are addressing a natural question: Which spines are of the gradient type? The main result of the section, Theorem 8.1,

[5]By definition, a spine of a closed manifold Y is a spine of the punctured Y, that is, of $Y \setminus D^3$.

claims that any $\vec{Y}$-spine $K \subset X$ can be approximated by a gradient spine $K(v)$; moreover, $c(K(v)) \leq c(K)$. Furthermore, one can get K from $K(v)$ by controlled elementary collapses of certain 2-cells. Theorem 8.1 depends on some results from Section 9 about possible cancellations of cusps from $\partial_3 X$. Theorem 8.2 establishes the equality $c_{\vec{Y}}(X) = gc(X)$ and, when $c(X) > 1$, the crucial inequality

$$c(X) \leq gc(X) \leq 6 \cdot c(X).$$

We proceed to classify oriented 3-folds X with $gc(X) \leq 1$. We have a complete description of such 3-folds with $\partial_1 X = \coprod S^2$. We prove that when $\partial_1 X$ is simply-connected and $gc(X) \leq 1$, then X is a connected sum of several 3-balls and $S^2 \times S^1$'s, a 3-fold of gradient complexity zero (Theorem 8.5). Therefore, assuming $\partial_1 X = \coprod S^2$, there is no X with $gc(X) = 1$. In contrast, the combinatorial complexity of punctured lens spaces $L^\circ_{4,1}$, $L^\circ_{5,2}$ is one. We also get some partial results about general oriented 3-folds X with $gc(X) \leq 1$. They form a rich family with complex, but constrained fundamental groups (Theorem 8.6).

These results testify that the theory of gradient spines is quite different from the theory of generic spines.

Section 9 contains many of our main results. Here we analyze the effect of deforming Morse data (f, v) on the gradient spine they generate. Theorem 9.3 claims that, in the process, the gradient spine goes through a number of elementary expansions and collapses of two-cells mingled with so-called α- and β-*moves* (see Figures 31, 32). These are analogs of the second and third Reidemeister moves for link diagrams. Theorem 9.1 describes possible cancelations of cusps from $\partial_3 X$ that accompany generic deformations of v. One of our main results, Theorem 9.4, is a combination of Theorem 9.3 with a special case of Phillips' Theorem [31]. We prove that when two nonsingular functions f_0 and f_1 on X produce the same invariants $h(f_0), h(f_1) \in H^2(X; \mathbb{Z})$–the same $Spin^c$-structures in the sense of Turaev [35]—, then their gradient spines are linked by a sequence of elementary 2-expansions, 2-collapses, and α- and β-moves (See the proof of Corollary 9.1 for the definition of the invariant $h(f) \in H^2(X; \mathbb{Z})$).

Deformations of (f, v) that cause jumps in the value of $h(f)$ (in the v-induced $Spin^c$-structures) manifest themselves as a "disk-supported surgery on the preferred spine orientation". We call them *mushroom flips* (see Figure 37).

Theorem 9.5 claims that, for any generic Morse data (f, v), there is a deformation of (f, v) so that, in the end, all the cusps from $\partial_3 X$ are eliminated, but the number of double-tangent trajectories $gc(f, v)$ is preserved. This is an important ingredient in the proof of our approximation Theorem 8.1.

Finally, it should be said that the Morse theory on stratified spaces, in general, and on manifolds with boundary, in particular, has been an area of an active advanced and interesting research. For a variety of perspectives on this topic see [28], [10], [11], [12], [4], [33]. Our intension is to bring the stratified Morse theory and the complexity theory of 3-folds under a single roof.

2. The Morse Stratification on Manifolds with Boundary

Let X be a compact 3-fold with boundary ∂X. Let $f : X \to \mathbb{R}$ be a generic smooth function. Then f has non-degenerate critical points in the interior of X and the restriction of f to the boundary ∂X is also a Morse function. Let v be a gradient-like vector field for f, that is, $df(v) > 0$ away from the f-critical points. Instead of working with such pairs (f, v), we can pick a Riemannian metric on X and choose $v = \nabla f$, the gradient field. Both points of view are equivalent, but we prefer the first.

The singularities of $f|_{\partial X}$ come in two flavors: *positive* and *negative*. At a positive singularity, the field v is directed inward X, and at a negative singularity, — outward. This distinction between positive and negative critical points of $f|_{\partial X}$ depends on f, not on v. At a positive singularity and in an appropriate coordinate system $\{x_1, x_2, x_3\}$ with $\{x_1 = 0\}$ defining ∂X and $x_1 > 0$ — the interior of X,

$$f(x) = c + x_1 + a_2 x_2^2 + a_3 x_3^2,$$

where c and $a_i \neq 0$ being constants. At a negative singularity, one has

$$f(x) = c - x_1 + a_2 x_2^2 + a_3 x_3^2.$$

Let $\Sigma_1^{\pm}$ be the set of positive/negative singularities of $f|_{\partial X}$ and let Σ_0—the set of f-singularities in the interior of X. Denote by $X_{\leq c}$ the set $\{x \in X \,|\, f(x) \leq c\}$.

Crossing the critical value $c_\star$ of a positive singularity causes the topological type of $X_{\leq c}$ to change, while crossing $c_\star$ of a negative singularity has no effect on the topology of $X_{\leq c}$ as illustrated in Figure 1.

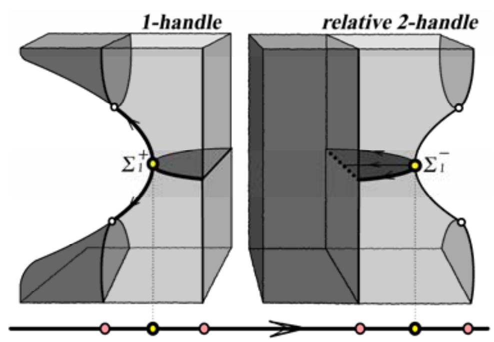

Figure 1. A positive singularity of index 1 and a negative singularity of index 2 on the boundary of a solid. The gradient-like field v is horizontal.

For a generic field v, the locus L where the field is tangent to ∂X is a 1-dimensional submanifold of the boundary; L divides ∂X into two domains: $\partial^+ X$ where v is directed inwards of X, and in $\partial^- X$, where it is directed outwards.

Morse noticed that, for a generic vector field v, the tangent locus L inherits a structure in relation to $\partial^+ X$ analogous to that of ∂X in relation to X [28]. To explain this point we need to revise our notations in a way which will be amenable to recursive definitions.

Let $\partial_0 X := X$, and $\partial_1 X := \partial X$. Denote by $\partial_2 X \subset \partial_1 X$ the locus, where v is tangent to $\partial_1 X$. For a generic v, $\partial_2 X$ divides $\partial_1 X$ into a domain $\partial_1^+ X$, where v is directed inwards X and a domain $\partial_1^- X$, where v is outwards X. Evidently, $\partial_1^\pm X \supset \Sigma_1^\pm$. Consider the set $\partial_2 X$, where v is tangent to $\partial_2 X$. The set $\partial_3 X$ divides $\partial_2 X$ into a set $\partial_2^+ X$, where v is directed inwards $\partial_1^+ X$ and a set $\partial_2^- X$, where v is directed outwards $\partial_1^+ X$. Finally, $\partial_3 X = \partial_3^+ X \coprod \partial_3^- X$, where v is directed inwards $\partial_2^+ X$ at the points of $\partial_3^+ X$.

From now and on, we call (f, v) *generic* if (1) all the strata $\{\partial_j X\}_{1 \le j \le 3}$ are regularly embedded smooth manifolds and (2) all the restrictions $f|_{\partial_j X}$ are Morse functions. Most of the time, the second property will be irrelevant, but when we need it, we do not want to modify our definition. At some point, the word "generic" will mean an additional general position requirement imposed on the field v (see Definition 5.2). When we say that a Riemannian metric is generic, we imply that $(f, \nabla f)$ is generic.

We introduce critical sets $\Sigma_j^\pm \subset \partial_j^\pm X$ of $f|_{\partial_j X}$ in a way similar to the one we used to define $\Sigma_1^\pm$. With a generic metric in place, let v_j be the orthogonal projection of v onto $\partial_j X$, and let n_j denote the field normal to $\partial_j X$ in $\partial_{j-1} X$ and pointing inside of $\partial_{j-1}^+ X$. Note that, away from the singularities from Σ_j, $df(v_j) > 0$.

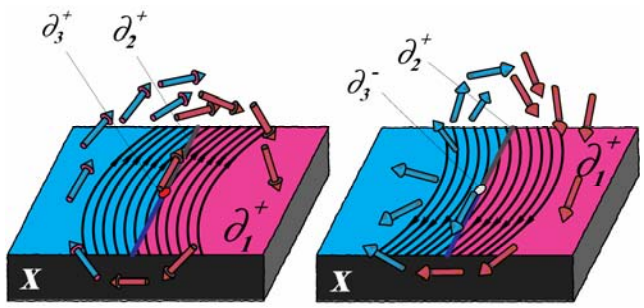

Figure 2. The patterns of fields v (the 3D-arrows) and v_1 (the parabolic flow) in vicinity of a point from $\partial_3^+ X$ (on the left) and a point from $\partial_3^- X$ (on the right).

For a vector field v_k as above on X_k with isolated singularities $\{x_\star \in \Sigma_k \subset Int(X_k)\}$, denote by $Ind_{x_\star}(v_k)$ its index at $x_\star$, and by $Ind^+(v_k)$ – the sum $\sum_{x_\star \in \Sigma_k^+} Ind_{x_\star}(v_k)$. Then, according to [28], one has two sets of equivalent relations:

Theorem 2.1 (Morse Law of Vector Fields)**.** *For any generic metric and* $0 \le k \le 3$,

- $\chi(\partial_k^+ X) = Ind^+(v_k) + Ind^+(v_{k+1})$ [6]
- $Ind^+(v_k) = \sum_{j=k}^{3} (-1)^j \chi(\partial_j^+ X)$.

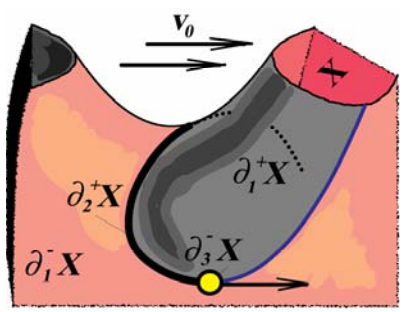

Figure 3. A more realistic picture of the boundary $\partial_1 X$ in vicinity of $\partial_3^- X$ in relation to the horizontal gradient field v.

[6]By definition, $Ind^+(v_3) = \#(\Sigma_3^+)$, and $Ind^+(v_4) = 0$.

Corollary 2.1. *For generic vector field v on X,*

$$Ind(v) = \sum_{k=0}^{3} (-1)^k \chi(\partial_k^+ X).$$

For an engaging discussion of the Morse Theorem 2.1 see the paper of Gottlieb [13].[7] In particular, it describes a link between the Morse stratification $\{\partial_j^+ X\}_j$ and the geometry (normal curvature K) of $\partial_1 X$:

Theorem 2.2. *Let $\Phi : X \to \mathbb{R}^3$ be a smooth map with a nonzero Jacobian on the boundary ∂X and $p : \mathbb{R}^3 \to \mathbb{R}$ a generic linear function, so that the function $f : p \circ \Phi$ has only isolated singularities in $Ind(X)$. Then the degree of the Gauss map $g : \partial X \to S^2$ can be calculated either by integrating the normal curvature K of $\Phi(\partial X) \subset \mathbb{R}^3$ (Gauss-Bonnet Theorem), or in terms of the v-induced stratification $\partial_3^+ X \subset \partial_2^+ X \subset \partial_1^+ X \subset X$:*

$$\begin{aligned}\deg(g) &= \frac{1}{4\pi}\int_{\partial X} K d\mu = \chi(X) - Ind(v)\\ &= \chi(\partial_1^+ X) - \chi(\partial_2^+ X) + \chi(\partial_3^+ X). \qquad (2.1)\end{aligned}$$

We notice that formulas from Theorem 2.1 and Corollary 2.1 admit *equivariant generalizations* [22]. For any compact Lie group G, a G-manifold X, an equivariant function $f : X \to \mathbb{R}$, and a generic G-equivariant gradient-like field v, the invariants $\{\chi(\partial_k^+ X)\}$, as well as the degree $\deg(g)$, can be interpreted as taking values in the Burnside ring $\mathcal{B}(G)$ of G.

There is another degree-type invariant of (X, f) linked to generic Morse data (f, v). The set $\partial_2 X \subset \partial_1^+ X$ carries two non-zero vector fields: the normal field n_2 that points inside $\partial_1^+ X$ and trivializes the oriented tangent bundle of $\partial_1 X$ along $\partial_2 X$, and the field $v = v_1$. Therefore, v

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[7]That nice paper attracted my attention to the topic of Morse theory on manifolds with boundary.

defines a *map* $h : \partial_2 X \to S^1$. We view h as an element in the one-dimensional oriented bordism group $\Omega_1(S^1)$ of the circle. This group splits as $\Omega_1(pt) \oplus \Omega_0(pt) \approx \Omega_0(pt)$ (see [5]), i.e., an element $h : M^1 \to S^1$ in $\Omega_1(S^1)$ is determined by the degree class $\deg(h) = [h^{-1}(pt)] \in \mathbb{Z}$.

Any deformation of v preserves the class of $h : \partial_2 X \to S^1$ in $\Omega_1(S^1)$ and thus the degree $[h^{-1}(pt)]$. Deformations of f that change the singularity set Σ_1 do change the degree class. This degree can be easily computed in terms of the cusp sets $\partial_3^+ X$ and $\partial_3^- X$.

Lemma 2.1. *For a fixed f, the number* $\#(\partial_3^+ X) - \#(\partial_3^- X)$ *equals to twice the degree of the map* $h : \partial_2 X \to S^1$ *and is independent of the fields* v, n_2.

Proof. Each loop from $\partial_2 X$ either entirely belongs to one of the two sets $\partial_2^+ X$ and to $\partial_2^- X$, or the arcs of belonging to $\partial_2^+ X$ and to $\partial_2^- X$ alternate. In the first case, the contribution of γ to $\deg(h)$ is zero. In the second case, the contribution of each arc with the ends of opposite polarity is also zero. Each arc with two positive ends contributes a rotation of v by $+\pi$, while each arc with two negative ends contributes a rotation by $-\pi$ (see Figure 7). Hence the total rotation along γ is $\pi[\#(\partial_3^+ X) - \#(\partial_3^- X)]$. □

By Corollary 9.2, a more refined count of the cusps from $\partial_3 X$ will produce a very different formula for the degree of $h : \partial_2 X \to S^1$.

For a given nonsingular $f : X \to \mathbb{R}$, each choice of a gradient-like field v *locally* gives rise to a map $p : X \to \mathbb{R}^2$. Let us outline the construction of p. Add an external collar W to X and extend the Morse data (f, v) into $Y := X \cup W$ without adding new singularities. At each point $x \in Int(W)$ the $(-v)$-flow defines a surjection p_x of a neighborhood $U_x \subset Y$ onto a neighborhood V_x of x in $f^{-1}(f(x))$. Consider the restriction $p_x : U_x \cap \partial_1 X \to V_x, x \in \partial_1 X,$ to the boundary $\partial_1 X.$ According to Whitney [36], generic smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ have only folds and cusps as their stable singularities. Therefore, for generic Morse data (f, v) and $x \in \partial_1 X \setminus \partial_2 X,$ $p_x : U_x \cap \partial_1 X \to V_x$ is a surjection, for $x \in \partial_2 X \setminus \partial_3 X,$ p_x is a folding along an arc of $\partial_2 X,$ and at $x \in \partial_3 X,$ p_x is a cusp map with $p_x(\partial_2 X)$ being the cuspidal curve. Note that along $\partial_2^+ X,$ $p_x : X \to V_x$ is locally onto, while along $\partial_2^- X,$ it is not.

It is especially easy to visualize the stratification $\{\partial_j X\}$ when X is embedded or immersed in $\mathbb{R}^3$ and f is induced from a generic linear function l on $\mathbb{R}^3.$ In such a case, a global surjection $p : X \to \mathbb{R}^2$ is available. Its fibers are parallel to the gradient vector $v = \nabla l.$ Now, $\partial_2 X$ can be identified with the folds of the map $p : \partial_1 X \to \mathbb{R}^2$ and $\partial_3 X$ with its cusps.

As we deform a nonsingular field v within generic one-parameter families, the local structure of the projections p_x can be described in terms of a few canonical forms. One of them, the *cusp*,

$$F(x, y) = (x^3 + xy, y) \tag{2.2}$$

is a stable singularity of a map from $\mathbb{R}^2$ to itself.[8]

The dove tail t-parameter family

$$F_t(x, y) = (x^4 + x^2 t + xy, y) \tag{2.3}$$

describes a cancellation of two cusps that will play a significant role in Section 9.[9]

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[8]It comes from the universal unfolding of the A_3 -singularity $f(x) = x^3.$

[9]It is the universal unfolding of the codimension 1 singularity of a mapping $\mathbb{R}^2$ to $\mathbb{R}^2$ coming from the universal (two-parameter) unfolding of A_4 -singularity.

3. Surgery on the Morse Stratification

Let X be a compact 3-fold X with boundary $\partial_1 X$. Given a smooth function $f : X \to \mathbb{R}$ with isolated singularities, we can construct a new function with *no singularities inside* X: just cut from X a number of tunnels. Each tunnel starts at the boundary $\partial_1 X$ and has a dead end which engulfs a singularity. Denote by T the interior of the tunnels. Then f, being restricted to $X \setminus T \approx X$, is nonsingular, and its perturbation can be assumed to be of the Morse type on $\partial(X \setminus T)$.

Lemma 3.1. *Let X be a compact 3-fold with boundary $\partial_1 X$. Let $f : X \to \mathbb{R}$ be a smooth function with no singularities in a regular neighborhood N of $\partial_1 X$. Denote by v be its gradient-like field. Let $\gamma \subset \partial_1^{\pm} X$ be a simple path that connects two points from $\partial_2 X$ and has an empty intersection with the critical set $\Sigma_1^{\pm}$.*

Then one can deform v in N to a new f-gradient-like vector field $\tilde{v}$ for which the new set $\partial_1^{\mp} X$ will be obtained from the original one by the one-surgery along γ. Outside of N, $v = \tilde{v}$.

A similar statement holds for any field v[10] *which is nonsingular along $\partial_1 X$ and in general position to it.*

Figure 4. Performing 1-surgery on $\partial_1^+ X$.

[10]not necessarily of the gradient type.

Proof. Let v_1 be the orthogonal projection of v onto $\partial_1 X$ in the metric on X in which $v = \nabla f$. The idea is to perform surgery on $\partial_1^+ X$ by a homotopy of the field $v - v_1$, while keeping f and v_1 fixed. We start with "1-surgery" on the fields along a band H which connects two arcs, say $A \subset \partial_2 X$ and $B \subset \partial_2 X$. The band, with the exception of small neighborhoods of its two ends, resides in $\partial_1^+ X$ (alternatively, in $\partial_1^- X$) as shown in Figure 4. The band avoids the singularities of the function $f|_{\partial_1 X}$, so that $v_1 \neq 0$ everywere in the band. Denote by Q a smller band which is contained in H (see Figure 4).

Let n denote the interior normal to $\partial_1 X$. We decompose the field v as $v_1 + h \cdot n$, where the function h is positive in the open domain U — the shaded area without the handle (it is bounded on the left and right by the two dotted segments) — and is negative in the interior of the complement to U. In fact, we can assume that 0 is a regular value of h.

At each point $x \in X$, the differential df picks a particular open half-space $T^+_{f,x}$ in the tangent space T_x, and $v \in T^+_f$. Along the boundary $\partial_1 X$, another family of half-spaces is available: let T^+_x denote the set of tangent vectors at $x \in \partial X$ which point inside of X. Note that, away from the singularities of $f|_{\partial_1 X}$, the cone $T^+_{f,x} \cap T^+_x$ is open.

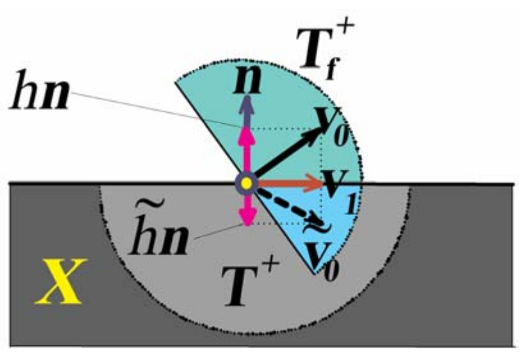

Figure 5. Changing the field $v = v_1 + h \cdot n$ at a point $x \in \partial_1^- X$ into a field $\tilde{v} = v_1 + \tilde{h} \cdot n$ for which $x \in \partial_1^+ X$.

Consider a smooth function $\widetilde{h} : H \to \mathbb{R}$ which satisfies the following properties: (1) $\widetilde{h}^{-1}([0, +\infty)) = V$, (2) zero is a regular value of $\widetilde{h}$ and $\widetilde{h}^{-1}(0) = \partial V$, (3) the field $v_1 + \widetilde{h} \cdot n \in T_f^+$. The last property can be achieved by starting with any $\widetilde{h}$ subject to (1) and (2) and rescaling it by a variable factor $a > 0$, so that $v_1 + (a\widetilde{h})n \in T_f^+ \cap T^+$ (see Figure 5). At each point, the existence of an appropriate a follows from the fact that $v_1 \in T_f^+$. Because the cone $T_f^+ \cap T^+$ is open and the domain H is compact, the global existence of such an a, by a partition-of-unity argument, follows from its existance at each point of H.

We extend the field $v_1 + \widetilde{h} \cdot n$ inside X to get a smooth f-gradient-like field w in a small regular neighborhood W of V. We use a smooth partition of unity $1 = \alpha + \beta$ subordinate to the cover W, $X \setminus Z$. The function α vanishses in Z and β in $X \setminus W$. Now consider the field $\widetilde{v} := \alpha v + \beta w$. Since T_x^+ is convex, $\widetilde{v} \in T_f^+$. Moreover, in V, $\widetilde{v}$ points inside X and, in $H \setminus V$, outside X. Also, outside W, $\widetilde{v} = v$.

Finally, for a fixed f, the set of all f-gradient-like fields is open and convex. Hence, any modification of a f-gradient-like field can be obtained by its deformation. The arguments for generic (non-gradient) fields v are similar and simpler. □

Corollary 3.1. *Under hypotheses and notations of Lemma* 3.1, *the following claims are valid. There is a deformation of a given gradient-like field in the neighborhood N of* $\partial_1 X$ *so that, for the new gradient-like field, both portions* $\partial_1^{\pm} X_j$ *of* $\partial_1^{\pm} X$ *residing in each connected component* $\partial_1 X_j$ *of* $\partial_1 X$ *are nonempty, and* $\partial_1^{+} X_j$ *is homeomorphic to any given domain in* $\partial_1 X_j$ *with a nonempty complement.*

In particular, for a given generic f and all j's, there exists a gradient-like field v such that any of the two properties is satisfied:

- $\partial_1^{+} X_j$ *is homeomorphic to a disk.*
- $\partial_1^{+} X_j$ *and* $\partial_1^{-} X_j$ *are homeomorphic surfaces.*

A similar statement is valid in a category of generic nonsingular vector fields.

Proof. Let $\partial_1 X_j$ be a component of $\partial_1 X$. If $\partial_1 X_j = \partial_1^+ X_j$ (or $\partial_1 X_j = \partial_1^- X_j$), then we can pick a point $x \in \partial_1 X$, where $v_1 \neq 0$. Employing an argument depicted in Figure 5, we can deform the field v in the vicinity of x so that, with respect to the modified gradient-like field, $x \in \partial_1^- X$ ($x \in \partial_1^+ X$, correspondingly). Thus we can assume that $\partial_1^- X_j, \partial_1^+ X_j \neq \varnothing$.

Now, by one-surgery on both $\partial_1^+ X$ and $\partial_1^- X$, we can change the topology of $\partial_1^+ X$ at will, as long as we keep the sets of both polarities nonempty. No matter how we change the two sets, we must keep $\Sigma_{1,j}^+$ inside $\partial_1^+ X$ and $\Sigma_{1,j}^-$ inside $\partial_1^- X$. In particular, we can deform the field so that $\partial_1^+ X_j$ is a 2-disk or, say, to insure that $\partial_1^+ X_j$ is homeomorphic to $\partial_1^- X_j$.

Note that typically surgery on $\partial_1^+ X$ will change the sets $\partial_3^{\pm} X$. □

4. Morse Strata and Convexity

Definition 4.1. Given generic Morse data (f, v) on a manifold X with boundary, we say that v is s-convex (concave), if $\partial_s^+ X = \varnothing$ ($\partial_s^- X = \varnothing$, correspondingly). If $\partial_2^+ X = \varnothing$ ($\partial_2^- X = \varnothing$), then we simply say that v is convex (concave).

An existence of convex Morse data has strong topological implications. Let Σ be a surface with boundary. We denote by $\mathcal{L}(\Sigma)$ a smooth 3-fold with boundary obtained from the product $\Sigma \times [-1, 1]$ by rounding its corners $\partial\Sigma \times \{\pm 1\}$ and by replacing a narrow cylindrical band $\Sigma \times [-\varepsilon, \varepsilon]$ with a "curved parabolic" one as shown in Figure 6. The projection $\mathcal{L}(\Sigma) \to [-1, 1]$ defines a nonsingular function f. The vertical field v in $\Sigma \times [-1, 1]$ is of the f-gradient type. With respect to it, $\partial_2^- \mathcal{L}(\Sigma) = \partial\Sigma$. We call the triple $(\mathcal{L}(\Sigma), f, v)$ a *lens* based on Σ.

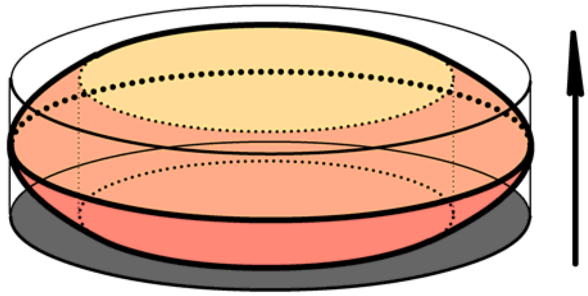

Figure 6. A lens $\mathcal{L}(\Sigma)$ is convex with respect to the vertical field: $\partial_2^+\mathcal{L}(\Sigma) = \emptyset$. The set $\partial_2^-\mathcal{L}(\Sigma)$ is the equator of the lens.

Lemma 4.1. *A connected* 3*-manifold X admits convex nonsingular data (f, v) if and only if X is diffeomorphic to a handlebody $\mathcal{L}(\partial_1^+ X)$.*

In particular, if an acyclic 3*-manifold X admits convex nonsingular data, it is a* 3*-disk.*

Proof. The first claim is straightforward (see [2], Proposition 4.2.2). Note that the convexity on a connected X implies that $\partial_1^+ X$ is connected.

When X is acyclic, a homological argument implies that $\partial_1 X \approx S^2$. Thus $\partial_1^+ X$ must be a contractible domain in S^2, that is, a 2-disk. Therefore, the manifold X must be shaped as a lens, one face of which is that disk. □

Figure 7 shows a typical behavior of a vector field v_1 in a neighborhood of $\partial_2 X$. The arcs of $\partial_2^+ X$ come in tree flavors: type A is bounded by a pair of points from $\partial_3^+ X$, type B is bounded by a pair of points from $\partial_3^- X$, and type C is bounded by a pair of mixed polarity.

This time, we play our convexity game in dimension 2, not 3. At points of $\partial_3^- X$ the field v_1 in $\partial_1^+ X$ is *convex*, at points of $\partial_3^+ X$ it is *concave*.

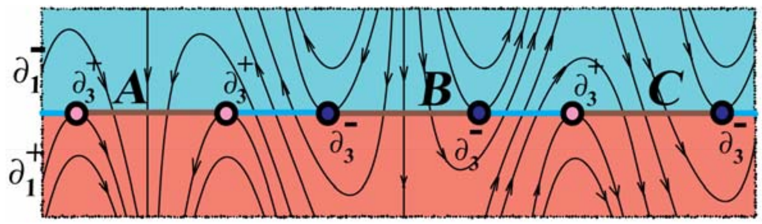

Figure 7. The arcs from $\partial_2^+ X$ of the types A, B, and C.

According to Theorem 9.5, we can deform v (in the space of nonsingular gradient-like fields) so that $\partial_3 X = \emptyset$, in other words, *there are no topological obstructions to the* 3-*convexity and* 3-*concavity of* (*gradient*) *fields*! On the way to establishing this fact, we need to perform 0-surgery on $\partial_3 X \subset \partial_2^+ X$.

Lemma 4.2. *Let v be a gradient-like field for $f : X \to \mathbb{R}$, $v \neq 0$ along $\partial_1 X$. Let $C \subset \partial_2^+ X$ be an arc with one of its ends $a \in \partial_3^+ X$, the other end $b \in \partial_3^- X$, and no other points of $\partial_3 X$ in its interior. Assume also that $f|_C$ has no critical points.*[11] *Then we can deform v in the vicinity of $C \subset X$ in such a way that*:

• *with respect to the new f-gradient-like field $\tilde{v}$ the strata $\partial_1^+ X$ and $\partial_2 X$ remain the same,*

• *arc C changes its polarity* (*from being in $\partial_2^+ X$ to being in $\partial_2^- X$*), *and the points a, b are eliminated from the set $\partial_3 X$.*

Proof. Let v_2 be an orthogonal projection of v on $\partial_2 X$. The argument is analogous to the one in Lemma 3.1. However, this time, we will keep both the direction of n_1-component $v - v_1$ of the field v and the field $v_2 \neq 0$ fixed in the vicinity of C, while deforming the field v_1. Because the direction of the normal component $v - v_1$ remains the unchaged, the stratum $\partial_1^+ X$ and its boundary $\partial_2 X$ will be preserved, but the arc C will change its polarity. □

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[11]See Figure 7, arc C. Note that the absence of critical points of $f|_C$ implies that $f(a) < f(b)$.

Corollary 4.1. *For any generic function* $f : X \to \mathbb{R}$ *with no critical points in* $\partial_1 X$, *there exists a gradient-like field* v *with the following property. Each arc* $C \subset \partial_2 X$, *which connects a minimum* x *of* $f|_{\partial_2 X}$ *with a consecutive maximum* y, *has*:

- *a single point from* $\partial_3^- X$, *provided* $x \in \Sigma_2^+$ *and* $y \in \Sigma_2^-$,
- *a single point from* $\partial_3^+ X$, *provided* $x \in \Sigma_2^-$ *and* $y \in \Sigma_2^+$, *and*
- *no points from* $\partial_3 X$ *when* $x,\ y \in \Sigma_2^{\pm}$. *In that case, the polarity of* C *is the same as the polarity of* x *and* y *in* Σ_2.

Proof. We can assume that $f|_{\partial_2 X}$ is Morse and its maxima and minima alternate. By Lemma 4.2, one can change the polarity of arcs $C \subset \partial_2 X$ between consecutive points a, b from $\partial_3 X$, provided $v_2|_C \neq 0$. Note that the polarity of a and b must be opposite. □

The Morse formula for the vector fields (Theorem 2.1) helps to link the topology of $\partial_1^+ X$ with the distribution of arcs from $\partial_2^+ X$ and points from $\partial_3^{\pm} X$ along $\partial_2 X$.

The next lemma is similar in spirit to Theorem 4.8 from [6]. That theorem is a very special case of the Eliashberg general surgery theory of folding maps (see [6, 7]). However, we cannot apply Eliashberg's results directly: our v-generated foldings $p_x : \partial_1 X \to f^{-1}(f(x))$ have a nice target space only *locally*; a natural target space in our setting is a 2-complex, typically with singularities.

Lemma 4.3. *Let* X *be a compact* 3*-fold with a generic nonsingular vector field* v. *Then the* v*-generated stratification* $\{\partial_k^+ X\}_{0 \le k \le 3}$ *of* X *has the following properties*:

- $\chi(\partial_1^+ X) - \chi(\partial_1^- X) = \#(\partial_3^+ X) - \#(\partial_3^- X) = 2[\#(B \text{ arcs}) - \#(A \text{ arcs})]$
- $\chi(\partial_1^+ X) = \chi(X) + [\#(\partial_3^- X) - \#(\partial_3^+ X)]/2.$

Proof. Since $v \neq 0$ in X, the index $I(v) = 0$. By the Morse formula, $\chi(X) - \chi(\partial_1^+ X) + \chi(\partial_2^+ X) - \chi(\partial_3^+ X) = 0$. Thus $\chi(\partial_1^+ X) = \chi(X) + \chi(\partial_2^+ X) - \chi(\partial_3^+ X)$.

Loops in $\partial_2^+ X$ do not contribute to $\chi(\partial_2^+ X)$, so $\chi(\partial_2^+ X) = \#(\text{arcs in } \partial_2^+ X)$. Hence, $\chi(\partial_1^+ X) = \chi(X) + \#(\text{arcs in } \partial_2^+ X) - \chi(\partial_3^+ X)$. Note, that the C-arcs and their ends do not contribute to the difference $\#(\text{arcs in } \partial_2^+ X) - \chi(\partial_3^+ X)$: such arcs have a single end in $\partial_3^+ X$. On the other hand, the B-arcs do not contribute to $\partial_3^+ X$. Therefore the difference $\chi(\partial_2^+ X) - \chi(\partial_3^+ X)$ is equal $[\#(A \text{ arcs}) + \#(B \text{ arcs})] - 2\#(A \text{ arcs}) = \#(B \text{ arcs}) - \#(A \text{ arcs}) = [\#(\partial_3^- X) - \#(\partial_3^+ X)]/2$.

Recall that for any 3-fold X, $\chi(X) = 1/2 \cdot \chi(\partial_1 X) = 1/2[\chi(\partial_1^+ X) + \chi(\partial_1^- X)]$. Replacing $\chi(X)$ with $1/2[\chi(\partial_1^+ X) + \chi(\partial_1^- X)]$ in the formulas above, leads to the relation $\chi(\partial_1^+ X) - \chi(\partial_1^- X) = \chi(\partial_3^+ X) - \chi(\partial_3^- X)$. □

Combining Lemma 4.3 with Corollary 3.1, we get

Corollary 4.2. *$\#(\partial_3^+ X) = \#(\partial_3^- X)$ if and only if $\chi(\partial_1^+ X) = \chi(\partial_1^- X) = \chi(X)$.*

When $\partial_1 X$ is connected, by deforming v, we can arrange for $\partial_1^+ X$ to be a 2-disk. For any such choice of Morse data (f, v),

$$\#(\partial_3^+ X) - \#(\partial_3^- X) = 2 - 2 \cdot \chi(X).$$

Corollary 4.3. *If a 3-fold X with $\chi(X) > 0$ admits a nonsingular function f with $\partial_3^+ X = \varnothing$, then the restriction $f|_{\partial_1^+ X}$ must have at least $\chi(X)$ extrema.*

Proof. The hypotheses $\partial_3^+ X = \varnothing$ implies that only B-arcs could be present in $\partial_2^+ X$. The positive contribution to $\chi(\partial_1^+ X)$ comes from the components of $\partial_1^+ X$ shaped as disks. We divide disks into two types: (1) disks with no B-arcs in their boundary (which entirely belongs to $\partial_2^+ X$ or to $\partial_2^- X$) and (2) the rest of the disks. Any disk of the first type must contain at least one local extremum of $f|_{\partial_1^+ X}$. Any disk of the second type contains at least one B-arc. Since $\partial_3^+ X = \emptyset$, we get $\chi(\partial_1^+ X) = \chi(X) + \#(B - \text{arcs})$. Now the lemma follows from writing down $\chi(\partial_1^+ X)$ as the Euler class of all disks of the first type plus the Euler class of the rest of $\partial_1^+ X$. □

Corollary 4.4. *For every* 3*-fold X with a connected boundary and Euler number χ, there exist nonsingular Morse data (f, v) so that $\partial_1^+ X$ is a disk D^2. For such data, we get $\#(\partial_3^+ X) \geq 2\chi - 2$ and $\#(\partial_3^- X) \geq 2 - 2\chi$. As a result, when $\chi > 1$, the disk cannot be convex with respect to the field v_1; as χ grows, the disk $\partial_1^+ X$ becomes more "wavy". Similarly, when $\chi < 1$, the disk cannot be concave with respect to v_1, that is, $\partial_3^- X \neq \emptyset$.*

Proof. By Lemma 3.1, appropriate deformations of v will produce $\partial_1^+ X \approx D^2$. In view of Lemma 4.3, the claim follows. □

The following theorem shows that convexity of Morse data is equivalent to the possibility of special *ordering* of the $f|_{\partial_1 X}$-critical points by their critical values, and thus, in general, fails. However, if we formally attach index $i + 1$ to each critical points $x \in \Sigma_1^-$ of classical index i, the new self-indexing of $f|_{\partial_1 X}$ becomes possible.

Theorem 4.1. *Let (f, v) be Morse data whose restriction (f_1, v_1) to the boundary $\partial_1 X$ is also of the Morse type. If $\partial_2^+ X = \emptyset$, then there is no ascending trajectory $\gamma(t) \subset \partial_1 X$ of the vector field v_1, such that $[\lim_{t\to+\infty} \gamma(t)] \in \Sigma_1^+$ and $[\lim_{t\to-\infty} \gamma(t)] \in \Sigma_1^-$.*

Conversely, if no such $\gamma(t)$ exists, one can deform the gradient-like vector fields $\{v, v_1\}$ (equivalently, the metric g in which $v = \nabla f$) to a new gradient-like pair $\{\tilde{v}, \tilde{v}_1\}$ (to a new metric $\tilde{g}$), in such a way that, with respect to the new fields, $\partial_2^+ X = \emptyset$. In particular, if $f(\Sigma_1^+) < f(\Sigma_1^-)$, then f admits convex Morse data (convex metric $\tilde{g}$).

In contrast, no nonsingular Morse data (f, v) *are* 2-*concave*: $\partial_2^- X \neq \varnothing$. [12]

Proof. For a generic metric g, consider the vector field $v = \nabla f$ and its orthogonal projection v_1 on $\partial_1 X$. The function $h_1 : \partial_1 X \to \mathbb{R}$, defined via the formula $v = v_1 + h_1 \cdot n_1$, where n_1 is the inward normal field, has zero as a regular value. Then $\partial_1^+ X = h_1^{-1}([0, +\infty))$, $\partial_1^- X = h_1^{-1}((-\infty, 0])$ and $\partial_2 X = h_1^{-1}(0)$. Now, if an ascending v_1-trajectory $\gamma(t)$ which links Σ_1^- with Σ_1^+ does exist, it must cross somewhere the boundary $\partial_2 X$ of $\partial_1^- X$. By definition, such crossing belongs to $\partial_2^+ X$ which must be non-empty.

On the other hand, if no such $\gamma(t)$ exists, then we claim the existence of codimension 1 closed submanifold $N \subset \partial_1 X$, which separates $\partial_1 X$ in two domains $A \supset \Sigma_1^+$ and $B \supset \Sigma_1^-$ $(\partial A = N = \partial B)$ and, in addition, has the following property. The vector field v_1 is *transversal* to N and points *outward* of A. Indeed, one can take a small regular neighborhood (in $\partial_1 X$) of the union of descending v_1-trajectories of all critical points from Σ_1^+ for the role of A. Here we are employing the fact that no descending trajectory originating at Σ_1^+ reaches Σ_1^-.

Since, away from $\Sigma_1^+ \cup \Sigma_1^-$, $v_1 \neq 0$, in the tangent space T_x of X $(x \in \partial_1 X)$, there is an open cone $T_{f,x}^+$ containing v_1 and comprised of gradient-like vectors.

With such a separator N in place, consider a smooth function $\tilde{h}_1 : \partial_1 X \to \mathbb{R}$ with the properties: (1) zero is a regular value of $\tilde{h}_1$ and $\tilde{h}_1^{-1}(0) = N$; (2) $\tilde{h}_1^{-1}(-\infty, 0) = A$, $\tilde{h}_1^{-1}[0, +\infty) = B$; (3) $\tilde{h}_1 = h_1$ in the vicinity of $\Sigma_1^+ \cup \Sigma_1^-$; and (4) $v_1 + \tilde{h}_1 \cdot n \in T_f^+$. Note that the field $\tilde{v} := v_1 + \tilde{h}_1 \cdot n$ points inside X along A and outside along B.

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[12]Note that any X admits a field $v \neq 0$ with respect to which $\partial_2^- X = \varnothing$ (cf. [2]).

Now we can find a metric $\tilde{g}$, in which $\tilde{v}$ is the gradient of *f*. In $\tilde{g}$, $\tilde{v}$ is orthogonal to the plane $K_x := Ker(df)$. Denote by $\tilde{v}_1$ the $\tilde{g}$-orthogonal projection of $\tilde{v}$ on T_x. Since $\tilde{h}_1$ vanishes on *N*, $\tilde{v}_1 = v_1$ along *N*, and therefore, is transversal to *N* and points outward *A*. As a result, with respect to $\tilde{g}$, $\partial_2^+ X = \emptyset$.

We can deform the original metric *g* into $\tilde{g}$, thus deforming the gradient fields v, v_1 into the gradient fields $\tilde{v}$, $\tilde{v}_1$.

The last claim follows from the observation that since $v \neq 0$ the absolute maximum (minimum) of *f* on *X* must be realized at a point from Σ_1^- (from Σ_1^+). □

In view of Theorem 4.1 and Lemma 4.1, we get

Theorem 4.2. *A connected* 3*-fold X with a connected boundary is a handlebody if and only if one of the following properties is valid*:

• *X admits a smooth nonsingular function f whose restriction on the boundary* $\partial_1 X$ *is Morse*; *moreover*, *no ascending trajectory of a gradient-like field* v_1 *links in* $\partial_1 X$ *a singularity from* Σ_1^- *to a singularity from* Σ_1^+.

• *X admits a smooth nonsingular function f*, *and* $\partial_1 X$ *is convex with respect to a gradient-like field v*, *that is*, $\partial_2^+ X = \emptyset$.

Given the remarkable proof of the Geometrization Conjecture [29], [30], the proposition below must be viewed just as an illustration. It shows what advances towards the Poincaré Conjecture are possible by modest means of the Morse Theory alone. We get the following criteria for recognizing standard 3-balls in terms of Morse data:

Corollary 4.5. *An acyclic* 3*-fold X is a* 3*-ball if and only if one of the following properties is valid*:

(1) *X admits a smooth function f with no critical points in X whose restriction on the boundary* $\partial_1 X \approx S^2$ *is Morse*; *moreover*, *no ascending trajectory of a gradient-like field* v_1 *links in* $\partial_1 X$ *a singularity from* Σ_1^- *to a singularity from* Σ_1^+.

(2) *X admits a smooth function f with no critical points in X whose boundary $\partial_1 X$ is convex with respect to a gradient-like field v.*

(3) *X admits a smooth function f with no critical points in X, so that $\partial_1^+ X \subset S^2$ is connected and $\partial_3^+ X = \varnothing$ (i.e., the Morse data are 3-convex).*

Proof. Evidently, a standard 3-ball admits Morse data with the properties described in (1)-(3).

By a homological argument based on the Poincaré duality, an acyclic 3-fold X has a spherical boundary.

By Theorems 4.1, 4.2, properties (1) and (2) are equivalent, and by Lemma 4.1, (2) implies that X is a 3-ball. To prove (3) we use the last formula from Lemma 4.3. Since $\chi(X) = 1$ and $\partial_1^+ X$ is connected, it follows that $\#(\partial_3^+ X) - \#(\partial_3^- X)$ is twice the number of holes in $\partial_1^+ X$. Hence, $\#(\partial_3^+ X) = 0$ implies that $\#(\partial_3^- X) = 0$ and, therefore, $\partial_1^+ X$ must be a 2-disk with no points from $\partial_3 X$ along its boundary. However, the boundary of the disk $\partial_1^+ X$ cannot belong entirely to $\partial_2^+ X$: f must attend its absolute maximum and minimum in S^2. We have seen already that this implies that $\partial_2^- X \neq \varnothing$. Hence, $\partial_3 X = \varnothing$ which implies that $\partial_2^+ X = \varnothing$. Thus, under the hypotheses, the 3-convexity implies the 2-convexity. In turn, the 2-convexity implies that the manifold is a 3-disk. By Corollary 3.1, we can find Morse data so that $\partial_1^+ X$ is a disk. □

Example 4.1. Let X be the Poincaré homological 3-sphere from which a 3-ball has been removed. It follows from the theorems above that, for any smooth function $f : X \to \mathbb{R}$ with no singularities in X, there is an ascending v_1-trajectory which links a singularity from Σ_1^- to a singularity from Σ_1^+. Also, no nonsingular Morse data (f, v) can insure both the connectivity of $\partial_1^+ X$ and the 3-convexity. Therefore, a "mild" connectivity restriction on $\partial_1^+ X$ turns the obstructions to 3-convexity into a topological phenomenon.

5. Cascades, 2-spines and Concavity

Definition 5.1. Let K be a finite two-dimensional polyhedron (cellular complex) embedded as a subcomplex in a compact 3-fold X with boundary $\partial_1 X$. We say that $K \subset X$ is a spine, if $X \setminus K$ is homeomorphic (diffeomorphic) to the product $(\partial_1 X \setminus (K \cap \partial_1 X)) \times [0, 1)$.

It follows that X is collapsible onto K, in particular, K is a strong deformation retract of X. Furthermore, X is homeomorphic to a cylinder of a map $g : \partial_1 X \to K$ which is an identity on $K \cap \partial_1 X$ (cf. [25], Theorem 1.1.7).

Our next goal is to use nonsingular Morse data (f, v) in order to construct rather special spines that we call *gradient*. First, we focus on the complications arising from the concave locus $\partial_2^+ X$. It is comprised of a finite number of disjoint arcs or loops $\{E_j\}$. For each connected curve $E_j \subset \partial_2^+ X$, consider the set of points in X which can be reached from E_j moving down along the trajectories of $-v$. Denote by W_j the closure in X of this set. We call such a set W_j a *waterfall*. The union $\bigcup_j W_j$ of all waterfalls is called a *cascade* and is denoted by $\mathcal{C}(\partial_2^+ X)$. We denote by $\mathcal{C}(\partial_3^- X)$ the finite union of the downward trajectories through the points of $\partial_3^- X$. These trajectories are called *free*.

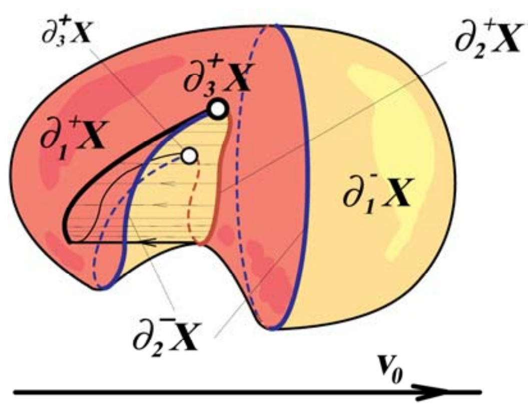

Figure 8. Killing a 1-cycle in $\partial_1^+ X$ by attaching the 2-cell $\mathcal{C}(\partial_2^+ X)$.

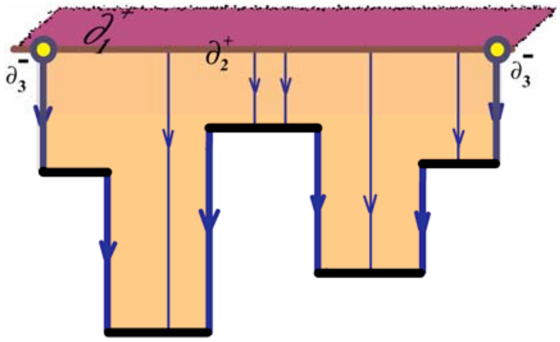

Figure 9. A waterfall of an A-arc in $\partial_2^+ X$.

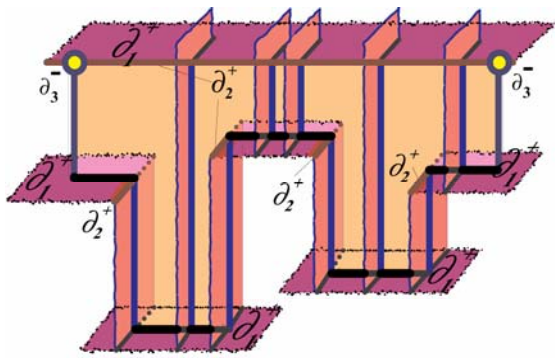

Figure 10. The same waterfall in a cascade.

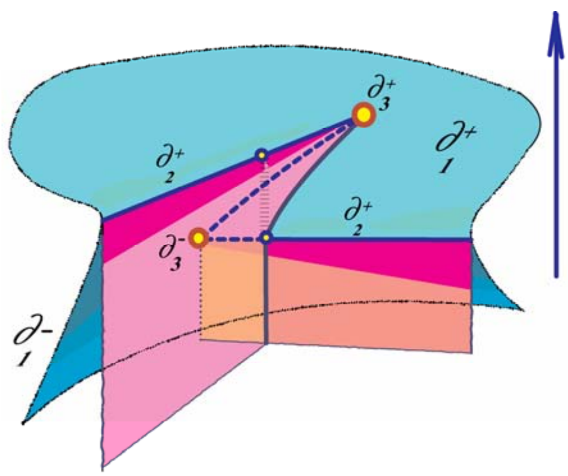

Figure 11. A cascade of the Dove Tail singularity with two cusps from $\partial_3^+ X$ and $\partial_3^- X$. Note the trajectory shared by the two waterfalls.

Definition 5.2. We say that the gradient-like field v is ∂_2^+-generic if

• for each $x \in \partial_2^+ X$ there is an open interval V_x centered on x such that the surface W_x, formed by the downward trajectories through V_x, away from V_x, has only transversal intersections with $\partial_2^+ X$;

• the downward trajectories of points from the finite set $\partial_3 X$ are all distinct and each trajectory belongs to a single waterfall W_j.

Figures 8, 10, 11 show mechanisms by which ∂_2^+-generic waterfalls are created, as well as their typical shapes.

Lemma 5.1. *A small pertubation of a gradient-like field v turns it into a ∂_2^+-generic field. For such a field, the downward trajectories of points from $\partial_3^- X$ terminate in the interior of the surface $\partial_1^+ X$.*

Proof. For v being ∂_2^+-generic is an open dense property established by standard transversality arguments. □

The following proposition describes how any generic v canonically generates its gradient spine.

Theorem 5.1. *Let X be a compact 3-fold. We assume that a smooth function $f : X \to \mathbb{R}$ has no singularities and its gradient-like field v is ∂_2^+-generic. Then the 2-complex $K = \partial_1^+ X \cup \mathcal{C}(\partial_2^+ X)$ is a spine of X.*

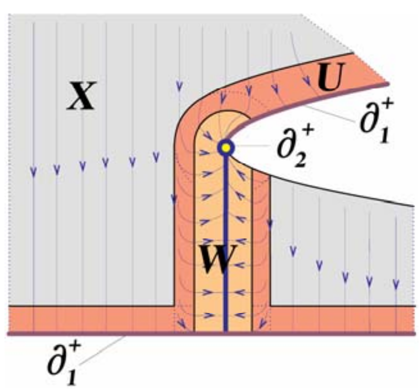

Figure 12. A section of the retraction $p : X \to K$ by a plane transversal to $\partial_2^+ X$.

Proof. We can modify the gradient-like field v by a nonnegative conformal factor $\phi : X \to \mathbb{R}_+$ which is zero on $\partial_1^+ X$ and positive everywhere else. Abusing notations, denote this modification by v as well. Let $\mathcal{U}$ be a regular neighborhood of the complex K in X which is invariant under the flow produced by the modified $-v$. The existence of $\mathcal{U}$ is implied by the invariancy of the set K. For example, one can use a metric in X, invariant under the flow, in order to choose $\mathcal{U}$ as an ε-neighborhood of K. Then we pick a regular $\varepsilon/2$-neighborho od $\mathcal{W} \subset X$ of the cascade $\mathcal{C}(\partial_2^+ X)$. It is properly contained in $\mathcal{U}$. Let w be a field that defines a retraction of $\mathcal{U}$ onto K. It vanishes on K, is tangent to $\mathcal{U} \cap \partial_1^- X$, as well to $\mathcal{W} \cap \partial_1 X$, and inward transversal to the portion of $\partial\mathcal{W}$ that is not in $\partial_1 X$. Moreover, as Figure 12 testifies, w can be chosen so that, away from K, it is never positively proportional to v. Consider a smooth partition of unity $1 = \alpha + \beta$, where α is supported in $X \setminus (\mathcal{W} \cup \partial_1^+ X)$ and β in $\mathcal{U}$. Form the vector field $u = -\alpha v + \beta w$. It is defined globally and vanishes only on K. By the construction of $\mathcal{U}$, $u = -v$ on the boundary of $\mathcal{U}$. Therefore, u or is tangent to $\partial\mathcal{U}$ or points inside $\mathcal{U}$ (see Figure 12). As a result, positive u-trajectories of points $x \in \mathcal{U}$ must reach either the cascade, or the set $\partial_1^+ X$. Hence, the u-flow governs the retraction of X on K, and K is a strong deformation retract of X. Moreover, the $(-u)$-flow gives a product structure $\partial_1^- X \times [0, 1)$ to $X \setminus K$. $\square$

Remark 5.1. For a given field v, one can introduce an equivalence relation $\sim_v$ among points of X: two points are defined to be equivalent if they both belong to the closure of the same v-trajectory. The quotient space $X/\sim_v$ with the quotient topology is called the *orbit-space* of v. For a generic $v \neq 0$, $X/\sim_v$ can be given the structure of a 2-dimensional CW-complex which is homotopy equivalent to X. In fact, for a generic $v \neq 0$, the obvious maps $P : X \to X/\sim_v$ and $P| : \partial_1^+ X \cup \mathcal{C}(\partial_2^+ X) \to X/\sim_v$ are Serre fibrations. Thus, for such v, the gradient spine can be also regarded as a homotopy substitute for the space of v-trajectories.

Remark 5.2. The 2-cells in which K is subdivided admit a preferred orientation induced by the preferred orientation of $\partial_1 X$, so that they form an integral 2-chain [21]. Its boundary $\partial[K]$ consists of the 1-chain $\partial_2^- X \cup \mathcal{C}(\partial_3^- X)$ together with the singularity locus $s(K)$ of K. This locus is comprised of curves from the intersection $\partial_1^+ X \cap \mathcal{C}(\partial_2^+ X)$ and the orbits shared by pairs of waterfalls. In short, each edge from the support of $\partial[K]$ contributes to the cycle $\partial[K]$ with multiplicity ± 1 (and not ± 3)–an important property which the gradient spines share with the branched spines (see Corollary 3.1.7, [2]). This property was originally studied in [8] and [9].

Remark 5.3. Note that changing f to $-f$ and v to $-v$ exchanges the strata $\partial_1^+ X \Leftrightarrow \partial_1^- X$, $\partial_3^+ X \Leftrightarrow \partial_3^- X$, and keeps the strata $\partial_2^+ X$, $\partial_2^- X$ fixed. Therefore, the gradient spines $K(f, v)$ and $K(-f, -v)$ "complement" each other in X: they share $\partial_2 X$ and their cascades $\mathcal{C}(f, v)$ and $\mathcal{C}(-f, -v)$ complement each other in the set spanned by all the trajectories through $\partial_2^+ X$. Evidently, the topologies of $K(f, v)$ and $K(-f, -v)$ could be radically different. However, their complexities (see Definition 7.1) are equal.

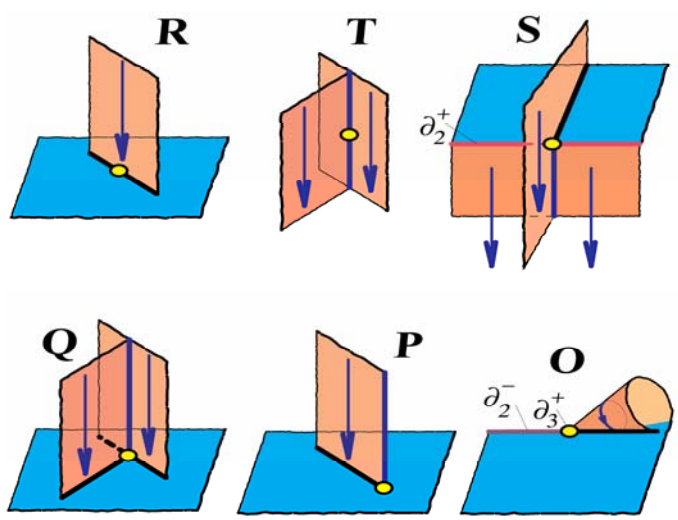

Figure 13. Singularity types of gradient spines.

By its very construction, any generic gradient spine K has a very particular stratified local geometry. In fact, as abstract cellular 2-complexes, gradient spines are *simple* spines in the sense of [25]. However, in our context, (as well as in the case of branched spines) *the smooth structure of* $\partial_1^+ X$ and waterfalls *breaks the* 3-*fold symmetry* of simple combinatorial spines: in other words, we distinguish between the "Y" and "T-shapes".

The stars of points in K are shown in Figure 13 and will be referred by their letter labels. The six types of links of points in K are depicted in Figure 20. A generic point from the set $\partial_2^- X \cup \mathcal{C}(\partial_3^- X)$ has a neighborhood shaped as a half-disk and has a link of type (2). The singular set $s(K)$ consists of points of types R, T, and S, all having links of type (5). The T-type is produced by the trajectories that belong to two distinct waterfalls or to two branches of the same waterfall. The R-points are generic to loci where waterfalls hit the ground $\partial_1^+ X$. The S-points are generated when a waterfall transversally hits an arc from $\partial_2^+ X$. They are hybrids of T and R types. Topologically R, T, and S types are indistinguishable. The Q-type is generated where two waterfalls hit the ground. Stars of Q-points are shaped as a union of a disk with a half-disk with a quoter-disk, and their links are of type (6). The Q-type singularities are isolated in K. Singularities of the P-type are located where a free trajectory through $\partial_3^- X$ hits the ground. Hence, they are in 1-to-1 correspondence with points of $\partial_3^- X$. A neighborhood of a P-singularity is a union of a disk with a quater-disk which share a common radius; its link is of type (4). Finally, the singularities of the O-type are just points from $\partial_3^+ X$. They also have stars shaped as cones over a circle union a radius (see Figure 11). Topologically the O and P-types are the same.

Next, we prove that 3-folds have gradient spines which are rather special "*origami*" folded from a 2-disk (see Figure 14). This result is very similar to Theorem 1.2 from [18], where an origami is built from a normal pair. However, one technical difference is evident: in [18] the v-flow is transversal to the normal disk, while in our construction, the gradient flow is not necessarily transversal to $\partial_1^+ X$ along its boundary $\partial_2 X$.

Theorem 5.2. *The spine* $K = \partial_1^+ X \cup \mathcal{C}(\partial_2^+ X)$ *of a* 3*-fold* X*, produced from generic Morse data is an image under a cellular map* $\Phi : S \to K$ *of a cellular* 2*-complex* S *homeomorphic to the surface* $\partial_1^+ X$*. The map* Φ *is* 1*-to-*1 *in the interiors of the* 2*-cells in which* S *is subdivided, at most* 2*-to-*1 *on the* 1*-skeleton without vertices, and at most* 3*-to-*1 *on the set of vertices. The local geometry of* Φ *can be described by the four identification patterns in Fig*ure 15.

Moreover, any 3*-fold* X *has a gradient spine* K *which is an image of a* 2*-disk* D^2 *under a cellular map* $\Phi : D^2 \to K$ *which is* 1*-to-*1 *in the interiors of the* 2*-cells in* D^2*, at most* 2*-to-*1 *on the* 1*-skeleton of* D^2 *without vertices, and at most* 3*-to-*1 *on the set of vertices.*

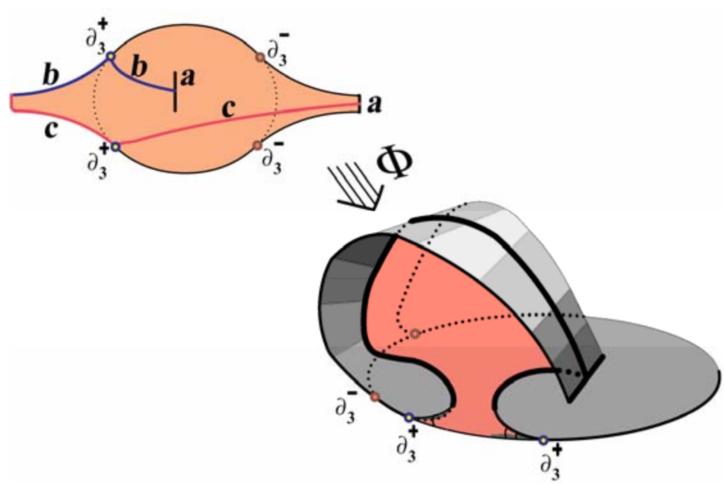

Figure 14. An "origami" map $\Phi : D^2 \to K$ with a collapsable K.

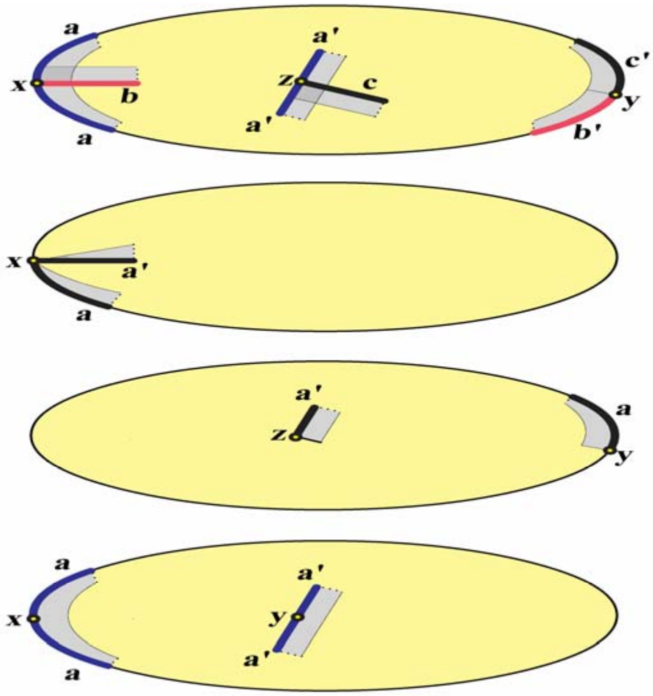

Figure 15. Identification patterns for gradient spines.

Proof. One can resolve K along its singular set $s(K)$ so that the resulting space is a celluar 2-complex $S := \partial_1^+ X \cup_{\partial_2^+ X} \left(\coprod_j W_j\right)$. It is obtained from $\partial_1^+ X$ by attaching individual waterfalls $\{W_j\}$ along the loops and arcs forming $\partial_2^+ X$. As a result, S is homeomorphic to the surface $\partial_1^+ X$. In particular, when $\partial_1^+ X$ is a 2-disk, so is the resolution S.

This resolution can be done by performing cuts of several types. First, at each singularity of any type, but the T-type, (see Figures 13, 16) the cut separates the cascade from the "ground" surface $\partial_1^+ X$. Then at each singularrity of the T-type (see Figure 17), there is a preferred "half-waterfall" which is separated from the adjacent waterfall by a cut along a trajectory that they both share.

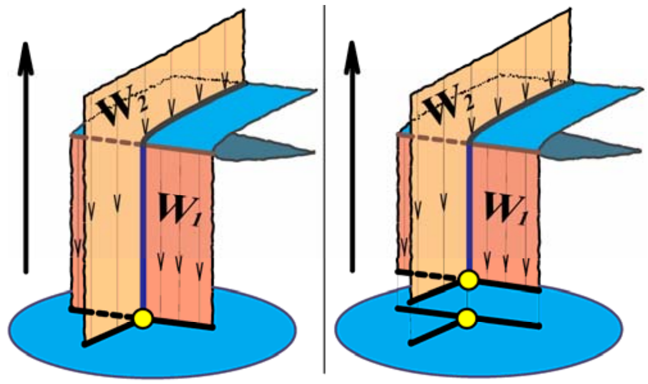

Figure 16. A singularity of the Q-type and its resolution.

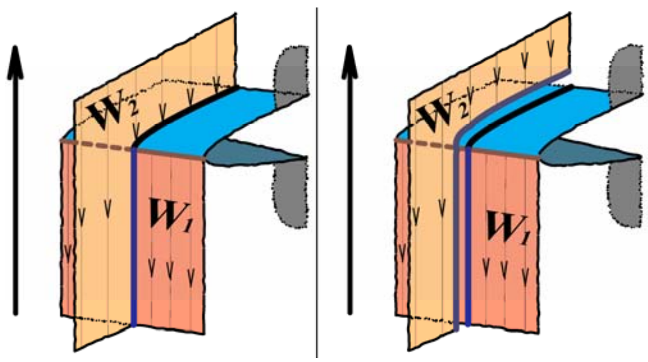

Figure 17. A singularity of the T-type and its resolution.

The shape of the waterfalls W_j's depends on the type of the arc or loop in $\partial_2^+ X$ from which it falls. Recall that, after an appropriate change of the vector fields, the arcs come in three flavors: A, B and C. The A-type waterfall has two "free" edges which are not affected by the gluing Φ. The edges emanate from the two ends $x, y \in \partial_3^- X$ of an A-arc. The B-type waterfall has no "free" edges at all. It falls from a B-arc whose edges are in $\partial_3^+ X$. The C-type waterfall has one "free" edge which emanates from a point in $\partial_3^- X$—the end of the arc. Figure 29 depicts the topological models of A, B, and C types in vicinity of an arc τ where they hit the ground $\partial_1^+ X$.

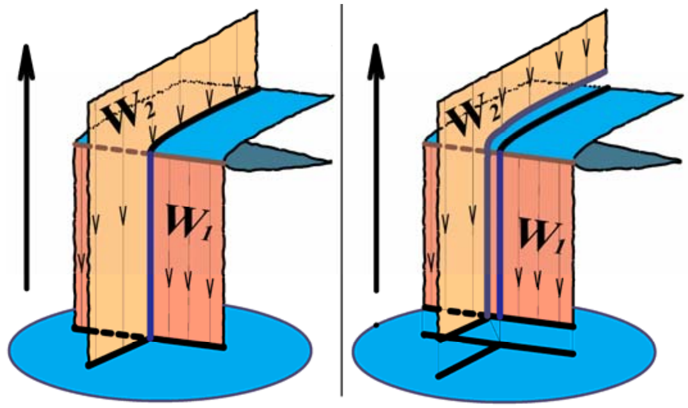

Figure 18. A complete resolution of singularities.

By Corollary 3.1, for appropriate Morse data, we can assume that $\partial_1^+ X$ is a disk.

Evidently, there is a mapping $\Phi : S \to K$ which glues the complex K back. The map Φ is 1-to-1 on the compliment to the 1-skeleton of S, it is at most 2-to-1 on the interior of its edges and at most 3-to-1 on its vertices (see Figures 15-18). With the local topology of K being restricted by the list in Figure 13, one can verify that Figure 15 lists all possible gluing patterns for Φ (see Appendix A in [21] for details). □

Globally, the gluing patterns of Φ can be quite intricate as shown in Figure 19 which depicts an example of a map $\Phi : D^2 \to K$ as in Theorem 5.2 with a contractible, but not collapsable K. The second diagram in Figure 19 is obtained from the first one by elementary collapses performed through the free trajectories. The third diagram suggests contracting the shaded disk to a point, thus ignoring the subtleties of the Φ-gluing pattern inside the disk, but keeping the presentation of $\pi_1(K) \approx 1$.

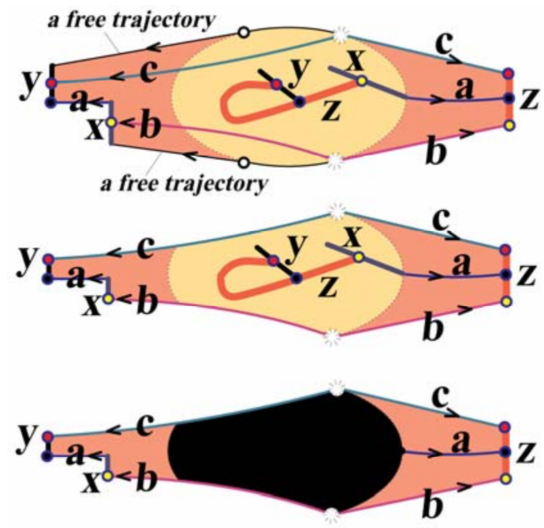

Figure 19. The combinatorics of a map $\Phi : D^2 \to K$ which gives rise to a non-collapsable K with $\pi_1(K) = 1$ presented as $\{a, b, c \,|\, ba = c^{-1}, ab^{-1} = 1, ac^{-1} = 1\}$.

When $\partial_1^+ X$ is a 2-disk, Morse data (equivalently, gradient spines K) produce a *presentation* of the fundamental group $\pi_1(X) \approx \pi_1(X/\partial_1^+ X)$. Its generators are the oriented v-trajectories that are shared by pairs of waterfalls, or by two branches of the same waterfall, together with the free trajectories that emanate from $\partial_3^- X$ (each waterfall has at most two free trajectories, and two waterfalls can share a number of trajectories). Let $\{\omega_{i\beta}\}_\beta$ be the set of shared trajectories in W_i and $\{\alpha_{i\delta}\}_\delta$ be the set of free ones. Evidently, the homotopy class of each loop γ in $K/\partial_1^+ X$ is characterized by its traces in the waterfalls (see Figures 10 and 19). In each waterfall W_i, the trace of γ can be replaced by a word in $\{\omega_{i\beta}\}_\beta$ and $\{\alpha_{i\delta}\}_\delta$. The relations are produced by looking independently at each waterfall marked with its shared trajectories. If we cut W_i along the $\{\omega_{i\beta}\}_\beta$'s, it will break into a number of polygons. Each polygon has certain shared and free trajectories in its boundary, and the rest of the boundary consists of arcs that belong to $\partial_1^+ X$. A free trajectory belongs to a single polygon, and each polygon has two free trajectories at most. Each polygon is oriented and contributes a single relation: moving along its oriented boundary produces a word in the alphabet that contains free and shared trajectories and their inverses (the arcs of the boundary that were attached to $\partial_1^+ X$ are ignored). In fact, when a polygon contains a free trajectory, we can delete it from the list of generators and the polygon itself from the list of relations. Eventually, this will eliminate all free trajectories with the exception of the pairs that belong to a single waterfall free of shared trajectories in its *interior*. One of the free trajectories in each of such pairs can be dropped from the list of generators and its polygon from the list of relations.

In general, the same recipe produces a presentation of $\pi_1(X/\partial_1^+ X)$. Hence,

Theorem 5.3. *Let X be a connected* 3*-fold. Then generic Morse data* (f, v) *such that* $\partial_1^+ X$ *is a* 2*-disk give rise to a finite presentation of* $\pi_1(X)$. *Thus,* (f, v) *determine the* Q^{**}*-class* (*as defined in* [27]) *of that presentation. By* [32], *this* Q^{**}*-class is a topological invariant of X.*

6. Abstract Gradient Spines

We already noticed that the geometry of gradient spines K provides us with a particular way of orienting their 2-cells. This orientation is induced by a preferred orientation of $\partial_1 X$ and is spread along the cascade by the v-flow. Furthermore, not only the surfaces in K° acquire a *preferred orientation*, but they also have *preferred sides* in the ambient X, the sides picked by the inner normals to $\partial_1^+ X \subset X$. The inner normals uniquely extend by continuity to each waterfall W and pick its side in X.

These properties of gradient spines can be captured in the notion of an *abstract gradient spine*. We start with a few preliminary definitions.

Definition 6.1. A *simple polyhedron* is a compact 2-dimensional polyhedron such that each of its points has a link homeomorphic to one of the shapes in Figure 20[13] and the corresponding star from Figure 13.

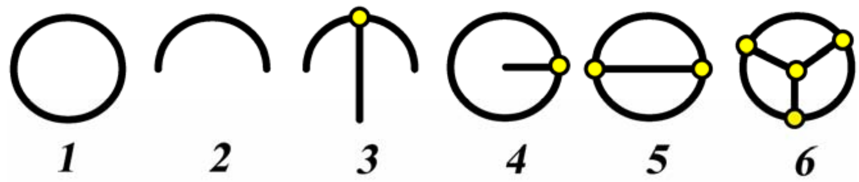

Figure 20. Link types of points in simple polyhedrons.

The six types have a partial order induced by the inclusions of closures of the appropriate strata:

$$(1) > (2),\ (1) > (5),\ (2) > (4),\ (5) > (3).$$

Note that generic gradient spines are simple polyhedra. However, typically they admit 2-collapses; thus the need for considering link types (2), (3), and (4).

The "topological boundary" $s_\bullet(K)$ of K is formed by points with links of type (2) degenerating into types (3) and (4). The graph $s_\dagger(K)$ is formed by the points with links of type (5) degenerating into types (3), (4), and (6). Thus a generic point from $s_\bullet(K)$ belongs locally to the boundary of a single surface in K, while a generic point from $s_\dagger(K)$ belongs to the boundaries of three surfaces. Let $s(K) = s_\bullet(K) \cup s_\dagger(K)$. The finite set of points of types (3), (4), and (6) is denoted $ss(K)$. The bivalent vertices from $ss_{\bullet\dagger}(K) = s_\bullet(K) \cap s_\dagger(K)$ are of type (4), and the rest of the vertices from $ss(K)$ are the four-valent ones of type (6). The set formed by the points of type (6) is also denoted $Q(K)$.

Recall that a simple spine K is called *special* in [25] or *standard* in [2] if the stratification $ss(K) \subset s(K) \subset K$ gives K a structure of cellular

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[13]Actually, type (3) does not appear in generic gradient complexes, but is present in their 1-parameter deformations.

2-complex, i.e., if $K^\circ := K \setminus s(K)$ is a disjoint union of open 2-cells and $s(K)^\circ := s(K) \setminus ss(K)$ is a disjoint union of open 1-cells.

If K is a simple special spine of X whose points have local models of types (1), (5), and (6), then X can be uniquely reconstructed just from a regular neighborhood $\mathcal{N}(s(K)) \subset K$ of the singular set $s(K)$ ([25], Theorem 1.1.17). For the reconstruction to work, it is important that $K \setminus s(K)$, the disjoint union of disks, does not support non-trivial line bundles. For any gradient spine K, the bundle, normal to $K \setminus s(K)$ in X, is also trivial. This property of gradient spines K will permit a unique reconstruction of X from K as well.

In order to distinguish intrinsically the T-shaped configurations from the Y-shaped ones, we use a particular *system of markers* placed along the edges of the graph $s_\dagger(K)$, K being a simple polyhedron. The marker is a short segment emanating from a generic point $x \in s_\dagger(K)$. It is transversal to $s_\dagger(K)$ and is contained in one of the three surfaces (pages) that join at x. We call such segments *T-markers*. A T-marker m, the vertical leg of letter T, tells us that the two pages that do not contain m are thought "to form 180° angle" in the ambient X. In the category of *branched* spines K, the local geometry of K along $s(K)$ also picks one page out of three: recall that only two out of three tangent pages form a cusp, and the preferred page is the third one [2]. It suffices to place a single T-marker at each edge or loop of the graph $s_\dagger(K)^\circ := s_\dagger(K) \setminus ss(K)$: by continuity, the marker spreads itself along the edge or loop until it reaches an isolated singularity from $ss(K)$. There the marker's pattern requires an additional explanation to be provided below.

The N-marker is attached to generic points $x \in s_\dagger(K)$ and is contained in one of the two pages that do not carry a T-marker. Informally, one can think of the pages with T-markers as waterfalls. The N-markers reside both in the "ground" and in the waterfalls, and can be regarded as substitutes for the preferred normals to the oriented pages that contain T-markers.

For gradient spines K, v provides us with a T-marker at points of $\partial_1^+ X \cap s(K)$. For points on a trajectory shared by two waterfalls, the T-marker resides in the page complementary to the two pages that form 180° angle. For each waterfall W, the vectors tangent to $\partial_1^+ X$ and pointing to the preferred side of W, produce the rest of N-markers. Effectively, the T and N-markers generate a *coloring* of $\mathcal{N}(s_\dagger(K)) \setminus \mathcal{N}(ss(K))$ with three colors, the colors of the tree pages which share an edge being distinct. Here $\mathcal{N}(\sim)$ denotes a regular neighborhood in K of an appropriate set.

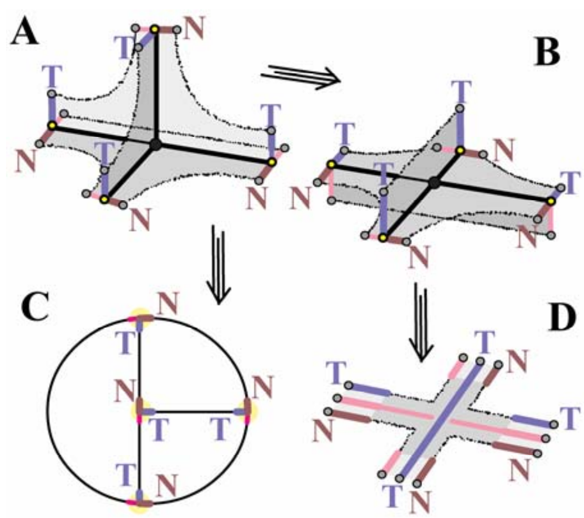

Figure 21. *TN*-markers in the vicinity of a *Q*-singularity and the plane pattern they generate. Diagram C is a view of configuration *A* from the top, Diagram D is a view of configuration *B* from the top.

A single pair of *TN*-markers approaches each singularity of types (3) or (4). In fact, the markers distinguish between the spines of type (4) in the vicinity of a point from $\partial_3^+ X$ (*O*-type in Figure 13) and a point y where a trajectory through $\partial_3^- X$ hits the ground $\partial_1^+ X$ (*P*-type in Figure 13): in the first case, moving along the loop in (4) in the direction of N, we return from the direction marked by T; in the second case, moving along the loop in the direction of N, we return from the unmarked direction.

Four pairs of *TN*-markers approach each singularity *x* of the type (6) from four different directions. The link L_x of *x* in *K* is shown in Figure 21, A and C, and the markers reflect the "plane – union half-plane – union quater-plane geometry" (as seen from the North Pole). Among the six edges of L_x, there is a single edge α with two *T*-markers and a single edge β with two *N*-markers, α and β sharing a vertex. The rest of the markers are determined by Figure 21, A and C. There configuration B is obtained from configuration A by opening the quoter-plane (the higher waterfall) into a half-plane and bending the ground plane (along the bold line) so that its two halves form a right angle. We introduce diagrams B and D to make some of our drawings more in the spirit of the plane calculus of 4-valent graphs as in [2], [18], [19], [23].

The *TN*-pattern in L_x is completely determined by an ordered pair of edges (α, β) that share a vertex. Hence there are $4 \times C_3^2 = 12$ ways of attaching a pattern of *TN* markers as in Figure 21 to a complete graph on four vertices. On the other hand, there are $(3!)^4 = 1296$ ways to mark the four *Y*-shaped beams that join at the singularity *x*. So, the majority of the four beam patterns will not match with the local geometry of a *Q*-singularity.

Definition 6.2. An *abstract TN-polyhedron* is a simple polyhedron *K* with *T* and *N*-markers along the edges of the graph $s_\dagger(K)$. We insist that, at each vertex *x* whose link is a complete graph in four vertices, these markers satisfy Figure 21 pattern or its mirror image.

Given an abstract *TN*-polyhedron *K*, using the *T*-markers, we can cut it open along $s_\dagger(K)$, so that locally each *T*-marked page is separated from the rest. For the resolution to work, it is important that the *TN* markers at the isolated singularities will obey the combinatorial rules depicted in Figure 21. The result of this *T*-resolution is a compact surface (not necessarily connected) which we denote by $res_T(K)$. It is equipped with the canonical map $\Phi : res_T(K) \to K$.

Definition 6.3. An abstract *TN*-polyhedron *K* is said to be *oriented* if its resolution $res_T(K)$ is an oriented compact surface.

Definition 6.4. An *abstract gradient spine* is an abstract oriented *TN*-polyhedron.

Hence, any generic gradient spine is an abstract gradient spine.

Theorem 6.1. *If two compact* 3*-manifolds* X_1 *and* X_2 *with boundaries have homeomorphic gradient spines* K_1 *and* K_2, *the homeomorphism* $h : K_1 \to K_2$ *being a diffeomorphism along the strata of the spines and respecting the TN-markers*, *then the manifolds are diffeomorphic.*

Proof. The arguments are similar to the ones used in [25], Lemma 1.1.15 and Theorem 1.1.17. Throughout this proof, an "embedding" of a stratified space Z in a smooth manifold means an embedding which is an immersion of the smooth strata of Z and which preserves their transversality.

Let D^3_+ be the half-disk $\{x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}$ with the equatorial disk $D^2 = \{z = 0\}$. Let $\mathcal{Q} \subset D^3_+$ be the union of D^2 with the half disk $D^2_+ = \{y = 0,\ z \geq 0\} \bigcap D^3_+$ with the quater disk $D^2_{++} = \{x = 0, y \geq 0, z \geq 0\}$ $\bigcap D^3_+$. Up to diffeomorphisms of D^3_+ that preserve D^2, there is a unique way to embed the pattern Γ_4 in Figure 21, C, (without the *TN*-markers) in ∂D^3_+ in such a way that the circle in Γ_4 is mapped onto ∂D^2 and the rest of the graph into $\partial D^3_+ \setminus D^2$.

Note that map h must not only respect the stratifications in K_1 and K_2 by the link types in Figure 20, but h also discriminates between the isolated singularities of types O, P, and Q in Figure 13. Indeed, the patterns of *TN*-markers are different for the O-cusps and the P-intersections of free trajectories: in the case of cusps, moving from the singular point y of the link (see Figure 20, pattern (4)) in the N-marked direction one returns to y through the T-marked direction, while in the P-case, leaving y in the N-marked direction results in the return to y through the unmarked direction. Gradient spines of O and P-type singularities admit preferred embeddings in D^3_+. In the case of a cusp,

$D^2 = \partial_1^+ X \cup \partial_1^- X$ and the waterfall is a triangle whose interior resides in $Int(D_+^3)$ and whose two sides are attached to D^2; one of the two sides is realized as the arc $\partial_2^+ X$, along the other side, the triangle is transversal to D^2. Let O be the gradient spine of a cusp, that is, sector $\partial_1^+ X$ union with this triangle. In the case of a P-singularity, the germ of the spine can be identified with $\mathcal{P} := D^2 \cup D_{++}^2 \subset D_+^3$. Note that each of the preferred embeddinds $\mathcal{Q}, \mathcal{O}, \mathcal{P} \subset D_+^3$ admits two distinct N-markings that pick the preferred side of the waterfall (they are mirror images of each other); the T-markings are uniquely determined by the geometry of the embeddings. In what follows, we fix one of the two choices for the N-markers in $\mathcal{Q}, \mathcal{O}, \mathcal{P}$.

For each isolated singularity $x \in ss(K_1)$ and $h(x) \in ss(K_2)$, consider sufficiently small regular neighborhoods $U_x \subset X_1$ and $V_{h(x)} \subset X_2$. Depending on the type of x, both pairs $(U_x, K_1 \cap U_x)$ and $(V_{h(x)}, K_2 \cap V_{h(x)})$ are diffeomorphic to one of the three models $(D_+^3, \mathcal{Q})$, $(D_+^3, \mathcal{O})$, and $(D_+^3, \mathcal{P})$, via the diffeomorphisms which respect the markings. Thus, there exist a diffeomorphism $\alpha : U_x \to V_{h(x)}$ which maps $K_1 \cap U_x$ to $K_2 \cap V_{h(x)}$ and respects the markings. Locally (at x), the composition $h^{-1} \circ \alpha : K_1 \cap U_x \to K_1 \cap U_x$ can be represented by the germ of a vector field w tangent to K_1 (integrating w over a unit of time produces $h^{-1} \circ \alpha$). This field w extends to a field $\hat{w}$, defined in some regular neighborhood of x in U_x. We use $\hat{w}$ to define a germ of a diffeomorphism $\beta : X_1 \to X_1$ at x. Evidently, the germ of $\alpha \circ \beta^{-1}$ is an extension $\hat{h}$ of h into a regular neighborhood of x in X_1. Let $\hat{U}_x$ be a regular neighborhood of x contained in the domain of $\hat{h}$ and $\hat{V}_{h(x)} := \hat{h}(\hat{U}_x)$. Therefore, we have constructed a diffeomorphism $\hat{h} : K_1 \cup (\bigcup_x \hat{U}_x) \to K_2 \cup (\bigcup_x \hat{V}_{h(x)})$ of stratified spaces which preserves their TN-markings.

Let $D^0 := (0, 0, 0)$, $D^1 := D^2 \bigcap D^2_+$, and $D^1_+ := D^2_+ \bigcap D^2_{++}$.

In a similar way, after "fattening" of h in the vicinity of $ss(K_1)$, we can extend $\hat{h}$ into a regular neighborhood $\mathcal{N}(s(K_1))$ of $s(K_1) \subset X_1$, while respecting the markings in the source and the target. To accomplish this we need the following models:

- $\{(D^1 \cup D^1_+) \times [0, 1]\} \cup D^2_+ \times \{0\} \cup D^2_+ \times \{1\} \subset D^2_+ \times [0, 1]$
- $\{(D^1 \cup D^1_+) \times S^1\} \subset D^2_+ \times S^1$
- $D^1_+ \times [0, 1] \cup D^2_+ \times \{0\} \cup D^2_+ \times \{1\} \subset D^2_+ \times [0, 1]$
- $D^1_+ \times S^1 \subset D^2_+ \times S^1$

The first couple models the vicinity of an arc or a loop from $s_\dagger(K_i)$, $(i = 1, 2)$, the second one of an arc or a loop from $s_\bullet(K_i)$.

The third, most problematic, extension of h occurs into a regular neighborhood W of $K_1^\circ := K_1 \setminus \mathcal{N}(s(K_1))$ in $X_1 \setminus \mathcal{N}(s(K_1))$. It is possible because the surfaces from K_i° have preferred normals, which results in the normal bundles $\nu(K_i^\circ, X_i)$ being trivial. This crucial observation is valid due to the gradient nature of the two spines.

Recall that any smooth regular neighborhood of a spine $K \subset X$ is diffeomorphic to X; thus, X_1 is diffeomorphic to X_2. □

Lemma 6.1. *Let $G_4(c)$ denote the number of connected regular four-valent graphs with c vertices, taken up to a homeomorphism. Then the number of connected special abstract gradient spines whose points are of the types* (1), (5), *and* (6) *does not exceed $G_4(c) \cdot 12^c$.*[14] *In turn, $G_4(c)$ can be crudely estimated from above by the number of elements in the symmetric group S_{4c} that move every symbol in* $(1, 2, 3, ..., 4c)$.

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[14]By Theorem 7.3, the same number $G_4(c) \cdot 12^c$ gives an upper bound on the number of irreducible with no essential annuli 3-folds.

Proof. A special spine K, with all its points modeled after types (1), (5), and (6), is completely determined by a regular neighborhood $\mathcal{U}$ of its one-skeleton, a regular 4-valent graph Γ; to reconstruct K from $\mathcal{U}$ we just attach a disk to every circular component of $\partial\mathcal{U}$. There are $G_4(c)$ such graphs Γ. Each edge $\gamma \subset \Gamma$ is a core of a beam $B_\gamma \subset \mathcal{U}$ with a Y-shaped section. Let V be the disjoint union of c copies of the star $V_\star$ of a Q-singularity. With the TN markers on both ends $a, b \in \partial V$ of the beam in place, intrinsically, there is a unique way to attach the beam B_γ to V. Therefore, *any* TN pattern as in Figure 21 assigned to each copy of $V_\star$ in V will determine the rules for attaching the beams, and thus the reconstruction of K. Since V supports 12^c TN-patterns, the total number of special abstract gradient spines does not exceed $G_4(c) \cdot 12^c$.

Fix a graph Γ as above. Then, $\mathcal{U}$ is determined by assigning, for each edge γ of Γ a pairings between the two Y-shaped plugs in V that correspond to γ. For each γ, there are six pairings, so that the total number of non-homeomorphic U's is bounded from above by 6^{2c}. This number $2^{2c} \cdot 3^{2c}$ should be compared with the estimate $12^c = 2^{2c} \cdot 3^c$ of the $\mathcal{U}$'s that are consistent with the abstract gradient spine structure. □

For a simple spine K with K° being a union of orientable surfaces, consider the set of integral 2-chains $C_2^\square(K)$ that are combinations of the fundamental cycles of connected surfaces that form K°, the coefficients in the combinations being ± 1. Also consider the 1-chains $C_1^\square(K)$ that are combinations of the fundamental cycles of arcs that form $s(K)^\circ :=$ $s(K) \setminus ss(K)$, the coefficients in the combinations being ± 1. Following [9], [8] and [2], we introduce the following notion:

Definition 6.5. An oriented branching on a simple complex K with an orientable K° is a 2-chain $\alpha \in C_2^\square(K)$ such that its boundary $\partial\alpha \in$ $C_1^\square(K)$.

In other words, an oriented branching is a special choice of orientations for each of the components of K°; note that, for an arbitrary $\alpha \in C_2^{\square}(K)$, some edges of $s(K)$ can contribute to $\partial\alpha$ with multiplicity ± 3.

Next, we introduce the notion of $\vec{Y}$-*structure* for spines $K \subset X$. It resembles to the notion of branched spines (see Definition 6.6 and Lemma 6.3) and plays a significant role in the sections to follow.

Consider a configuration $\mathcal{Y}$ in $\mathbb{R}^3$ of three distinct half-planes that share a line l. For any nonzero vector $w \in \mathbb{R}^3$ that is not parallel to l, consider a linear surjection $p_w : \mathbb{R}^3 \to \mathbb{R}^2$ with the kernel generated by w. There are two possibilities for the map $p_w : \mathcal{Y} \to \mathbb{R}^2$: (1) generic points in $\mathbb{R}^2$ have preimages of cardinalities one and two, and p_w is onto; or (2) the cardinalities are zero and three (p_v is not onto). We attach symbol $\vec{Y}$ to the first situation and symbol $\vec{W}$ to the second one.[15] The $\vec{Y}$-configurations are generated when w and $-w$ point into distinct chambers in which the three half-planes divide $\mathbb{R}^3$.

At each point $x \in s_\dagger(K)^\circ$ of a simple spine $K \subset X$, the linearization of the three surfaces that join at x generates a configuration $\mathcal{Y}_x$ in the tangent space T_xX of X. At each point $x \in Q(K)$, the linearization of the four surfaces that meet at x generates a 2-complex $\mathcal{X}_x$ which divides T_xX into four pyramids. We will prefer configurations $\mathcal{X}_x$ for which w and $-w$ point into distinct pyramids. Such configurations are said to be of the $\vec{X}$-type. For them, the fibers of p_w will be of cardinality 1, 2, and 3.

Definition 6.6. Let X be an oriented 3-fold. We say that a spine $K \subset X$ is a $\vec{Y}$-*spine* if there exists a vector field w along $s_\dagger(K)$ in X

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[15]The shapes of the letters are mimicking the desired properties of the half-plane configuration with respect to a horizontal w.

which is transversal to each of the surfaces that form K and join along $s_\dagger(K)$. Moreover, for $x \in s_\dagger(K)^\circ$, the configuration $\mathcal{Y}_x \subset T_x X$ is of the $\overrightarrow{Y}$-type, and, for any $x \in Q(K)$, the configuration $\mathcal{X}_x$ is of the $\overrightarrow{X}$-type with respect to $w(x)$. In addition, we require that the global orientations of surfaces in K° will agree with their local orientations induced by w in the vicinity of $s_\dagger(K)$.[16]

For example, consider a union Y of three radii in a disk D^2, Y being symmetric under the rotation ϕ on the angle $2\pi/3$. Let X be the mapping torus of $\phi : D^2 \to D^2$ and K be the mapping torus of $\phi : \mathsf{Y} \to \mathsf{Y}$. Evidently, K is a spine of X, but not a $\overrightarrow{Y}$-spine.

Lemma 6.2. *Any gradient spine $K = K(f, v)$ in X is a $\overrightarrow{Y}$-spine.*

Proof. For a ∂_2^+-generic v, the waterfalls and the ground $\partial_1^+ X$ are transversal along their intersections. At a generic point $x \in s_\dagger(K)$ two out of three half-planes form an angle 180°. Thus, for an open and dense cone of vectors $w(x) \in T_x X$ the configuration $\mathcal{Y}_x$ is of the $\overrightarrow{Y}$-type. Consider the two sheets S_T and S_N of K at x that are marked with the T and N-markers and a cone $\mathcal{C}_x \subset T_x X$—the convex closure of the two half-spaces tangent to S_T and S_N at x. Since the unmarked tangent half-space is never in the interior of $\mathcal{C}_x$, the cone $\mathcal{C}_x$ picks a unique chamber C among the three chambers in which K divides X in the vicinity of $x \in s_\dagger(K)^\circ$.

At each Q-singularity x, K divides the star of x in X into four pyramids. For a gradient spine K, the NT markers pick one of these four pyramids: its triangular base is built out of three edges that are marked

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[16]This requirement is redundant for surfaces whose boundary mets $s_\dagger(K)$ along connected curves.

with *TT*, *NN* and *TN*-markers in Figure 21, diagram C. This choice is consistent with the choice of chambers $\mathcal{C}$ of the four beams that merge at x.

We already noticed that for vectors $w(x) \in \mathcal{C}_x^\circ$ the configuration $\mathcal{Y}_x$ is of the $\vec{Y}$-type. By partition of unity and convexity arguments, we conclude that there is a vector field w along the graph $s_\dagger(K)$ such that w belongs to the *NT*-preferred chamber along $s_\dagger(K)^\circ$ and to the preferred pyramid P at the points of $Q(K) \subset ss(K)$. At the same time, $-w$ does not belong to P. Moreover, w (which has a nonzero projection on the N-marked normal to the waterfalls) extends to a field which is transversal to K along $s_\bullet(K)$ as well. As Figure 12 and Figure 22 testify, w points into the half-spaces picked by the inner normals to $\partial_1^+ X$ which are spread by continuity along the waterfalls. Thus, w locally induces the same orientation of the surfaces in the spine, as the inner normals do. □

Lemma 6.3. *For an oriented X, the notions of an oriented branched spine* $K \subset X$ *and of a* $\vec{Y}$*-spine are equivalent.*

Proof. Assume that K is an oriented branched spine. Since X is oriented, the orientation of each component S of K° picks a particular normal ν_S to S in X. For any $x \in s_\dagger(K)$, an oriented branching α on K, picks a preferred surface S_x out of the three oriented surfaces that join at x. Here is the recipe for choosing S_x : if γ is the edge of $s_\dagger(K)$ through x, locally, there are exactly two surfaces, say S_1 and S_2, such that contributes to ∂S_1 and ∂S_2 with the same sign. Then γ contributes to ∂S_x with the opposite sign.

Let U be a small regular neighborhood of $s_\dagger(K)$ in X, equipped with a Riemannian metric. For some $x \in s_\dagger(K)$, the normals ν_{S_x} and $-\nu_{S_x}$ belong to the same chamber in $X \setminus K$. For those x, we employ an isotopy which is an identity in complement to U and in $S_x \cap U$ to “widen” the angle between S_1 and S_2, so that the ν_{S_x} and $-\nu_{S_x}$ will reside in different chambers. As a result, S_x, S_1, S_2 will form a $\vec{Y}$-configuration with respect to the normal field ν_{S_x}. In a sense, ν_{S_x} is a piece-wise smooth surrogate of the desired field $w(x)$.

By Proposition 3.1.6, [2], any oriented branching extends to $Q(K)$, that is, for any $x \in Q(X)$, the preferred chambers of the four Y-beams that approach x "share" a vector $w_x \in T_x X$ transversal to K. More accurately, the infinitesimal parallel shifts of this w_x along the four edges $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \subset s_\dagger(K)$ that emanate from x belong to the preferred chambers $\mathcal{C}_{\gamma_1}, \mathcal{C}_{\gamma_2}, \mathcal{C}_{\gamma_3}, \mathcal{C}_{\gamma_4}$ of the beams, while the infinitesimal shifts of $-w_x$ do not belong to the preferred chambers. Next, we smoothly interpolate between the parallel shifts of $\{w_x\}_{x \in Q(X)}$ in the vicinity of $Q(X)$ and the fields ν_{S_x} along the edges of $s_\dagger(K)$. This interpolation gives the desired field w along $s(K)$.

Conversely, if K admits a $\vec{Y}$-structure, then K is an oriented branched spine. Indeed, the field w, transversal to K along $s(K)$, picks a particular orientation of each component S of K° along its boundary ∂S. Since S is orientable, according to Definition 6.6, it gets a particular global orientation. Thus, w generates an element $\alpha \in C_2^{\square}(K)$. Because of the very nature of the $\mathcal{Y}_x$-configurations, α has the required property $\partial\alpha \in C_1^{\square}(K)$. □

7. Combinatorial and Gradient Complexities of 3-manifolds

Following Matveev [26], the *complexity* $c(K)$ of a simple 2-polyhedron K is defined to be the cardinality of the set $Q(K)$ formed by points of type (6) in Definition 6.1. This definition of $c(K)$ can be applied to gradient spines as well.

Here and on, by a *trajectory* of a vector field we mean an integral curve that does not admit a continuation.

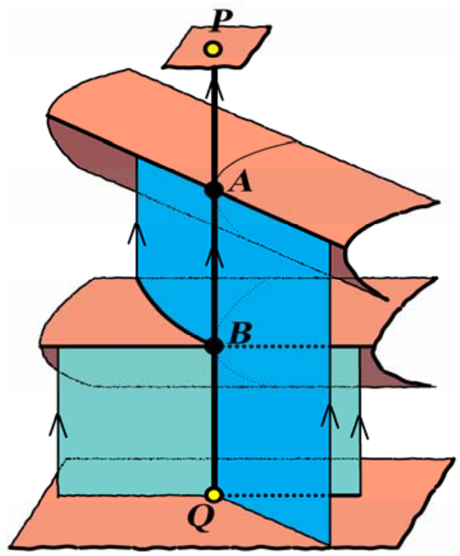

Figure 22. Any v-trajectory QP, tangent to $\partial_1 X$ at A and B, generates a singularity of type (6) at Q.

We notice that the Q-singularities of a gradient spine are in 1-to-1 correspondence with segments $[BQ]$ of the v-trajectories that are shared by either a pair of waterfalls or, locally, by two branches of the same waterfall (see Figure 22). The number of such shared segments in a cascade can be given another, less technical, interpretation. We notice that, for a ∂_2^+-generic field v, a shared segment $[BQ]$ corresponds to a unique pair of distinct points $A, B \in \partial_2^+ X$ that are linked by a v-trajectory $[PQ]$. In turn, such trajectories are exactly the ones that link distinct points $A, B \in \partial_1 X$ and that are *tangent* to $\partial_1 X$ at A and B. Indeed, $\partial_2 X$ is the locus, where v is tangent to $\partial_1 X$, and points of $\partial_2^{-1} X$ do not communicate through the bulk X. We call such trajectories $[PQ]$ *double-tangent*.

Generic Morse data (f, v) provides us with an *oriented tangle* $\mathcal{T}(v) \subset X$ of segments $[AB]$ of double-tangent trajectories, the orientation of $\mathcal{T}(v)$ being induced by v. When $\partial_1^+ X$ is a disk, $\mathcal{T}(v)$ produces a coupling of points in its circular boundary $\partial_2 X$.

Question. How does the tangle $\mathcal{T}(v)$ and the coupling change as v deforms in the space of nonsingular (gradient-like) fields?

Inspired by [25], we propose the following two definitions.

Definition 7.1. The *complexity* $gc(f, v)$ *of generic Morse data* (f, v) is defined to be the number of double-tangent v-trajectories.[17]

For gradient spines, *polarities* $\oplus, \ominus$ can be given to the isolated singularities of Q-type. We have seen that any $x \in Q(K)$ corresponds to a trajectory γ tangent to $\partial_1 X$ at a pair of points $A, B \in \partial_2^+ X$. Morse data (f, v) help to break symmetry between A and B: indeed, $f(A) > f(B)$. Recall that the preferred orientation of $\partial_1^+ X$ induces an orientation of $\partial_2 X$. Consider a vector v_A tangent to $\partial_2 X$ at A and a vector v_B tangent to $\partial_2 X$ at B, the directions of both vectors agreeing with the orientation of $\partial_2 X$. The $(-v)$-flow spreads v_A and v_B and produces an ordered normal frame $(\tilde{v}_B, \tilde{v}_A)$ along the trajectory $\gamma = [P, Q]$. At $x = Q$, the orientation induced by $(\tilde{v}_B(x), \tilde{v}_A(x))$ can agree or disagree with the preferred orientation of $\partial_1^+ X$. In the first case, the polarity of x and is defined to be positive $(\oplus)$, in the second case, it is negative $(\ominus)$. We denote by $Q(K)^{\oplus}$ and $Q(K)^{\ominus}$, respectively, the sets of positively and negatively polarized points in $Q(K)$. The same polarities $\oplus, \ominus$ can be assigned to the double-tangent trajectories γ. Note that reversing the orientation of $\partial_1^+ X$, reverses the orientation of $\partial_2 X$, and thus the frame (v_B, v_A) is replaced by $(-v_B, -v_A)$. Therefore, the $(\oplus, \ominus)$ polarity is independent of the choice of an orientation in $\partial_1 X$.

Definition 7.2. The *polarized gradient complexity* $gc_{\ominus}^{\oplus}(f, v)$ of generic Morse data (f, v) is defined to be the difference between the number of positive and negative double-tangent trajectories.

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[17]By a general position argument, we can assume that each trajectory is tangent to $\partial_1 X$ at two points at most.

The polarized gradient complexity $gc^{\oplus}_{\ominus}(f, v)$ can be given another interpretation which has the flavor of a "self-linking number" for the 1-cycle $\partial[K]$.

By Lemma 6.2, the gradient spine $K = K(f, v)$ has a preferred vector field w along $s(K)$ which gives K its $\vec{Y}$-structure. This field is transversal to K and points inside X along $\partial_1^+ X$ and into the preferred side of each waterfall from the cascade.

We view the 1-cycle $\partial[K]$ with support in $s(K)$ as an oriented graph. Employing a preferred field w, we push $\partial[K]$ a bit in the direction of the field. Denote $\partial[K]_w$ the perturbed graph. By the construction of w, the 2-chain $[K]$ and the 1-cycle $\partial[K]_w$ are in a general position in X, and their intersection points occur only in the vicinity of $Q(K)$, a single intersection per each point of $Q(K)$. Therefore, $gc(f, v) := c(K) = \#(\partial[K]_w \cap [K])$. We claim that an algebraic count $\partial[K]_w \circ [K]$ of intersection points in $\partial[K]_w \cap [K]$ also makes sense[18], provided that X is oriented. Let us explain this observation. Each vertex of the oriented graph $\partial[K]$ whose multiplicity > 2 is a Q-singularity. It has valency four, a pair of incoming, and a pair of outgoing edges. Therefore, at each $x \in Q(K)$, there are exactly two oriented resolutions of the graph $\partial[K]$ into a pair of arcs. One of the resolutions is the boundary of the resolved surface $res_T(K) \subset X$ (see Figure 18, the right diagram). The other resolution of $\partial[K]$ is denoted by $\overline{res}(\partial[K])$. We denote by $\overline{res}(\partial[K])_w$ its w-shift. Consider the algebraic intersection $\overline{res}(\partial[K])_w \circ res_T(K)$ of the curve $\overline{res}(\partial[K])_w$ with the surface $res_T(K)$. It is easy to see that the points of $\overline{res}(\partial[K])_w \circ res_T(K)$ and of $\partial[K]_w \cap [K]$ are in 1-to-1 correspondence. *I* requires more effort to check that the standard orientation assigned to

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[18]Recall that intersection theories on spaces with singularities of codimension one fail miserably.

each point of $\overline{res}(\partial[K])_w \cap res_T(K)$ is positive if and only if the corresponding singularity $x \in Q(K)$ has positive polarity $\oplus$. Since the preferred sides of the surfaces from K that join at x depend only on Morse data (and not on the orientation of $\partial_1 X$) and so does the choice of w, one needs to consider only the interaction of a particular w-shift with the eight choices for the orientations of the two waterfalls and the ground. We leave to the reader the rest of the verification.

Define the self-linking number $lk_w(\partial[K], \partial[K])$ by the formula $\overline{res}(\partial[K])_w \circ res_T(K)$. By standard homological reasoning, $lk_w(\partial[K], \partial[K])$ depends only on the graph $\partial[K]$ and its preferred framing w (which, in turn, depends only on the marked germ of $s(K)$ in X).

Combining Definitions 7.1 and 7.2 with the arguments above, we get

Theorem 7.1. *For generic Morse data* (f, v) *and its gradient spine* $K = K(f, v)$ *carrying the preferred framing* w *along* $s(K)$,

$$gc(f, v) = \#(\partial[K]_w \cap [K]), \tag{7.1}$$

$$gc_{\ominus}^{\oplus}(f, v) = lk_w(\partial[K], \partial[K]). \tag{7.2}$$

Hence, the number of double-tangent v*-trajectories is at least* $|\, lk_w(\partial[K], \partial[K])\,|$.

We can refine the gradient complexity and view it as an ordered *pair* of nonnegative integers $(\#[Q(K)^{\oplus}], \#[Q(K)^{\ominus}])$, where $K = K(f, v)$. We will see that once a pair $(c_{\oplus}, c_{\ominus})$ has been realized by some Morse data, then $(c_{\oplus} + 1, c_{\ominus} + 1)$ can be realized as well (see Figure 31).

Contemplating about formula (7.2), we realize that it makes perfect sense for any simple $\vec{Y}$-spine $K \subset X$. This leads to

Definition 7.3. Let X be a compact 3-fold with boundary. Its $\vec{Y}$*-complexity*, $c_{\vec{Y}}(X)$, is the minimal combinatorial complexity $c(K)$ of $\vec{Y}$-spines K in X.

Alternatively, $c_{\vec{Y}}(X)$ can be defined as $\min_{\{K,w\}} \#(\partial[K]_w \cap [K])$, where (K, w) runs over the set of $\vec{Y}$-spines $K \subset X$ equipped with their preferred fields w.

Definition 7.4. Let X be a compact 3-fold with boundary. Its *gradient complexity*, $gc(X)$, is the minimum, over all nonsingular ∂_2^+-generic gradient-like fields v, of the number of double-tangent v-trajectories.

Alternatively, we can define $gc(X)$ as $\min_{\{K,w\}} \#(\partial[K]_w \cap [K])$, where $K = K(f, v)$ runs over generic gradient spines equipped with their preferred fields w.

Evidently, $gc(X) \geq c_{\vec{Y}}(X) \geq c(X)$. Our Theorem 8.1 implies that actually $gc(X) = c_{\vec{Y}}(X)$. In general, $gc(X) > c(X)$. For example, for the punctured lens space $\overset{\circ}{L}_{3,1}$, one gets $c(\overset{\circ}{L}_{3,1}) = 0$, while $gc(\overset{\circ}{L}_{3,1}) \geq 2$.

Since, for any handlebody X and appropriate Morse data, $\partial_2^+ X = \emptyset$, it follows that the gradient complexity of handlebodies equals to zero.

In general, computing $c(X)$ is hard; it is much easier to estimate it from above or below. For instance, if X admits a triangulation comprising n tetrahedrons, then $c(X) \leq n$ ([25], Proposition 2.1.6).

An embedded surface $\Sigma \subset X$ is called *proper*, if $\partial \Sigma = \Sigma \cap \partial_1 X$. Recall the X is *irreducible* if any embedded sphere $S^2 \subset X$ bounds a 3-ball; in such a case S^2 is called *inessential*. X is called *prime* if no essential embedded 2-sphere separates it[19]. We say that X is *boundary irreducible* if, for any proper 2-disk $(D, \partial D) \subset (X, \partial X)$, there is a 3-ball $B \subset X$ so that its boundary ∂B consists of D and the complementary disk $\partial B \setminus D \subset \partial_1 X$. A proper annulus $(A, \partial A) \subset (X, \partial X)$ is called *nonessential* if either A is parallel to $\partial_1 X$ or the core loop of A is contractible in X. The rest of annuli are called *essential*.

[19] $S^2 \times S^1$ is prime.

Note that if a 3-ball is deleted from an irreducible 3-fold, then the resulting manifold with boundary is boundary irreducible and has no essential annuli.

For boundary irreducible with no essential annuli 3-folds, we can reduce generic $\vec{Y}$-spines to special ones without compromising their combinatorial complexity (cf. Theorem 2.2.4 in [25]).

Theorem 7.2. *Let X be a boundary irreducible with no essential annuli 3-fold $(\partial_1 X \neq \emptyset)$, and let $K \subset X$ be a simple spine such that K° is orientable. Then X has another simple spine K' such that*

- *$K' \setminus s(K')$ is a collection of 2-disks,*
- *$s(K')$ is a connected graph whose vertices are of multiplicity 1 and 4,*
- *$c(K') \leq c(K)$,*
- *when K is a $\vec{Y}$-spine, so is K',*
- *when K is an abstract gradient spine, so is K'.*

If $c(K) > 0$, then K' is a special spine.

Proof. Recall that X is PL-homeomorphic to the mapping cylinder of a cellular map $q : \partial_1 X \to K$. Denote by $r : X \to K$ the retraction induced by the q-mapping cylinder structure. In fact, one can choose r so that: (1) for any $x \in K^\circ = K \setminus s(K)$, $r^{-1}(x)$ is homeomorphic to a closed interval, and (2) for any $x \in K \setminus s_\dagger(K)$, $r^{-1}(x)$ is homeomorphic either to a closed interval or to a singleton.

Pick a regular open neighborhood $\mathcal{N}$ of $s_\dagger(K) \setminus ss_{\bullet\dagger}(K)$ in K so that the singularities with the links of types 3 and 4 from Figure 20[20] are excluded from $\mathcal{N}$. Let S be a connected component of the compact surface $K \setminus \mathcal{N}$. The boundary ∂S of S consists of the set $\partial_\dagger S = \partial(\mathcal{N}) \cap S$

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[20]For the gradient spines, these are the cusps from $\partial_3^+ X$ and the loci where the trajectories through $\partial_3^- X$ hit $\partial_1^+ X$.

and its complement $\partial_\bullet S = \overline{\partial S \setminus \partial_\dagger S}$, each of the two sets being a union of simple arcs and loops.

When S is a closed surface, $K = S$ and X is a segment bundle over S.

Consider a simple closed path $\gamma \subset S$ which is not contractible in S and such that the restriction of the normal line bundle $\nu(S, X)|_\gamma$ is trivial (for gradient spines, $\nu|_\gamma$ is automatically trivial). In fact, such a loop γ always exists, unless $S = D^2, S^2$, or $\mathbb{R}P^2$. Since S is orientable, D^2 and S^2 are the only exceptions. If $S = S^2$, then $s(K) = \emptyset$. In such a case, $K = S^2$ and $X = S^2 \times [0, 1]$.

Denote by A the annulus $r^{-1}(\gamma)$. Since no essential annuli are permitted, either (1) A is parallel to the boundary $\partial_1 X$, or (2) γ is contractible in X.

In the first case, γ must divide S. Indeed, if it does not, we can find a loop $\delta \subset S$ which intersects with A at a singleton; this will imply that A is essential. Moreover, by the definition of A being parallel to $\partial_1 X$, there is a solid torus $T \subset X$ whose boundary is divided into A and the complementary annulus $A' \subset \partial_1 X$. As we delete T from X, we do not change the topological type of X but do change K to a new $K' = K \setminus (K \cap T^\circ)$. Again, due to the construction of A, deleting T preserves the r-induced product structure in $X \setminus K$, and thus K' is a spine. Now $\gamma \subset s_\bullet(K') = K' \cap \partial_1 X$ (so that each point of γ has a link in K' of type 2 from Figure 20).

Next, consider the case when γ is nullhomotopic in X. Then, by Dehn's Lemma, each of the two loops γ_1 and γ_2 comprising the boundary of A and residing in $\partial_1 X$ bounds a disk in X. Since X is boundary irreducible, γ_1 bounds a disk $D_1 \subset \partial_1 X$ and γ_2 bounds a disk $D_2 \subset \partial_1 X$. Push D_2 slightly inside X so that the loop ∂D_2 slides along A and denote by D_2' the pushed disk. Consider the proper 2-disk $D \subset X$ formed by D_2' and the portion of A between D_2' and D_1. Its boundary is ∂D_1. Note that $D \bigcap K = \gamma$. Recalling that X is boundary irreducible, we conclude that D bounds a 3-ball $B \subset X$ whose boundary is $D \cup D_1$. Thus γ must divide S into $S'' := B \bigcap S$ and its complement $S' := (X \setminus B^\circ) \bigcap S$, and K into $K'' := B \bigcap K$ and its complement $K' := (X \setminus B^\circ) \bigcap K$. Denote by A' the subannulus of A bounded by γ and $\partial D_2'$. We notice that $K^\sharp := K' \cup A' \cup D_2'$ is a spine of $X \setminus B^\circ$. In fact, we have replaced S by a new component $S^\sharp := S' \cup A' \cup D_2'$ in which γ is contractible.

We already noticed that any closed loop γ in S with $\nu|_\gamma$ being trivial, can be a source for one of the two previous spine modifications. By modifications of the second type, we can insure that each S is incompressible in X. Then, by modifications of the first type, we can make sure that all the nontrivial homotopy classes of simple loops in S will be represented in $\partial_\bullet S$ and thus will be disjoint from $s_\dagger(S)$. Therefore, we can assume that S has no handles, and any nontrivial loop in S is homotopic to a loop contained in $\partial_\bullet S$ (unless $S = S^2$ and $X = S^2 \times [0, 1]$). Thus, S must be either (1) a disk, or (2) a disk with a number of holes.

Next, we claim that any simple path with $\partial\gamma \subset \partial_\bullet S$ must separate S. Assume to the contrary that such non-separating γ exists. Since $r^{-1}(x)$ is a segment or a singleton for all $x \in S$, $r^{-1}(\gamma)$ is a disk $D \subset X$ whose boundary $\partial D \subset \partial_1 X$. Since X is boundary irreducible, there exists a 3-ball $B \subset X$ whose boundary ∂B comprises D union with another disk $D' \subset \partial_1 X$. The intersection $K \bigcap B$ bounds γ (on one side), contrary to the hypothesis about γ. Therefore, $\partial_\bullet S$ is present in no more than one loop from ∂S : otherwise a non-separating path γ as above must exist. On the other hand, we have already arranged for any non-trivial loop in S to be homotopic to a loop from $\partial_\bullet S$. Thus, S is either an annulus, or a disk.

If S is an annulus, one of its boundary loops must be in $\partial_\bullet S$ and the other one in $\partial_\dagger S$. So we can collapse S onto $\partial_\dagger S$ and further onto $s_\dagger(K)$, thus simplifying K. Therefore, we managed to construct a spine K with all the components S being homeomorphic to a disk. Some of these disks S could have the property $\partial_\bullet S \neq \emptyset$, in which case they also can be collapsed, further simplifying the spine.

The moment we arranged for K° to be a disjoint union of 2-disks, the graph $s(K)$ becomes connected. Suppose, to the contrary, that $s(K) = s'(K)\coprod s''(K)$, where $s'(K)$, $s''(K) \neq \emptyset$. Consider regular neighborhoods $\mathcal{N}'$ and $\mathcal{N}''$ of $s'(K)$ and $s''(K)$ in K, respectively. Then the boundaries $\partial\mathcal{N}'$ and $\partial\mathcal{N}''$ each is a disjoint union of circles. In order to form K, we attach disks to $\partial\mathcal{N}'$ and $\partial\mathcal{N}''$; however this will lead to a disconnected K, clearly a contradiction.

The vertices of $s(K)$ are isolated singularities of K of types (3), (4), and (6) from Figure 20. Types (3) and (4) have a single edge of $s_\dagger(K)$ of type (5) and at least one free edge of type (2) that terminates there; type (6) is a four-valent vertex. After all the collapses, the free edges will disappear and $s(K) = s_\dagger(K)$ will become a regular four-valent graph.

Let us examine how the spine modifications above have affected the given abstract gradient or $\vec{Y}$-structures of the original spine. Let w be a preferred vector field along $s(K)$. Note that all we did amounts to deleting from X a number of relative 3-balls B and solid tori T. In all cases, but one, the effect on a spine K was deleting its portion $K \cap B$ or $K \cap T$. Evidently, these operations neither increase the complexity of K, nor destroy the orientations of S and the TN markers (in the case when K is an abstract gradient spine). In the case of $\vec{Y}$-spines, the cuts do not change the fact that w along $s(K)$ is transversal to each surface S and that the configurations $\mathcal{Y}_x \subset T_xX$ are of the $\vec{Y}$-type with respect to v. The only less trivial case occurred when we replaced K with $K^\sharp := K' \cup A' \cup D_2'$, but again, the procedure could only eliminate a portion of $s(K)$ which was *disjoint* from the rest and thus did not affected the transversality of w to S or the *TN* markers. Note that the orientation of S' uniquely spreads over the cup $A' \cup D_2'$.

The only simplifying move that could harm an abstract gradient structure of a given spine is the collapse of some of the disks or annuli S in the very end of the game. So, if we want to keep the abstract gradient structure of K, we should stop there. □

An important warning. Note that elementary expansions and collapses of abstract gradient (or oriented branched) spines K are very different from the elementary expansions and collapses of K, viewed just as the underlying 2-dimensional complexes. The orientations can prevent us from executing some collapses which non-oriented complexes would support. Also, we need to pay close attention to the choice of *NT* markers, an integral part of the abstract gradient complex structure.

For example, take a plane on which a circular fence is erected. The fence divides the plane into two domains, the disk and its exterior. If the orientations of the two domains disagree, we cannot collapse the fence.

Because the *T*-resolution of an abstract gradient spine leads to an oriented surface, we can always collapse a fence marked with *T*. In fact, an unmarked fence also can be collapsed.

Our Theorem 8.2 claims that $gc(X) = c_{\bar{Y}}(X)$. In any case, the obvious inequality $gc(X) \geq c(X)$ can be combined with a number of results about $c(X)$ in order to get a lower bound on the number of double-tangent trajectories.

Matveev proved that, up to a homeomorphism, there are only finitely many compact boundary irreducible 3-folds X that have no essential proper annuli and with a bounded combinatorial complexity $c(X)$ (see [25], Theorem 2.2.5). We can be a bit more specific:

Theorem 7.3. *Let X be a compact 3-fold X with $\partial_1 X \neq \emptyset$. For any nonsingular ∂_2^+-generic gradient-like field v, the number of double-tangent v-trajectories is greater or equal to $c(X)$.*

The number of topological types of boundary irreducible X with no essential annuli and gradient complexity $gc(X) = c$ *does not exceed* $\Gamma_4(c)\cdot 12^c$, *where* $\Gamma_4(c)$ *is the number of topological types of four-valent connected graphs with c vertices at most*[21]. *In particular, there are no more than* $\Gamma_4(c)\cdot 12^c$ *distinct orientable hyperbolic* 3*-folds X with* $gc(X) = c$.

Proof. Since we established that gradient spines are special kind of spines and in view of the arguments centered on Figure 22, the first claim is clear.

In order to prove the second claim, consider a gradient spine $K(f, v) \subset X$ with $gc(f, v) = gc(X) = c$. Then by Theorem 7.2, we can simplify $K(f, v)$ to a special abstract gradient spine K with c Q-singularities at most. Examining the constructions that lead to the proof of Theorem 7.2, we see that deleting 3-disks and solid tori adjacent to $\partial_1 X$, did not change the combinatorics of the special abstract gradient spine in the vicinity of the remaining Q-singularities (as depicted in Figure 21). After collapsing all its 2-cells whose boundary touches the topological boundary $s_\bullet(K)$ of K, we could eliminate some of the Q-singularities and transform the Y-beams that connect them into I-shaped ones. We conclude that, in the end, $s_\dagger(K)$ must be a connected regular 4-valent graph with c vertices at most. By Lemma 6.1, the number of such abstract gradient spines has a upper boundary $\Gamma_4(c)\cdot 12^c$. By Lemma 1.1.15 in [25], any special simple spine $K \subset X$ determines the topological type of its ambient X. □

The v-flow through the "bulk" X and the v_1-flow trough its boundary $\partial_1 X$ are intimately linked. For instance, we get the following proposition:

Corollary 7.1. *Let* (f, v) *and* $(f|_{\partial_1 X}, v_1)$ *be generic Morse data on X and* $\partial_1 X$, *respectively, and* $v \neq 0$. *If there is no ascending* v_1*-trajectory*

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[21]The number of *labeled* regular four-valent graphs with c vertices is less than $(4c-1)!!$ where $k!!$ denotes the product of all odd numbers that do not exceed k.

that connects in $\partial_1 X$ *a point of* Σ_1^- *to a point of* Σ_1^+, *then* v *has no double-tangent trajectories*; *in other words*, $gc(X) = 0$.

On the other hand, $c(X) \neq 0$ implies that, for each nonsingular f, there is an ascending v_1-trajectory that connects in $\partial_1 X$ a point of Σ_1^- to a point of Σ_1^+.

Proof. By Theorem 4.1, the absence of an ascending v_1-trajectory γ that connects in $\partial_1 X$ a point in Σ_1^- to a point in Σ_1^+ implies covexity of the Morse data. On the other hand, if $\partial_2^+ X = \emptyset$, then no v-trajectory links a pair of points $x, y \in \partial_1 X$ and is tangent to $\partial_1 X$ at x and y. □

Given manifolds X' and X'' with boundary, two types of connected sum operations are available: $X' \# X''$ and $X' \#_\partial X''$. In the first construction, a 1-handle is attached to the interiors of X' and X''; in the second one, a 1-handle is attached so that the boundary $\partial_1 X' \coprod \partial_1 X''$ is subjected to 1-surgery as well.

Theorem 7.4. *The Morse complexity is a semi-additive invariant*:

$$gc(X' \# X'') \leq gc(X') + gc(X''),$$

$$gc(X' \#_\partial X'') \leq gc(X') + gc(X'').$$

In particular, *attaching a solid* 1-*handle to the interior of* X *or to its boundary does not increase its gradient complexity*. *Also deleting a* 3-*ball from the interior of* X *does not change its gradient complexity*. *Therefore*, *only* 2-*surgery on* X *has the potential to increase its gradient complexity*.

Proof. The semi-additivity $gc(X' \#_\partial X'') \leq gc(X') + gc(X'')$ is easy to validate. Let (f', v') delivers $gc(X')$ and (f'', v'') delivers $gc(X'')$, where X', X'' are 3-folds with boundary. Assume that f'' attains its minimum at $a \in \partial_1^+ X''$ and f' attains its maximum at $b \in \partial_1^- X'$. By adding a positive constant to f'', we may assume that $\min(f'') > \max(f')$. By perturbing v'' a bit, we can arrange that the cascade in X'' has an empty intersection with a small disk $D_a^2 \subset \partial_1^+ X''$, centered at a, without changing the original topology of the gradient spine K''. Similarly, by perturbing v' if necessary, we can pick a sufficiently small disk $D^2_b \subset \partial_1^- X'$ so that the $-v'$ trajectories through D^2_b do not intersect the cascade in X'. Then we attach an one-handle $H \approx D^2 \times [0, 1]$ to $X' \cup X''$ at a and b and extend the Morse data from the top disk D^2_a and the bottom disk D^2_b inside H. The neck $\gamma := \partial D^2 \times 1/2$ belongs to the set $\partial_2^+(X' \# X'')$ and the annulus $\partial D^2 \times [1/2, 1]$ to the set $\partial_1^+(X' \#_\partial X'')$. Thus, the cylindrical waterfall streaming from γ does not intersect with the cascade in X' and hits $\partial_1^+ X'$ without producing new shared trajectories. By the choice of D^2_a, the cascade in X'' does not fall through H. Thus $gc(X' \#_\partial X'') \leq gc(X') + gc(X'')$.

Deleting a 3-ball B from the interior of X does not change its gradient complexity. Indeed, if (f, v) delivers $gc(X)$, then we pick a sufficiently small ball B whose center lies on a v-trajectory γ that is not in the cascade. Then ∂B is concave with respect to v, and the trajectories tangent to ∂B are separated from the old cascade. As a result, they do not contribute to the set of double-tangent trajectories of v in $X \setminus B$. Hence $gc(X \setminus B) \leq gc(X)$. On the other hand, if (f, v) delivers $gc(X \setminus B)$, then there is a trajectory which connects a point $x \in \partial B$ to a point in $y \in \partial_1 X$ and is transversal at x and y to the boundary of $X \setminus B$. Drilling a narrow tunnel $W \subset X \setminus B$ centered on γ and with a concave bottleneck produces a manifold $\hat{X}$ homeomorphic to X (we need to smoothen $\hat{X}$ at both ends of the tunnel). This can be done in such a way that $X \setminus B$ and $\hat{X}$ will share the same cascade. Hence, $gc(X \setminus B) \geq gc(\hat{X}) = gc(X)$.

Now $X' \# X''$ can be obtained by the following operations: (1) deleting 3-balls B', B'' from X' and X'', respectively, (2) forming a connected sum $(X' \setminus B') \#_\partial (X'' \setminus B'')$ by attaching a 1-handle $H = D^2 \times [0, 1] \subset S^2 \times [0, 1]$ to $\partial B' \cup \partial B''$, and (3) by attaching a 3-ball to the boundary of

$(S^2 \times [0, 1])\backslash H.$ By the previous arguments, these operations lead to the inequality $gc(X' \# X'') \leq gc(X') + gc(X'').$

Let T denote a solid torus. Since $gc(T) = 0,$ we have $gc(X \#_{\partial} T) \leq gc(X)$ and $gc(X \# T) \leq gc(X).$ Therefore, only 2-surgery on X has the potential to increase its gradient complexity. □

To establish the additivity of $gc(X)$ seems to be much harder. The additivity of $c(X)$ is a nontrivial fact which relies on Haken's theory of normal surfaces ([25], Theorem 2.2.9). We notice that the defect $gc(X) - c(X)$ is also semi-additive under the connected sum operation.

Corollary 7.2. *For any non-negative integer n, there exists a gradient-like flow on X with $gc(X) + n$ double-tangent trajectories.*

Proof. First we notice that the ball $D^3 = \{x^2 + y^2 + z^2 \leq 1\}$ admits a gradient flow with a single double-tangent trajectory. Indeed, start with convex non-singular function $z : D^3 \to \mathbb{R}$ and form two indents in D^3: the first one being formed by inserting in D^3 the half-plane $P_1 = \{x \geq 0, z = 0.5\},$ the second one by inserting the half-plane $P_2 = \{y \geq 0, z = -0.5\}$ (four cusps are formed as a result of this deformation). More accurately, one deletes the parabolic cylinders $\{(z - 0.5)^2 < \varepsilon x\}$ and $\{(z + 0.5)^2 < \varepsilon y\}$ (ε being sufficiently small) from D^3 and then smoothens the corners. Next, use the Morse data (f, v) which deliver $gc(X)$ and form a connected sum $X \#_{\partial} D^3,$ equipped with the new Morse data as in the proof of Theorem 7.4. Evidently, the number of new double-tangent trajectories in $X \approx X \#_{\partial} D^3$ is $gc(X) + 1.$ Recycling this construction proves the claim. □

Each time we have a lower bound on the combinatorial complexity $c(X),$ for any gradient-like nonsingular flow, the number of double-trajectories must be at least $c(X)$ (Theorem 7.3). Fortunately, [25] contains a complete list of punctured closed irreducible 3-manifolds of combinatorial complexity $\leq 6.$ To assemble this list is a labor-intense accomplishment. For example, using computations in [25], pp. 77 and

407-408, for *punctured* elliptic manifolds we get: $gc(S^3/P^{\circ}_{24}) \geq 4$, $gc(S^3/P^{\circ}_{48}) \geq 5$, $gc(S^3/P^{\circ}_{120}) \geq 5$. In fact, for $2 \leq n \leq 6$, $gc(S^3/Q^{\circ}_{4n}) \geq n$. For punctured lense spaces $L^{\circ}_{p,q}$, one has: $gc(L^{\circ}_{4,1}) \geq 1$, $gc(L^{\circ}_{5,2}) \geq 1$; $gc(L^{\circ}_{5,1}) \geq 2$, $gc(L^{\circ}_{7,2}) \geq 2$, $gc(L^{\circ}_{8,3}) \geq 2$; $gc(L^{\circ}_{6,1}) \geq 3$, $gc(L^{\circ}_{9,2}) \geq 3$, $gc(L^{\circ}_{10,3}) \geq 3$, $gc(L^{\circ}_{11,3}) \geq 3$, $gc(L^{\circ}_{12,5}) \geq 3$, $gc(L^{\circ}_{13,5}) \geq 3$, etc.

Reinterpreting Theorem 2.6.2 in [25] (see also [26]), one gets a lower homological bound on the number of double-tangent gradient-like trajectories in manifolds with simply-connected boundary.

Corollary 7.3. *Let X be a 3-fold obtained from a closed irreducible and orientable 3-fold Y, different from the lens space $L_{3,1}$, by removing a number of 3-balls. Then any generic gradient-like nonsingular flow on X has at least*

$$2 \cdot \log_5 |\, Tor(H_1(X;\, \mathbb{Z}))\,| + rk(H_1(X;\, \mathbb{Z})) - 1$$

double-tangent trajectories. Here $|\, Tor(H_1(X;\, \mathbb{Z}))\,|$ denotes the order of the torsion subgroup $Tor(H_1(X;\, \mathbb{Z})) \subset H_1(X;\, \mathbb{Z})$.

For hyperbolic X, both $c(X)$ and $gc(X)$ exhibit at least linear growth as functions of the hyperbolic volume.

Theorem 7.5. *Let X be a compact 3-fold obtained from a closed hyperbolic 3-fold Y by removing a number of 3-balls. Let $V(Y)$ denote the hyperbolic volume of Y, and V_0 the volume of a regular ideal tetrahedron in the hyperbolic space $\mathbf{H}^3$. Then, any generic gradient-like nonsingular flow v on X has at least $V(Y)/V_0$ double-tangent trajectories.*

Proof. The statement follows from [25], Lemma 2.6.7 and Corollary 2.6.8, coupled with the inequality $c(X) \leq gc(X)$. □

Let G be a finitely presented group. The *length of a relation* is the number of generators and their inverses that are present in the relation. The *complexity of a presentation* is defined to be the sum of lengths of all its relations (for example, the complexity of the presentation in Figure

19 is $3 + 2 + 2 = 7$). *Presentation complexity* $c(G)$ is the minimum complexity among all G-presentations.

Let us return to the description of the presentation of $\pi_1(X/\partial_1^+ X)$ given by a generic gradient cascade and described prior to Theorem 5.3. Its generators are shared segments of trajectories of the cascade together with some free trajectories of waterfalls that have two free trajectories and no shared segments at all. Each waterfall of this kind contributes a single "free" generator and no relations. We notice that each shared segment in a cascade belongs to three polygons (see Figure 10). Therefore, it is present in the relations three times at most.

As a result, the complexity of the presentation of $\pi_1(X/\partial_1^+ X)$ induced by a cascade does not exceed three times the number of shared segments of the v-trajectories.

This leads to the following analogue of Proposition 2.6.6 from [25] which claims $c(X) \geq -1 + 1/3 \cdot c(\pi_1(X))$ for any closed irreducible orientable 3-fold X, different from S^3, $\mathbb{R}P^3$, and $L_{1,3}$. In fact, we can drop the assumption of irreducibility of X:

Corollary 7.4. *For any* 3*-fold X and generic Morse data* (f, v),

- $gc(f, v) \geq 1/3 \cdot c(\pi_1(X/\partial_1^+ X))$;
- *for any generic Morse data with a disk-shaped* $\partial_1^+ X$,

$$gc(f, v) \geq 1/3 \cdot c(\pi_1(X));$$

- *for any* 3*-fold X whose boundary is a union of spheres*,

$$gc(X) \geq 1/3 \cdot c(\pi_1(X)).$$

Proof. In view of the discussion above, we need to clarify only the statement in the last bullet. Since the image $\pi_1(\partial_1^+ X) \to \pi_1(\partial_1 X) \to \pi_1(X)$ is trivial, attaching a cone with the base $\partial_1^+ X$ to X can only add new free generators to a representation of $\pi_1(X/\partial_1^+ X)$ given by an optimal (that is, $gc(X)$-realizing) cascade. Hence $\pi_1(X)$ and $\pi_1(X/\partial_1^+ X)$ share the same set of relations which implies that $gc(X) \geq c(\pi_1(X))/3$. □

Each self-indexing Morse function on a closed manifold Y provides an upper bound for $gc(X)$, where X is obtained from Y by removing a number of 3-balls.

Let Y be a closed 3-fold with a self-indexing Morse function $h : Y \to \mathbb{R}$ which has a single minimum. Then h and its gradient-like field v give rise to a presentation $\mathcal{P}_v$ of $\pi_1(Y)$: its generators are in 1-to-1 correspondence with the h-critical points x of index one and its relations are generated by some of the h-critical points y of index two. Recall that, for all x, y, we have $h(x) = 1$ and $h(y) = 2$. Denote by D_x the ascending 2-manifold of x and by D_y the descending 2-manifold of y. Each relation $R_y = 1$ is obtained by moving along the boundary ∂_y of the unstable disk $h^{-1}[1.5, 2] \cap D_y$ and recording oriented transversal intersections of the loop ∂_y with the disks $\{h^{-1}[1, 1.5] \cap D_x\}_x$ (in the surface $h^{-1}(1.5)$) in the order they appear along ∂_y. This generates a word R_y in the alphabet $\{x, x^{-1}\}$ and thus a presentation $\mathcal{P}_v$.

Consider the number $c_M(y) = \min_{\{h, v\}} length(\mathcal{P}_v)$, the minimum being taken over all pairs (h, v) as above. Evidently, $c_M(Y) \geq c(\pi_1(Y))$.

Theorem 7.6. *Let X be a 3-fold obtained from a closed manifold Y by removing a number of 3-balls. Then $1/3 \cdot c(\pi_1(X)) \leq gc(X) \leq 4 \cdot c_M(Y)$.*

Proof. In view of Corollary 7.4, we need to validate only the second inequality. Consider (h, v) as above. Delete from Y some balls centered on the critical points of h. Their size is picked so that, in the vicinity of each critical point of indices 1 and 2, the variation of h in the balls is less than $\varepsilon < 0.5$. Also we assume that, in the balls, h admits a canonical Morse form. Denote by X the complement to the balls in Y and restrict (h, v) to X. Then each critical point x of index 1 contributes an "equatorial" annulus A_x, and each critical point y of index 2 contributes a pair of "polar" disks $B_y^{\pm}$ to the set $\partial_1^+ X$. The boundaries of A_x and $B_y^{\pm}$ form the set $\partial_2^+ X$. These observations are based on the quadratic nature of h in the Morse coordinates. Moreover, $\partial_1^+ X.$ consists of $\{A_x\}_x$ and $\{B_y^{\pm}\}_y$ together with the sphere $S = h^{-1}(1 - \varepsilon)$ centered on the point $h^{-1}(0)$ of the absolute minimum 0. The cascade $\mathcal{C}(h, v)$ falls from the union of loops $(\bigcup_x \partial(A_x)) \bigcup (\bigcup_y \partial(B_y^{\pm}))$ on the "ground" S. Note that this particular choice of Morse data (h, v) on X has the property $\partial_3 X = \varnothing$ and thus are 3-*convex*!

Denote by $W_x^{\pm}$ the two waterfalls from ∂A_x. By choosing ϵ small enough we insure that each of the two waterfalls $W_y^{\pm}$ from $\partial B_y^{\pm}$ follow the unstable manifold D_y very closely. In particular, we make sure that, for each x, the two loops $\gamma^{\pm} := cl\{W_y^{\pm} \bigcap [S \cup A_x \cup W_x^{\pm}]\}$, are "parallel" and close to the loop $\gamma := cl\{D_y \bigcap [S \cup A_x \cup W_x^{\pm}]\}$, where "$cl$" stands for the closure. Again, by choosing a small ε, each intersection point from $D_y \bigcap h^{-1}(1 + \varepsilon) \bigcap D_x$ corresponds to a unique point in $\gamma \bigcap D_x$, the corresponding points acquiring similar orientations. Thus, each point from $D_y \bigcap h^{-1}(1 + \varepsilon) \bigcap D_x$ corresponds to four points in $\gamma^{\pm} \bigcap S \bigcap W_x^{\pm}$. These 4-tupples are exactly the Q-singularities of the spine in X generated by (h, v).

Therefore $gc(h, v) = 4 \cdot \#\{(\bigcup_x D_x) \bigcap h^{-1}(1 + \varepsilon) \bigcap (\bigcup_y D_y)\}$—four times the length of the representation $\mathcal{P}_v$. Employing Theorem 7.4, we conclude that the same holds for any other manifold $\widetilde{X}$ obtained from Y by deleting any positive number of disks.

Note that locally the picture of the spine is symmetric with respect to the planes of D_x (and D_y), so that the Q-singularities of the spine occur in pairs of *opposite* polarities $\oplus, \ominus$. As a result, $gc_{\ominus}^{\oplus}(h, v) = 0$. □

As an example, consider a matrix $A \in SL_2(\mathbb{Z})$ whose first row is $(-q, s)$ and the second one is (p, r), $(qr + ps = 1)$. We employ A to form a closed manifold $L_{p,q}$ from two solid tori $\mathbf{T}_1$ and $\mathbf{T}_2$. Their boundaries $T_1 = S^1 \times S^1$ and $T_2 = S^1 \times S^1$ are glued via A. Here we assume that the first multipliers in the two products are, respectively, the meridians of $\mathbf{T}_1$ and of $\mathbf{T}_2$. We denote by X any manifold obtained from $L_{p,q}$ by removing a number of balls.

Corollary 7.5. *For any* $X = L_{p,q} \setminus (\bigcup_i B_i^3)$ *as above,* $gc(X) \leq 4 \cdot \min\{q, q'\}$, *where* $qq' = \pm 1 \operatorname{mod} \cdot\, p$ *and* $1 \leq q,\, q' \leq p - 1$.

Proof. The corollary is obtained by constructing a Morse function $h : L_{p,q} \to \mathbb{R}$ such that: (1) on $T_1 = T_2$, h is constant, (2) h has a single minimum and a single critical point of index 1 in $\mathbf{T}_2$, (3) h has a single maximum and a single critical point of index 2 in $\mathbf{T}_1$. Then apply to h the construction in Theorem 7.6. Since $L_{p,q}$ and $L_{p,q'}$ are diffeomorphic, provided $qq' = \pm 1 \operatorname{mod} \cdot\, p$, the claim follows. □

8. Which Spines Are Gradient?

Next, we address a natural question: Which spines K are produced in the form of a cascade $\partial_1^+ X \cup \mathcal{C}(\partial_2^+ X)$ by appropriate generic Morse data (f, v)? In other words, *Which spines are gradient*?

It turns out that any $\vec{Y}$-spine, and thus any *branched* spine, can be approximated by a gradient spine without compromising its combinatorial complexity.

Theorem 8.1. *Let K be a simple* $\vec{Y}$*-spine in a* 3*-fold X. Then, there exist a nonsingular function* $f : X \to \mathbb{R}$ *and its gradient-like field v so that the Morse data* (f, v) *produce a gradient spine* $K(v) \subset X$ *with the following properties*:

- *K is homeomorphic to a* 2*-complex obtained from* $K(v)$ *by elementary collapses of some of its* 2*-cells*,
- $c(K(v)) = c(K)$.

Moreover, for any simple spine K and generic Morse data (f', v') in X, such that v' delivers K as a $\vec{Y}$-spine, and for any $\epsilon > 0$, there exist new Morse data (f, v) as above which satisfy two additional properties:

- *f and f' coincide when restricted to K,*
- *the f-controlled size of the 2-collapses $K(v) \to K$ does not exceed ε.*[22]

Proof. Let w be a field which delivers the $\vec{Y}$-structure to $K \subset X$. We start a construction of the appropriate Morse data (f, v) by extending w from the graph $s(K)$ to a smooth field v in a open neighborhood of $s(K)$. Since $s(K)$ is 1-dimensional and w is transversal to K along $s(K)$, for a sufficiently small $\varepsilon > 0$, the v-flow ϕ_t will have the property $\phi_t(s(K)) \cap \phi_{t'}(s(K)) = \varnothing$ for all $0 \leq t < t' \leq \varepsilon$. Hence, the set $L := \{\phi_t(x)\}_{x \in s(K),\ 0 \leq t \leq \varepsilon}$ can be given a product structure $H : L \approx s(K) \times [0, \varepsilon]$, H being a diffeomorphism of stratified spaces. We define a function $\hat{f} : L \to \mathbb{R}$ as the composition of H with the projection on $[0, \varepsilon]$. Next, we extend $\hat{f}$ to a smooth function $\tilde{f} : X \to \mathbb{R}$. We can assume that the singularities of $\tilde{f}$ are not located on K and thus can be removed by finger moves which originate at $\partial_1 X$ and are confined to $X \setminus K$. The resulting nonsingular smooth function $f : X \to \mathbb{R}$ and its gradient field v, $v|_{s(K)} = w$, give K its $\vec{Y}$-spine structure.

Consider a small open regular neighborhood $\mathcal{N}$ of $K \subset X$. The Regular Neighborhood Theorem (see [17], Theorems 2.11, 2.16) implies that K is a spine for $\mathcal{N}$ and that there is a PL-homeomorphism $g : \mathcal{N} \to X$ which is an identity on K. In dimension three, we can assume that $\partial\mathcal{N}$ is a smooth surface and that g is a diffeomorphism.

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[22] i.e., the f-images of the collapsing 2-cells are ε-small.

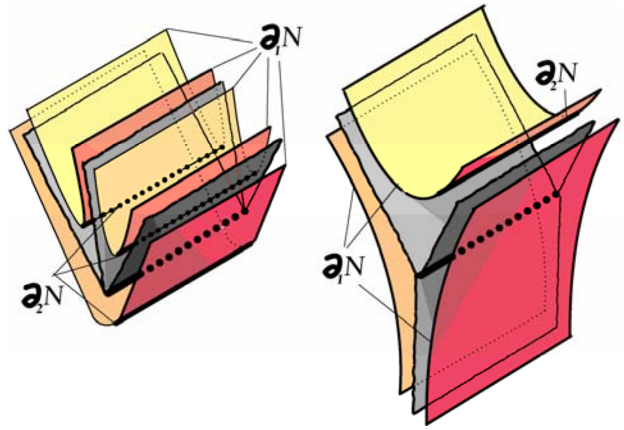

Figure 23. Field v is orthogonal to the plane of drawing Π. Let $p : \partial_1 N \to \Pi$ be the projection along $-v$. Note the three bold curves in the left diagram and the single bold curve in the right one, all marked as $\partial_2 N$. These are the p-images of the folds. On the right, $p(\partial_2 N)$ is a simple curve, so no double-tangent v-trajectories are present.

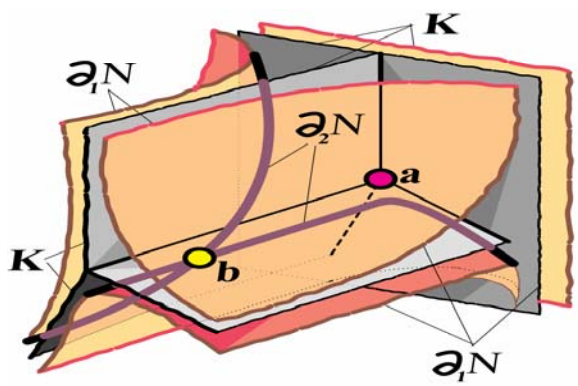

Figure 24. Note the intersection point b of the two bold curves—the p-images of the folds of the projection $p : \partial_1 N \to \Pi$ along $-v$. Point b is the p-image of unique double-tangent v-trajectory.

Using the smooth product structure in $\mathcal{N} \setminus K$, we can construct a smooth function $F : \mathcal{N} \to [0, 1]$ so that: (1) $F^{-1}(0) = K$, (2) $F^{-1}(1) = \partial\mathcal{N}$, (3) $(0, 1]$ being the set of regular values, and (4) K being the critical set. The minus gradient-like flow of F defines a collapse of $\mathcal{N}$ on K.

Next, we pick $\varepsilon > 0$, so small that $\mathcal{N}_\varepsilon := F^{-1}([0, \epsilon])$ interacts with the v-flow as is depicted in Figure 23, the right diagram, Figure 24, and Figure 27, diagrams 1 and 2. This depiction—a linearization at a point from $s_\dagger(K)$ of the surfaces that form K—is based on v being in general position with respect to K° and giving K its $\vec{Y}$-structure.

Let us take a closer look at the interaction of v with the boundary $\partial_1 \mathcal{N}_\varepsilon := \partial \mathcal{N}_\varepsilon$:

(1) in the vicinity of $s_\dagger(K) \setminus Q(K)$,

(2) in the vicinity of $Q(K)$,

(3) along the loops of v-tangency which are located in K°,

(4) in the vicinity of $s_\bullet(K) \setminus s_{\bullet\dagger}(K)$,

(5) in the vicinity of $s_{\bullet\dagger}(K)$.

In the first case (see the three-page book in Figure 23, (2), and its 2D-section in Figure 27, (2)), the gradient spine $K^\sharp_\varepsilon = K(\mathcal{N}_\varepsilon, f, v)$ generated by the Morse data (f, v) on $\mathcal{N}_\varepsilon$ is locally homeomorphic to the given spine K. This conclusion depends heavily on the $\vec{Y}$-nature of the field. Moreover, because of this nature, for a sufficiently small ϵ, the image of the fold is a simple arc—no double-tangent trajectories are present in the vicinity of $s_\dagger(K) \setminus Q(K)$ (see Figure 23, (2)). Compare diagrams 2 and 3 in Figure 27: diagram 2 reflects the fact that v is of the $\vec{Y}$-type, while in diagram 3 the field is of the $\vec{W}$-type.

In the second case (see Figures 24 and 27, diagram 1, where $\mathcal{N}_\varepsilon$ is abbreviated to N), the surface $\partial_1 \mathcal{N}_\varepsilon$ and the field v at a Q-singularity are transversal, and v points into two (out of four) chambers-pyramids in which X is divided by K. Each of the other two chambers provide a fold of $\partial_2 X$, shown in Figure 24 as a bold arc. The intersection b of the two arcs is the image of a single double-tangent v-trajectory for the v-generated gradient spine $K^\sharp_\varepsilon$ of $\mathcal{N}_\varepsilon$.

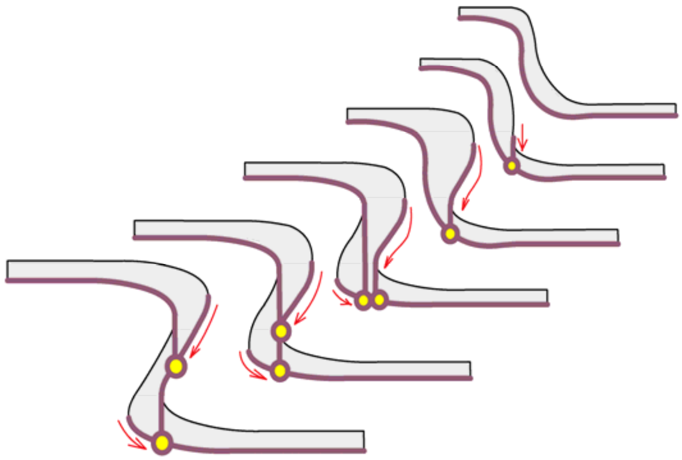

Figure 25. Slicing the gradient spine $K^\sharp_\varepsilon$ of $\mathcal{N}_\varepsilon$ in the vicinity of a cusp. The arrows indicate collapses of $K^\sharp_\varepsilon$ onto a 2-complex homeomorphic to the given K.

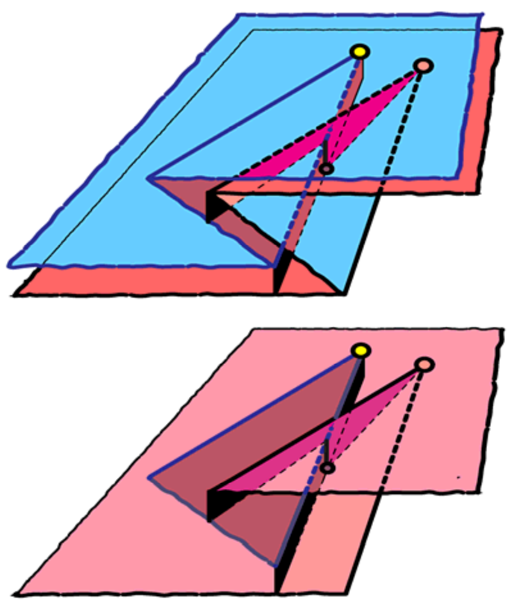

Figure 26. A regular neighborhood of a cusp from $\partial_3(K^\circ, v)$ and its gradient spine with a single double-tangent trajectory.

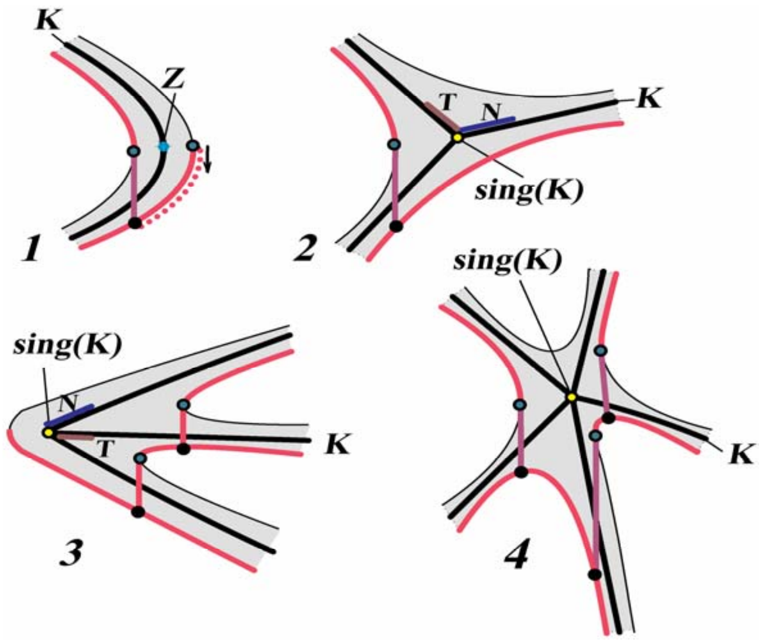

Figure 27. Sections of a spine K and its regular neighborhood by a plane transversal to $s(K)$ and containing (vertical) v. Diagrams 3 and 4 illustrate complications arising when v is of $\overrightarrow{W}$-type and when $K \subset X$ is not simple.

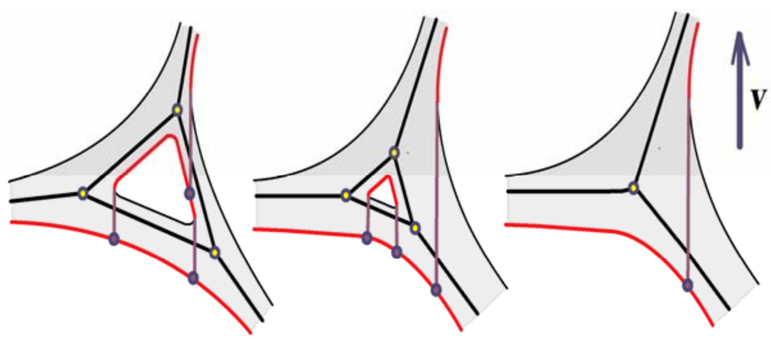

Figure 28. Sections of the gradient spine $K_\varepsilon^\sharp \subset \mathcal{N}_\varepsilon$ by a family of "parallel" surfaces in vicinity of a Q-singularity. The surfaces are invariant under the v-flow.

The third case (see Figure 27, (1), and Figure 25), the behavior of v with respect to K° is similar to the behavior of a generic field with respect to $\partial_1 X$ along $\partial_2 X$. Consider a smooth arc $L \subset K^\circ$ where v is tangent to the surface K° and transversal to L. In the vicinity of a point $x \in L$, $\mathcal{N}_\varepsilon$ and the flow are represented, up to a diffeomorphism, by the product of first diagram in Figure 27 with a segment. Note the portion of the cascade $K^\sharp_\varepsilon$ generated in $\mathcal{N}_\varepsilon$ by v and marked with a dotted line. Collapsing this portion produces a complex locally homeomorphic to the original K. Points $x \in L$ where v is tangent to L require special attension. There K° and v have local geometry similar to the geometry of $\partial_1 X$ in the vicinity of a cusp point from $\partial_3 X$ (see Figures 25 and 26). Figure 25 demonstrates how $K^\sharp_\varepsilon$ collapses onto the given K. We notice that each cusp from $\partial_3(K^\circ, v)$ contributes a single Q-singularity to $K^\sharp_\varepsilon$ (i.e., one double-tangent trajectory to $\mathcal{N}_\varepsilon$), a singularity which has no counterpart in the original K! Fortunately, according to Therem 9.6, we can modify our Morse data (f, v) away from a neighborhood of $s(K)$ (where v is transversal to $K \subset X$ and gives it its $\vec{Y}$-structure) so that the modified data will have no cusps in K°. Therefore, we get $gc(\mathcal{N}_\varepsilon, f, v)$ $= c(K^\sharp_\varepsilon) = c(K)$.

Simpler cases (4) and (5) (cf. Figure 25) are treated similarly. None of them contributes Q-singularities to the gradient spine of $\mathcal{N}_\varepsilon$.

Next, we use the diffeomorphism $g_\varepsilon^{-1} : X \to \mathcal{N}_\varepsilon$ to transplant the previously constructed Morse data (f, v) from $\mathcal{N}_\varepsilon$ to X. Evidently, the transplanted data and their gradient spine still have all the desired properties.

We leave to the reader to verify that the size of elementary collapses, as measured in terms of the f-variation, can be made arbitrary small. The argument uses the uniform continuity of the smooth functions F and f together with the fact that v is in general position with respect to K.

These imply that, for a sufficiently small $\varepsilon' > 0$, there is $\varepsilon > 0$ so that in $\mathcal{N}_\varepsilon := F^{-1}([0, \varepsilon])$ the variation of f along each v-trajectory does not exceed ϵ' (see Figure 27, (1)). Moreover, the f-controlled size of elementary collapses does not exceed the f-controlled size of the trajectories inside $\mathcal{N}_\varepsilon$. This property is preserved under the diffeomorphism g_ε^{-1}. □

Theorem 8.1 has an important implication:

Theorem 8.2. *For any compact* 3*-fold* X *with a nonempty boundary*, $c_{\vec{Y}}(X) = gc(X)$. *Moreover, for any boundary-irreducible with no essential annuli* X *with* $c(x) > 1$,

$$c(X) \leq gc(X) \leq 6 \cdot c(X).$$ [23]

Proof. Let $K \subset X$ be a simple spine with $c(X)$ Q-singularities. If such a spine admits a $\vec{Y}$-structure, by Theorem 8.1, $c(X) = c_{\vec{Y}}(X)$ $= gc(X)$, and we are done.

By Theorem 7.2 (cf. [25], Theorem 2.2.4), any boundary irreducible with no essential annuli X has a *special* spine K of complexity $c(X)$ at most, provided $c(X) > 1$. According to [2], Theorem 3.4.9, for any special spine K, there exists a sequence of not more than $5 \cdot c(K)$ oriented Matveev-Piergallini moves (see [25], Figure 1.12) which convert K into an *oriented branched* spine (this theorem is based on a rather intricate argument). Furthermore, each move increases the combinatorial complexity of the modified spine by one. Thus, these moves result in a branched spine of complexity $\leq 6 \cdot c(X)$. Next, by Lemma 6.3, any branched spine is a $\vec{Y}$-spine. By Theorem 8.1, it admits an approximation by a gradient spine of the same complexity. Thus $gc(X) \leq 6 \cdot c(X)$. □

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[23] Note that for $gc(L_{8,3})$ both Theorem 8.2 and Corollary 7.5 give the same upper boundary 12.

Corollary 8.1. *For any X which is a connected sum of boundary-irreducible with no essential annuli compact 3-folds whose complexity exceeds* 1, $gc(X) \leq 6 \cdot c(X)$.

Proof. The claim follows from the semi-additivity of $gc(\sim)$ (Theorem 7.4) and the additivity of $c(\sim)$ (Theorem 2.2.9, [25]), coupled with Theorem 8.2. □

For example, $gc((L_{5,1}\# \; S^3/Q_8)^\circ) \leq 6[c(L^\circ_{5,1}) + c(S^3/Q^\circ_8)] = 6(2 + 2) = 24$. However, using Corollary 7.5, this estimate can be improved: $gc((L_{5,1}\# \; S^3/Q_8)^\circ) \leq gc(L^\circ_{5,1}) + gc((S^3/Q_8)^\circ) \leq 4 + 6 \cdot 2 = 16$. Thus, $4 \leq gc((L_{5,1}\# \; S^3/Q_8)^\circ) \leq 16$.

Since every handlebody X admits a convex gradient flow, $gc(X) = 0$. By Theorem 7.4, deleting a few open 3-balls from X produces a new manifold X' also with zero gradient complexity. Note that $\pi_1(X)$ is free. It seems natural to expect that any X with $gc(X) = 0$ is a handlebody with holes. However, Theorem 8.3 below claims that there are other examples of 3-folds X with $gc(X) = 0$ and rather complex fundamental groups. We are going to describe a combinatorial scheme that generates all of them. At the same time, when $\partial_1 X$ is just a union of 2-spheres and $gc(X) = 0$, then the topology of X is very special (Theorem 8.4), and indeed $\pi_1(X)$ is free. Moreover, $gc(X) = 0$ is definitely a stronger requirement on X than $c(X) = 0$.

We start building a combinatorial scheme that captures the topology of abstract gradient spines of zero complexity.

Consider an oriented 3-fold X which admits a simple (abstract) gradient spine K with $Q(K) = \emptyset$. Let $p : X \to K$ be the collapsing map.

The singularity set $s_\dagger(K)$ consists of a number of disjoint simple loops and arcs. The orientation of X, coupled the NT-induced orientation of the normal to $s_\dagger(K)$ bundle, picks a preferred orientation of the arcs and loops in $s_\dagger(K)$. As a part of the abstract gradient spine structure, the surfaces in K° are oriented.

Consider a regular neighborhood $\mathcal{N}(s_\dagger(K)) \subset K$. Figure 29 shows the three patterns $\mathcal{N}_\tau$ of $\mathcal{N}(s_\dagger(K))$ in the vicinity of each arc τ from $s_\dagger(K)$ (cf. Figure 13). These patterns reflect the fundamental assumption $Q(K)=\emptyset$. The *TN*-markers along τ are not shown, but suggested by the shapes: the *T*-marker is normal to the ovals, the *N*-marker points towards the viewer.

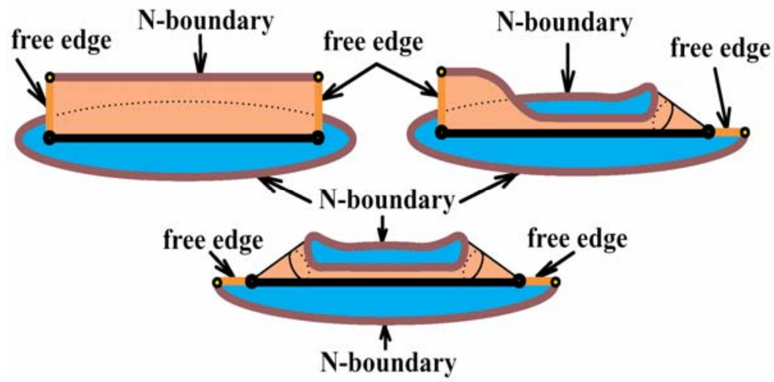

Figure 29. The three types of regular neighborhoods of arcs from $s_\dagger(K)$.

In the first pattern, the portion $\partial\mathcal{N}(s_\dagger(K))$ of the topological boundary of $\mathcal{N}(s_\dagger(K))$ that is shared by $\mathcal{N}(s_\dagger(K))$ and its complement $K \setminus \mathcal{N}(s_\dagger(K))$ (marked as "*N*-boundary" in Figure 29) is a loop and an arc; in the second pattern, it is a single arc; in the third case, it is again a loop and an arc.

Let $K^\odot := K \setminus \mathcal{N}(s_\dagger(K))$. Because $p : p^{-1}(\partial\mathcal{N}(s_\dagger(K))) \to \partial\mathcal{N}(s_\dagger(K))$ is a trivial fibration with a segment for a fiber, $K^\odot$ is an abstract gradient spine of the 3-fold $p^{-1}(\, K^\odot)$.

Each of the patterns in Figure 29 has two free arcs that belong to $\partial_\bullet K$, the topological boundary of *K*. In the first and third patterns, *K* is obtained from $K^\odot$ by attaching a 2-disk to the loop in $\partial\mathcal{N}_\tau$ and then a rectangular band *H* with its two opposite edges being the free edges shown in the figure. In the second pattern, we can collapse $\mathcal{N}_\tau$ onto $\partial\mathcal{N}_\tau \cup \tau \cup I$, where the segment *I* is a generator of the cone (to visualize this fact, perform two "finger moves" on *K* that originates at the free edges and terminates at I). Therefore, X is obtained from $p^{-1}(K^{\odot})$ by attaching to its boundary either (1) a 2-handle followed by a 1-handle, or (2) a 3-ball (the thickening of $I \cup \tau$) along its northern hemisphere. So, we can assume that X is obtained by attaching 2-handles[24], *followed* by 1-handles, to the boundary of an oriented 3-fold X'. It has an abstract gradient spine K' with $s_\dagger(K')$ being only a union of loops.

Next, consider the case when a component of $s_\dagger(K)$ is a simple loop ω. Because $\mathcal{N}(s_\dagger(K)) \subset K$ admits *TN*-markings, in the vicinity of each ω, $\partial\mathcal{N}(s_\dagger(K))$, consists of three simple loops parallel to ω and residing on the boundary of the solid torus T_ω, the regular neighborhood of ω in X. Let $\mathcal{N}_\omega := \mathcal{N}(s_\dagger(K)) \cap T_\omega$ and $\partial\mathcal{N}_\omega := \mathcal{N}(s_\dagger(K)) \cap \partial T_\omega$. Recall that ω contributes with the multiplicity ± 1 to the boundary of the fundamental 2-chain, so that the orientation of one of the three loops disagrees with the orientations of the other two.

Thus K is obtained from $\mathcal{N} := \left(\coprod_\omega \mathcal{N}_\omega\right) \coprod \left(\coprod_\tau \mathcal{N}_\tau\right)$ by attaching compact connected and oriented surfaces $\{S_\kappa\}_\kappa$ to $\partial\mathcal{N}$ along some disjoint loops and arcs in their boundaries. When an arc $\tau \subset s_\dagger(K)$ is of types 1 and 3 in Figure 29, we include the 2-disk, residing in $\mathcal{N}_\tau$ and bounding the loop from $\partial\mathcal{N}_\tau$, into the surface that is attached to that loop. If an arc or a loop in ∂S_κ remains unattached to $\partial\mathcal{N}$ (that is, free), it is possible to collapse S_κ onto a 1-dimensional graph $\Gamma_\kappa \subset S_\kappa$, so that homotopically attaching S_κ is equivalent to attaching Γ_κ to $\partial\mathcal{N}$. In particular, any S_κ that is attached to an *arc* in $\partial\mathcal{N}_\tau$ must have a free portion in its boundary. Therefore, attaching such a surface results in attaching a handlebody, the thickening of Γ_κ, to a portion X' of X. The attaching diffeomorphism must reverse the boundary orientations, so that X is orientable when X' is.

[24]Later on, we will incorporate these 2-handles into the combinatorial scheme.

We concentrate of surfaces S_κ whose entire boundary is attached to loops in $\partial\mathcal{N}$ (We already cupped the boundary components of S_κ's that where attached to loops in $\coprod \mathcal{N}_\tau$). Now we deal only with the surfaces whose boundary loops are attached to $\partial\mathcal{N}_\omega$.

We will use a graph $\Gamma = \Gamma(K)$ to organize and record the structure of a spine K, the result of attaching oriented surfaces S_κ to $\partial\mathcal{N}$ only along its loops. The vertices of Γ are divided into two groups $\{A_i\}_i$ and $\{B_j\}_j$. Each vertex from the first group depicts a loop ω_i from $s_\dagger(K)$, each vertex from the second group stands for one of the connected and oriented surfaces S_κ. The edges of Γ record the way the components of ∂S_κ are attached to the ω_i's. A typical edge which links A_i with B_j is denoted $\gamma_{ij,k}$. We require that Γ will satisfy the following properties:

(8.1)

(1) Γ is connected,

(2) for each vertex A_i, its multiplicity $\mu(A_i) = 3$,

(3) a number $\varepsilon(\gamma_{ij,k}) = \pm 1$ is assigned to each edge $\gamma_{ij,k}$ so that, for each i, $\sum_{j,k} \varepsilon(\gamma_{ij\,k}) = \pm 1$,

(4) an integer $\chi(B_j)$, the Euler number of S_j, is attached to each vertex B_j.

Next proposition describes all orientable 3-folds of zero gradient complexity.

Theorem 8.3. *Let X be an orientable* 3*-fold such that* $gc(X) = 0$. *Then X is a connected sum of a handlebody with an orientable* 3*-fold* X' *which has an abstract gradient spine K uniquely defined by the combinatorial data* (8.1). *If X has no essential annuli*, *then* Γ *is a tree*, *and each* $\chi(B_j) = 2 - \mu(B_j)$.

On the other hand, each Γ, subject to (8.1), gives rise to an orientable 3-fold X' of gradient complexity zero. Its Euler number is given by the

RHS of formula (8.2), and its fundamental group has a presentation described by formulas (8.3) and (8.4).

Proof. We keep notations and use conclusions introduced and derived prior to the formulation of Theorem 8.3 and centered on Figure 29.

As before, we fix an orientation for each loop ω_i in $s_\dagger(K)$ and an orientation for each surface S_j. In the induced orientation, each component $\partial S_{j,k}$ of ∂S_j, glued to ω_i in K, is attached via either an orientation-preserving homeomorphism, in which case $\varepsilon(\gamma_{ij,k}) = 1$ or an orientation-reversing homeomorphism, in which case $\varepsilon(\gamma_{ij,k}) = -1$. The branched orientation of K enforces $\sum_{j,k} \varepsilon(\gamma_{ij,k}) = \pm 1$.

The number of components in ∂S_j is the multiplicity $\mu(B_j)$ of the vertex B_j. Therefore, the topological type of S_j can be reconstructed from $\chi(B_j)$ and $\mu(B_j)$. Let $\hat{S}_j$ be the closed surface obtained from S_j by cupping each component of ∂S_j with a disk. Denote by g_j the genus of $\hat{S}_j$. Then $\chi(B_j) = 2 - 2g_j - \mu(B_j)$. We get

$$\chi(X) = \sum_j (2 - 2g_j - \mu(B_j)) = 2\sum_j (1 - g_j) - 3k, \tag{8.2}$$

where k is the number of loops in $S_\dagger(K)$, that is, the number of vertices $\{A_i\}$. Indeed, as we paste the S_j's along their boundaries to form K, we identify the vertices and the edges of the appropriate triangulations in triples, the number of the edge triples being the same as the number of the vertex triples. As a result, formula (8.2) is valid.

Recall that $\chi(\partial_1 X) = 2 \cdot \chi(X)$. Thus, if $\partial_1 X$ is connected, it must be an oriented surface of genus $2\sum_j (1 - g_j) - 3k + 1$. On the other hand, for X with $\partial_1 X$ being a union of q copies of S^2, $\chi(X) = q$. Thus,

Corollary 8.2. *For any X and any abstract gradient spine $K \subset X$ with $Q(K) = \varnothing$ (thus $gc(X) = 0$) and such that $s_\dagger(K)$ is a union of k*

loops, we have

$$\chi(X) = k \bmod\cdot\, 2, \quad \textit{and} \quad \chi(X) = \sum_j (g_j - 1) \bmod\cdot\, 3.$$

In particular, if $\partial_1 X$ *is a union of* q *spheres, then* $q = k \bmod\cdot\, 2$, *and* $q = -|\,\pi_0(K \setminus S_\dagger(K))\,| \bmod\cdot\, 3$.

Each edge $\gamma_{ij,k} \subset \Gamma$ encodes a particular component of ∂S_j. Let $\beta = \beta_{ij,k} \subset S_j^\circ$ be a loop parallel to that component. If $\gamma_{ij,k}$ is a part of a cycle μ in Γ, then X has an essential annulus $p^{-1}(\beta)$, where $p : X \to K$ is the collapsing map. Indeed, μ helps to construct a loop $\tilde{\mu} \subset K$ which intersects the annulus transversally at a single point. Therefore, Γ must be a *tree* for the 3-folds X with no essential annuli and $gc(X) = 0$. Similarly, if X has no essential annuli, no S_j can have a handle: for any pair of loops $\tau, \eta \in S_j^\circ$ which transversally meet at a singleton, the annulus $p^{-1}(\tau)$ must be essential. Moreover, τ and η can be pushed into $\partial_1 X$, where they still will intersect at a singleton: note that $p : p^{-1}(S_j^\circ) \to S_j^\circ$ is a trivial fibration with a closed segment for the fiber. So, when X has no essential annuli, each $\hat{S}_j$ is a 2-sphere, and $g_j = 0$.

For any X, note that both loops in the boundary of the annulus $p^{-1}(\beta)$ are homotopic in X to $\beta^{\pm 1}$; therefore, for an appropriate choice of the base point x_0, each loop $\omega_i \subset s_\dagger(K)$ is in the image of the inclusion-induced homomorphism $\pi_1(\partial_1 X, x_0) \to \pi_1(X, x_0)$.

Now we would like to investigate the case when $\partial_1 X = \coprod S^2$. Again, in such a case, no S_j has a handle. Since $p^{-1}(S_j^\circ)$ is a trivial fibration over the interior S_j°, any two loops in S_j° which intersect at a singleton can be pushed into the boundary $\partial_1 X = \coprod S^2$, where they still share only a singleton, a clear impossibility. So, each $\hat{S}_j$ must be a 2-sphere. Moreover, by the argument above, each loop $\omega_i \subset s_\dagger(K)$ is contractible in X ($\pi_1(\partial_1 X) \to \pi_1(X)$ is trivial), and thus is contractible in K.

In general, $\pi_1(K)$ is generated by the basic cycles in Γ, being lifted to K (these are free generators), by the oriented components $\beta_{j,k}$ of ∂S_j, and by the basic loops $a_{j,1}, b_{j,1}, \dots, a_{j,g_j}, b_{j,g_j}$ in S_j, where $2g_j = 2 - \chi(B_j) - \mu(B_j)$. These generators are subject to the following relations:

(1) for each j,

$$\mathcal{R}_j = \left\{ \prod_k \beta_{j,k} = \prod_{l=1}^{g_j} [a_{j,l},\, b_{j,l}] \right\}, \tag{8.3}$$

(2) for each i and the triple of edges $\gamma_{ij_1,k_1}, \gamma_{ij_2,k_2}, \gamma_{ij_3,k_3}$ that terminate at A_i,

$$\mathcal{T}_{i,s} = \{\beta_{ij_s,k_s} = \omega_i^{\epsilon(\gamma_{ij_s,k_s})}\}, \quad s = 1, 2, 3. \tag{8.4}$$

In the case of $\partial_1 X = \coprod S^2$, according to the previous arguments, all ω_i and $a_{j,1}, b_{j,1}, \dots, a_{j,g_j}, b_{j,g_j}$ are trivial, which forces $\pi_1(X) \approx \pi_1(\Gamma)$ to be free or trivial. Note that the same conclusion also follows from Corollary 7.4.

According to Matveev's classification Table 9.1 on page 407 in [25], the only *closed* oriented 3-folds Y with $c(Y \setminus D^3) = 0$ are the connected sums of the following prime blocks: (1) spheres S^3, (2) products of spheres $S^2 \times S^1$, (3) projective spaces $\mathbb{R}P^3$, and (4) lens spaces $L_{3,1}$. Since $gc(Y \setminus D^3) = 0$ implies $c(Y \setminus D^3) = 0$ and, as we have seen, $\pi_1(Y \setminus D^3)$ must be free or trivial, the presence of the $\mathbb{R}P^3$'s and $L_{3,1}$'s in the connected sums is ruled out. In particular, any generic gradient flow on $\mathbb{R}P^3 \setminus D^3$ and $L_{3,1} \setminus D^3$ must have at least one double-tangent trajectory[25].

[25]Soon we will show that any gradient flow on these 3-folds must have two double-tangent trajectories at least.

Therefore, if $\partial_1 X = \coprod S^2$ and $gc(X) = 0$, X must be a connected sum of 3-balls with a number of $S^2 \times S^1$, a claim of Theorem 8.4 below.

In fact, any such sum X is of zero gradient complexity. Indeed, consider a Morse function $h : S^2 \to \mathbb{R}$ with only two critical points, a and b, and a Morse function $g : S^1 \to \mathbb{R}$ with only two critical points, c and d, so that $h(a) = 0$, $h(b) = 2$, $g(c) = 0$, $g(d) = 3$. By an argument as in the proof of Theorem 7.6, applied to the Morse function $f = h + g$, we construct a gradient spine of $S^2 \times S^1$ from which four small balls, centered on the critical points of f, have been removed. Let us denote this 3-fold by M. Its boundary consists of four 2-spheres, two of which are $f^{-1}(1)$ and $f^{-1}(4)$. Since $f(b \times c) > f(d \times a)$, the f-critical point of index 1 lies above the f-critical point of index 2. Hence, these two critical points are not linked by a gradient trajectory in $S^2 \times S^1$. As a result, the waterfalls in M, generated in the vicinity of these two points, do not intersect each other. They hit the sphere $f^{-1}(1) \subset \partial_1 M$ along four disjoint simple loops. Therefore, $gc(M) = 0$. By Theorem 7.4, $gc(f^{-1}([1, 4])) = gc(f^{-1}([0, 4])) = gc(M) = 0$. Since any connected sum X is assembled from the blocks of the types $f^{-1}([1, 4])$, and $f^{-1}([0, 4])$, the semi-additivity of $gc(\sim)$ under connected sums implies the claim.

If we combine the previous arguments (or just Corollary 7.4) with [14], Exercise 5.3, and with the validated Poincaré Conjecture [29], [30], we could avoid using the classification in [25]. In fact, the following (well-known) proposition is valid:

If $\pi_1(X)$ is free of rank r and $\Sigma_1, \ldots, \Sigma_q$ are the components of $\partial_1 X$, then

$$X = H_1 \# \cdots \# H_q \# E_1 \# \cdots \# E_s,$$

where H_i is a handlebody with $1/2\,(2 - \chi(\Sigma_i))$ handles, E_j is a 2-sphere bundle over S^1 (trivial when X is orientable), and $s = r$

$-1/2\sum_{i=1}^{q}(2-\chi(\Sigma_i))$. In particular, when the boundary is a union of q copies of S^2, $X = D_1^3 \# \cdots \# D_q^3 \# E_1 \# \cdots \# E_r$.

Next, moving from spines K to the corresponding manifolds, we claim that any set of combinatorial data as in (8.1) generates an abstract gradient spine K and the corresponding (see Theorem 6.1) orientable 3-fold X with $gc(X) = 0$. Indeed, using data in (8.1), we assemble X from the oriented blocks $\{S_\kappa \times [0, 1]\}$ and the solid tori T_{ω_i}. In the process, we attach annuli in the oriented boundaries of the blocks by orientation-reversing diffeomorphisms. Depending whether $\epsilon(\gamma) = +1$ or $\epsilon(\gamma) = -1$, the attaching diffeomorphism must reverse or preserve the orientations the radial segments in the two annuli. Now the claim follows from our realization Theorem 8.1 and Lemma 6.3. Indeed, the oriented branched structure of $\mathcal{N}$ allows for the $\vec{Y}$-structure in the vicinity of the thickening of $s_\dagger(K)$. As a result, there is a gradient flow on X with no double-tangent trajectories.

This completes the proofs of Theorem 8.3 and of Theorem 8.4 below. □

Theorem 8.4. *Let X be a connected oriented 3-fold with its boundary being a union of spheres. If $gc(X) = 0$, then X is a connected sum of several 3-balls with several copies of $S^2 \times S^1$. In particular, if any essential 2-sphere in such an X is parallel to its boundary $\partial_1 X = S^2$, then X is a 3-ball.*

The group presentations as in (8.3) and (8.4) allow for a great variety of fundamental groups $\pi_1(X)$. Although I do not have a good conjecture describing all these groups intrinsically, such a description seems feasible.

Example 8.1. We consider a few basic examples of 3-folds X with $gc(X) = 0$. They are generated by gradient spines K with $s(K) = s_\dagger(K)$ being a single loop ω. There are three possibilities for Γ: it has (1) three vertices B_1, B_2, B_3, (2) two vertices B_1, B_2, and (3) one vertex B_1. In the first case, $\pi = \pi_1(X)$ is generated by three groups of generators

$\{a_1, b_1, \dots, a_{g_1}, b_{g_1}\}$, $\{c_1, d_1, \dots, c_{g_2}, d_{g_2}\}$, $\{e_1, f_1, \dots, e_{g_3}, f_{g_3}\}$, subject to the relations $\prod_{i=1}^{g_1}[a_i, b_i] = \prod_{j=1}^{g_2}[c_j, d_j] = \prod_{k=1}^{g_3}[e_k^{-1}, f_k^{-1}]$. Note that $\pi/[\pi, \pi]$ is free abelian of rank $2(g_1 + g_2 + g_3)$.

In the second case, there are two distinct options. Either $\pi = \pi_1(X)$ is generated by two sets of generators $\{a_1, b_1, \dots, a_{g_1}, b_{g_1}\}$, $\{c_1, d_1, \dots, c_{g_2}, d_{g_2}\}$ and an additional free generator f (it corresponds to the loop in Γ), subject to a single relation $\prod_{i=1}^{g_1}[a_i, b_i] = 1$, or π is generated by $\{a_1, b_1, \dots, a_{g_1}, b_{g_1}\}$, $\{c_1, d_1, \dots, c_{g_2}, d_{g_2}\}$ and f, subject to a single relation $\prod_{i=1}^{g_1}[a_i, b_i] = \left\{\prod_{j=1}^{g_2}[c_j^{-1}, d_j^{-1}]\right\}^2$. In both options, $\pi/[\pi, \pi]$ is free abelian of rank $2(g_1 + g_2) + 1$.

In the third case, π is freely generated by $\{a_1, b_1, \dots, a_{g_1}, b_{g_1}\}$ together with f_1, f_2 (they stand for two basic loops in Γ).

In all these descriptions of π we have assumed that $g_1, g_2, g_3 > 0$. Numerous exceptions to this assumption lead to groups π of a slightly different kind. For example, when Γ is as in the second case and $g_1 = 0$, we get a group π with the 2-torsion element $\prod_{j=1}^{g_2}[c_j, d_j]$; however, it is trivial in the homology $\pi/[\pi, \pi]$.

Next we show that there is no orientable 3-fold with simply-connected boundary and of gradient complexity one. In contrast, $c(L^\circ_{4,1}) = 1$ and $c(L^\circ_{5,2}) = 1$. Despite little evidence, we conjecture that $gc(X) \geq 4$, provided $\partial_1 X = \coprod S^2$.

Theorem 8.5. *No orientable* 3-*fold* X *with* $\partial_1 X = \coprod S^2$ *is of gradient complexity one. In fact, if an* X, *whose boundary is simply-connected, admits a gradient flow with a single double-tangent trajectory, it also admits a gradient flow with no double-tangent trajectories at all. In such a case,* X *is a connected sum of a number of* 3-*balls and products* $S^2 \times S^1$.

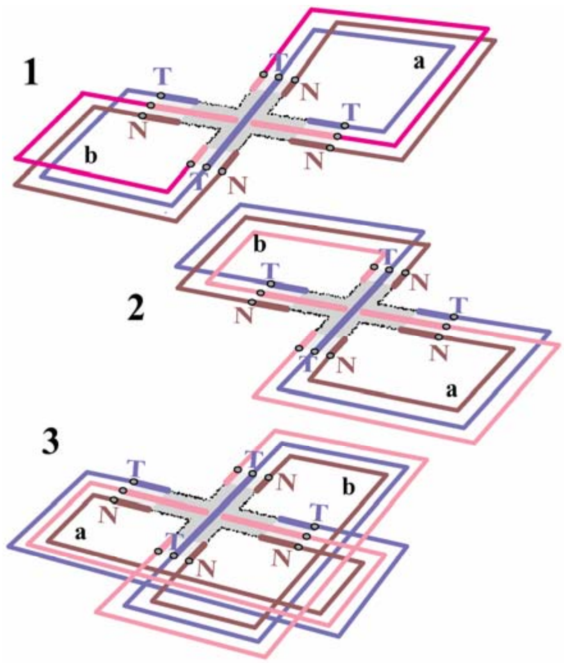

Figure 30. The three patterns for the graphs $\partial(\mathcal{N}(s_{\dagger}(K)))$ corresponding to abstract gradient spines K with $c(K) = 1$ and a connected singular set $s(K) = \infty$. The intersections of the lines should be ignored.

Proof. Assume that $gc(X) = 1$. Then $c(X) \leq 1$.

If $\partial_1 X = S^2$, according to the classification in [25], X can only be: (1) of combinatorial complexity zero, or (2) a connected sum of either $L^{\circ}_{4,1}$ or $L^{\circ}_{5,2}$ with a closed manifold of combinatorial complexity zero.

Let $K \subset X$ be an abstract gradient spine of complexity $c(K) = 1$. A neighborhood of its Q-singularity A is depicted in Figure 21. In the vicinity of A, there are four arcs in $s_{\dagger}(K)$ that emanate from A. If one of these arcs γ terminates at a point B of type P from Figure 13, then, in the vicinity of B, the topological boundary $\partial_{\bullet}K$ of K has a free arc that can be used to collapse K onto a subcomplex $K' \subset K$ such that $s_{\dagger}(K')$ does not contain $Int(\gamma)$, and hence $c(K') = 0$. This contradicts with the assumption $gc(X) = 1$. Similarly, if γ terminates at a point B of type O from Figure 13, then again a free arc in $\partial_\bullet K$ can be used to initiate a collapse $K \to K'$ which will result in K' of zero complexity. Therefore, each arc γ must join with another arc γ' emanating from A, that is, the A-component of $\partial_\dagger K$ must be homeomorphic to figure ∞. Examining Figure 21, we notice that there are only three (up to mirror symmetry) possible patterns for the connected component $\mathcal{N}_A$ of $\mathcal{N}(s_\dagger(K)) \subset K$ $(A \in \mathcal{N}_A)$, provided that $\mathcal{N}(s_\dagger(K))$ admits an oriented branching in the sense of Definition 6.5 (equivalently, admits TN-markings which must be consistent along the core ∞ of $\mathcal{N}_A$). These three patterns are shown in Figure 30. The figure does not show the singular set ∞. A regular neighborhood of $\mathcal{N}_A$ in X is a handlebody whose core is ∞ and whose boundary is a surface Σ of genus 2. The three graphs $\partial\mathcal{N}_A$ in Figure 30 should be imagined as residing in Σ. They are collections of simple loops (ignore the self-intersections). In fact, the cardinality of $\pi_0(\partial\mathcal{N}_A)$ in the three patterns in Figure 30 are 1, 2, and 3, correspondingly.

When X has no essential annuli, by Theorem 7.2, we can assume that the abstract gradient spine K as above is obtained by attaching 2-disks and annuli (with one of their two boundary components left free) to the boundary $\partial\mathcal{N}_A$. When only the disks are attached, the following three outcomes are available: (1) $|\pi_0(\partial\mathcal{N}_A)| = 1$, and $\Pi := \pi_1(X)$ is generated by a, b, subject to a single relation $a[a, b]b^{-1} = 1$ (so that $H_1(X) \approx \mathbb{Z}$); (2) $|\pi_0(\partial\mathcal{N}_A)| = 2$, and $\pi_1(X) = \{a, b \,|\, aba^{-2}b = 1, b = 1\} \approx 1$; (3) $|\pi_0(\partial\mathcal{N}_A)| = 3$, and again $\pi_1(X) \approx 1$. Here a, b are the two generators of the free group $\pi_1(\infty)$. In the second case, two 2-handles are attached to Σ so that $X = D^3$. In the third case, three 2-handles are attached so that $X = S^2 \times [0, 1]$. Recall that $gc(D^3) = 0 = gc(S^2 \times [0, 1])$.

Clearly, $\mathbb{R}P^{3^\circ}$ and $L^\circ_{3,1}$ have different fundamental groups. Hence $gc(\mathbb{R}P^{3^\circ}) > 1$ and $gc(L^\circ_{3,1}) > 1$. Table 9.2 from [25] implies that $L_{4,1}$ and

$L_{5,2}$ are the only two new possibilities for prime blocks in X with $c(X) \leq 1$; and again, they both fail because the fundamental groups do not match with the ones supported by the three previous *TN*-marked patterns in Figure 30.

Now consider an arbitrary 3-fold X with $\partial_1 X = \coprod S^2$ and $gc(X) \leq 1$. Let $K \subset X$ be an abstract gradient spine with a single Q-singularity, and $p : X \to K$ a collapsing map. As for abstract gradient spines of zero complexity, we can organize the information about K into a labeled graph Γ. If $gc(X) = 1$, we can assume that $s_\dagger(K)$ consists only of oriented loops ω_i and one copy of figure ∞ graph, with its two oriented loops a, b being attached to the isolated Q-singularity. A regular neighborhood $\mathcal{N}(\infty) \subset K$ can be only of one of the three types depicted in Figure 30. In each of the cases, the loops in $\partial\mathcal{N}(\infty)$ come with a preferred orientation. The oriented connected surfaces S_j in $K^{\odot} := K \setminus \mathcal{N}$, where $\mathcal{N} := [(\mathcal{N}(\infty) \cup (\bigcup_i \mathcal{N}(\omega_i))]$, each gives rise to vertex B_j of Γ. The Euler number χ_j of S_j and its genus g_j are recorded as a part of the combinatorial structure. Denote by $\hat{S}_j$ the closed surface obtained from S_j by attaching a 2-disk to each component of ∂S_j. As before, each loop ω_i is represented by a vertex A_i of multiplicity 3. The singularity set ∞ is represented by a special vertex A_∞. Depending on the three types of $\mathcal{N}(\infty)$ in Figure 30, the multiplicity of A_∞ in Γ is 1, 2, or 3, respectively. Each edge $\gamma \subseteq \Gamma$ represents a connected component $\partial S_{j,\gamma}$ of the boundary ∂S_j. Each γ is equipped with a label $\epsilon(\gamma) = \pm 1$. If attaching the oriented surface S_j to $\mathcal{N}$ preserves the preferred orientations of the loop $\beta_{j,\gamma} := \partial S_{j,\gamma} \subset \partial S_j$ and of the corresponding loop in $\partial\mathcal{N}$, we put $\epsilon(\gamma) = 1$, otherwise, $\epsilon(\gamma) = -1$.

Let us go back to the case $\partial_1 X = \coprod S^2$. Under this assumption, each loop $\beta_{j,\gamma}$ is parallel to the boundary of the annulus $p^{-1}(\beta_{j,\gamma})$ residing in $\partial_1 X = \coprod S^2$, and therefore, is null-homotopic in X. Since $K \subset X$ is a homotopy equivalence, $\beta_{j,\gamma}$ must be null-homotopic in K as well. Therefore, the quotient map $K \to K_\star$ which shrinks each loop $\beta_{j,\gamma}$ to a point $\beta^\star_{j,\gamma} \in K_\star$ is a homotopy equivalence of 2-complexes. Thus, $\pi_1(X) \approx \pi_1(K_\star)$. Now, $\pi_1(K_\star)$ can be easily computed from Γ. Let $\mathcal{N}_\star(\infty)$ be a 2-complex obtained from $\mathcal{N}(\infty)$ by capping each component of its boundary with a 2-disk. Let $T \subset \Gamma$ be a maximal tree. Then $\pi_1(K_\star)$ is a free product of the group $\pi_1(\mathcal{N}_\star(\infty))$ and the groups $\{\pi_1(\hat{S}_j)\}_j$ with a free group F whose rank is the number of basic cycles in Γ (the number of edges in $\Gamma \setminus T$). Recall that $\pi_1(\mathcal{N}_\star(\infty))$ is either $\Pi = \{a, b \mid a[a, b]b^{-1}\}$, or trivial. As before, $\partial_1 X = \coprod S^2$ implies that S_j has no handles (i.e., $\hat{S}_j = S^2$). As a result, $\pi_1(X) \approx \Pi * F$, or $\pi_1(X) \approx F$, provided $gc(X) \leq 1$ and $\partial_1 X = \coprod S^2$.

On the other hand, the additivity of $c(\sim)$ under the connected sums implies that X could only be a connected sum of several D^3's, $S^2 \times S^1$'s, $\mathbb{R}P^3$'s and $L_{3,1}$'s with a single copy of either $L_{4,1}$ or $L_{5,2}$. Thus, $\pi_1(X)$ must be a free product of a number of copies of the groups $\{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5\}$, and $H_1(X; \mathbb{Z})$ must have torsion elements, unless X is a connected sum of D^3's and $(S^2 \times S^1)$'s, a manifold of vanishing gradient complexity. None of the other prime blocks allow for the fundamental group $\pi_1(X) \approx \Pi * F$, or $\pi_1(X) \approx F$. □

In contrast, if one does not require $\partial_1 X$ to be simply-connected, then an argument similar to the one used in the proof of Theorem 8.3 (see the last paragraph of the proof) shows that plenty of 3-folds of gradient complexity one do exist. Here is a description of the main example. Let M be a 3-fold obtained from a handlebody H with ∂H of genus 2 and longitudes a, b by attaching a 2-handle along the simple non-separating loop $\rho = a[a, b]b^{-1}$ in ∂H. Since $\chi(M) = 0$ and ρ is non-separating, ∂M is a 2-torus. Recall that $\pi_1(M) \approx \Pi = \{a, b \mid a[a, b]b^{-1} = 1\}$.

Theorem 8.6. *Any labeled graph* Γ *as above with a single vertex* A_∞*, modeled after* $\mathcal{N}(\infty)$ *as in Figure* 30, *pattern* 1, *gives rise to an oriented* 3*-fold* X *of gradient complexity one. In fact,* $X = H\#_\phi X'$*, where an arbitrary oriented* 3*-fold* X' *of gradient complexity zero is attached along an annulus* $A' \subset \partial X'$ *to an annulus* $A \subset \partial H$*, the core of* A *being the loop* ρ*, by an orientation-reversing diffeomorphism* $\phi : A \to A'$*. Thus,* $\pi_1(X)$ *maps onto the group* $\prod * \pi_1(X'/A')$*. The group* $\pi_1(X')$ *admits a presentation as in* (8.3) *and* (8.4).

In particular[26]*, any connected sum* $X = M \# X'$*, where* $gc(X') = 0$*, is of gradient complexity one. Its fundamental group* $\pi_1(X) \approx \Pi * \pi_1(X')$.

9. How Deformations of Morse Data Affect the Spine

Now we will study the effect of deforming the Morse function f and its gradient-like field v on the gradient spines they generate. We start with deformations of nonsingular functions that do not introduce singularities in the process.

Since the gradient spines in our inquiry play an auxiliary role, and the true heros are the gradient flows, we cannot restrict our considerations only to special (“standard” in [2]) spines; there is nothing natural about the flows that produce special gradient spines. In this context, there is little of value in developing some combinatorial calculus of special gradient spines based on 1-dimensional graphs (appropriately decorated). In terminology of [3], the required version of the Combed Calculus is “not local”. It looks that we have no choice, but to keep track of the whole spine evolution as we deform (f, v). The reader can view this section as an attempt to build an analogue of the Benerdetti-Petronio Combed Calculus of concave traversing flows (cf. [2] Figure 4.7, and [3]) for the category of generic non-vanishing gradient fields (which are never concave).

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[26]the core loop of A' being contractible in X'

New *orientations* can be given to isolated singularities from $\partial_3 X$. These orientations depend on the Morse data and the preferred orientation of $\partial_1 X$[27], i.e. on the structure of an abstract gradient spine inherited by the cascades. Any point of $\partial_3 X$ comes equipped with one of the two polarities marked by "$\oplus$" and "$\ominus$": if at $x \in \partial_3 X$ the orientation of the arc from $\partial_2 X$ defined by the tangent vector $v(x)$ agrees with the orientation of that arc induced by the preferred orientation of $\partial_1^+ X$, then we assign $\oplus$ to x; otherwise, we assign $\ominus$. The polarities $\oplus$, $\ominus$ divide $\partial_3 X$ into two sets $\partial_3^{\oplus} X$ and $\partial_3^{\ominus} X$. As a result, the set $\partial_3 X$ is subdivided into four disjoint subsets: $\partial_3^{+\oplus} X$, $\partial_3^{-\oplus} X$, $\partial_3^{+\ominus} X$, $\partial_3^{-\ominus} X$. Put $\partial_3^A X := \partial_3^{+\oplus} X \cup \partial_3^{-\ominus} X$ and $\partial_3^B X := \partial_3^{-\oplus} X \cup \partial_3^{+\ominus} X$.

As we deform Morse data and watch the transformations of the corresponding spines, we will pay a close attention to the evolution of signs attached to the Q-singularities and to the points from $\partial_3 X$.

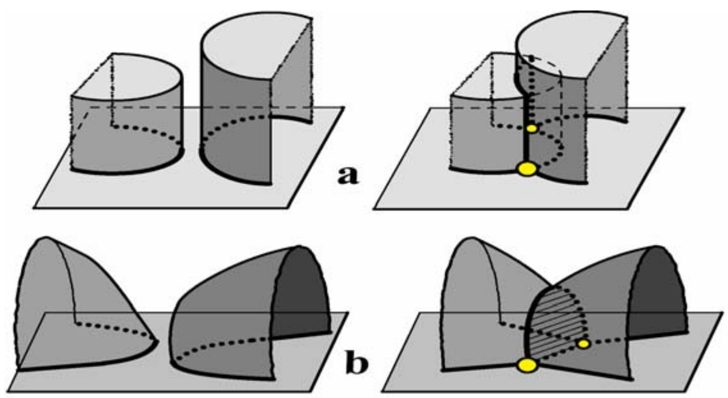

Figure 31. The α-move shown as a change in the shape of cascades (a) and as a change in the shape of 2-complexes (b).

Two types of elementary transformations of gradient spines are instrumental. The first one is depicted in Figure 31 as the passage from

[27]which, when X is orientable, is induced by an orientation of X.

the left to right diagrams and back. We call it an α-*move*. An α-move is similar to the second Reidemeister move, where the role of the link diagram is played by the folds $\partial_2^+ X$. The two Q-singularities generated in an α-move have opposite signs (that is, $\oplus$ and $\ominus$). If we forget the markers (which break the symmetry between the left and right surfaces in Figure 31, (b)), then α-moves make sense for any unmarked 2-complex K.

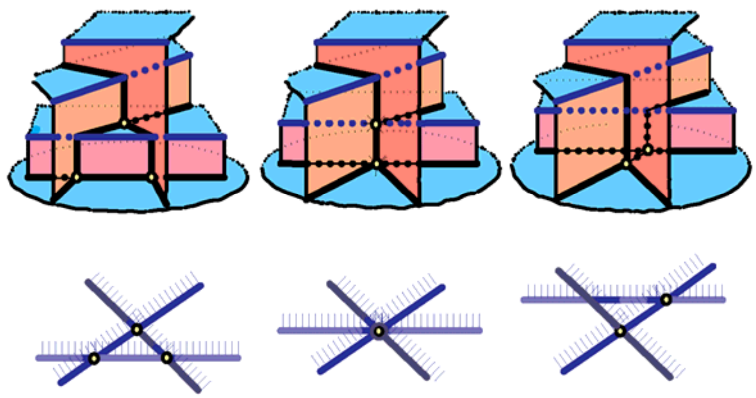

Figure 32. The β^{-1}-move shown as a change in the shape of cascades and as a v-projection of $\partial_2^+ X$ on $\partial_1^+ X$.

The β-move is an analogue of the third Reidemeister move (see Figure 32). In a β-move, three branches of $\partial_2^+ X$ form a triangular configuration, as viewed from the v-direction. In the deformation process, the configuration degenerates into one with "triple intersection", i.e. into a cascade that has a trajectory tangent to $\partial_1 X$ at *three* distinct points. The β-moves come in different flavors depending on the orientations and coorientations of the three folds from $\partial_2^+ X$ and their ordering by the function f (Figure 32 shows only one of the possible flavors).

Theorem 9.1. *Let* $\{f_t : X \to \mathbb{R}\}$, $t \in [0, 1]$, *be a continuous family of smooth nonsingular functions and* $\{v_t\}$ *be a corresponding family of gradient-like vector fields. Let* $K(v_t)$ *denote the spine* $\partial_1^+ X \cup \mathcal{C}(\partial_2^+ X)$

generated by v_t*. Assume that* v_0 *and* v_1 *are* ∂_2^+*-generic. Then the* 2*-complexes* $K(v_0)$ *and* $K(v_1)$ *are linked by a finite sequence of elementary expansions and collapses of two-cells, combined with a sequence of* α*- and* β*-moves and their inverses.*

In the deformation process, the cusps from $\partial_3 X$ *can cancel in pairs as shown in Figure* 34, *diagrams* 1-6, *and their mirror images. The change in topology of* $\partial_1^+ X$ *is accompanied by cancelations of pairs from* $\partial_3 X$ *with the opposite second polarity* $(\oplus, \ominus)$*. The cancellations of pairs with the opposite first polarity* $(+, -)$ *do not change the topology of* $\partial_1^+ X$*. The pairs sharing the same first and second polarities cannot be canceled.*

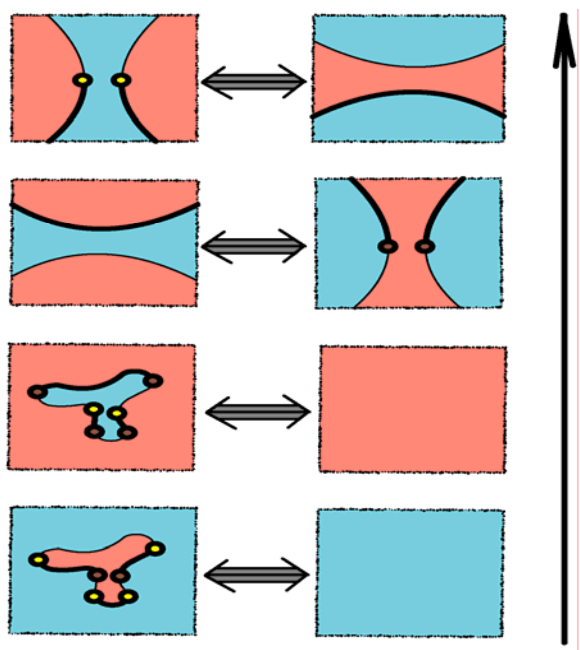

Figure 33. Elementary surgery on $\partial_1^+(X)$ and its effect on $\partial_2^+(X)$ (bold arcs) and $\partial_3^+(X)$ (bold dots).

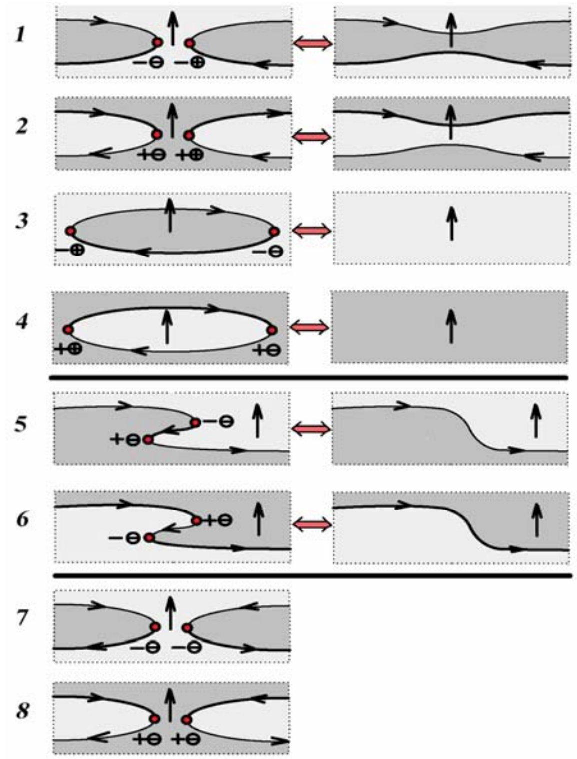

Figure 34. In diagrams 1-4 the topology of $\partial_1^+ X$ changes, in diagrams 5-6 it does not; diagrams 7-8 show "impossible cancellations".

Proof. We can assume that there are only finitely many $t \in (0, 1)$ for which v_t is not ∂_2^+-generic. As we deform Morse data (f_t, v_t), the topology of the sets $\partial_1^+ X := \partial_1^+(X, v_t)$ is changing by surgery which can be decomposed into a sequence of elementary surgeries depicted in Figure 33.[28] All these events take place in the vicinity of a point $x_\star \in \partial_1 X$ (where the two singularities from $\partial_3 X$ merge at a moment $t_\star$) and propagate inside of X via the waterfalls. In Figure 33, the vector field $v_1(x_\star, t_\star)$ is directed upward.

[28]Actually, the third and fourth moves can be decomposed into similar moves applied to a number of convex (round) holes with a single arc for $\partial_2^+(X, v_t)$ followed by 1-surgeries as in the first and second moves.

Diagrams 1 and 2 in Figure 34 depict 1-surgery, and diagrams 3 and 4 depict 0-surgery and 2-surgery on $\partial_1^+ X$. In each diagram, the curves from $\partial_2 X$ are oriented; so the points from $\partial_3 X$ acquire four flavors: $(+, \oplus)$, $(+, \ominus)$, $(-, \oplus)$, $(-, \ominus)$. Reversing the orientation of $\partial_2 X$ flips the polarities $\oplus \Leftrightarrow \ominus$. Such a flip leads to another four diagrams (not shown in Figure 34) which are the mirror images of the diagrams 1-4. They complete the elementary surgery list. We notice that in the process, the first polarities $(+, -)$ of the canceling singularities from $\partial_3 X$ are the same, and the second polarities $(\oplus, \ominus)$ are opposite. The orientation of $\partial_2 X$ prevents pairs of the types $(+, \oplus)$, $(+, \oplus)$ or of the types $(-, \ominus)$, $(-, \ominus)$ from cancelation (see diagrams 7 and 8). Again, we think about moves depicted in Figure 34 as localized events (occurring at a point of the boundary $\partial_1 X$) whose effect on the gradient spine propagates inside X.

Note that each pair of canceling singularities from $\partial_3 X$ has one point in $\partial_3^A X$ and the other in $\partial_3^B X$. Hence, the difference $\#(\partial_3^A X) - \#(\partial_3^B X)$ stays *invariant* under the transformations from Figure 34. Actually, by Theorem 9.5, this difference is always zero.

Diagrams 5 and 6 from Figure 34 show another generic mechanism by which the singularities from $\partial_3 X$ cancel, a mechanism which has no effect on the topology of $\partial_1^+ X$, but which modifies $\partial_2^+ X$. In fact, this type of cancellation is generated by the universal dove tail family (2.3). Again, reversing the orientations of the arcs from $\partial_2 X$ will produce two more diagrams not shown in Figure 34. This time, the cancelation occurs among the points of opposite first and same second polarities.

Let us first examine the effect of 1-surgery (the first and the second moves) on the shape of the cascades. Figure 35 shows what happens with cascades as two cusps merge (as in Figure 34, diagrams 1 and 2). Viewed in terms of the local projections $p_x : \partial_1 X \to f^{-1}(f(x))$, this transformation is generated via a generic one-parameter family of mirror-symmetric merging cusps. The upper diagram depicts the case when the “groovings” are located in $\partial_1^+(X)$ (Figure 35, diagram 2), and the lower diagram when they are in $\partial_1^-(X)$ (Figure 35, diagram 1).[29] In the upper diagram the cusp points a and b of opposite second polarities belong to the set $\partial_3^+(X)$.

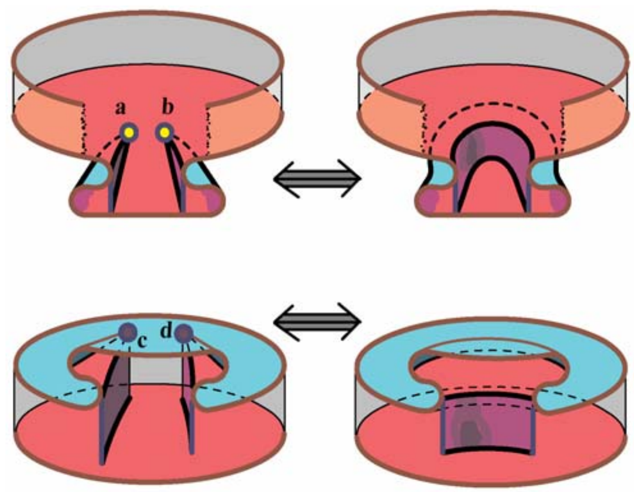

Figure 35. Change of cascades by elementary collapses and expansions of 2-cells that corresponds to 1-surgery on $\partial_1^+(X)$.

Consider the arc-shaped band marked with a dotted line in the upper-right diagram. The band belongs to $\partial_1^+(X)$.[30] In order to get the upper-left diagram, we collapse a 2-cell (the middle rectangle) in that band onto a segment. In the lower diagram, the cusp points c and d of opposite second polarities belong to the set $\partial_3^-(X)$ and are assumed to be very close to each other. Therefore, the $(-v)$-trajectories through c and d will hit the same plato (shown as the lower disk)—the cascades are assumed to be generic. Locally, the spine in the left-lower diagram can be obtained from this plato by two elementary expansions of 2-cells, while the spine in the right-lower diagram by a single elementary expansion.

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[29]In the first case, the f-controlled size of collapses and expansions is small, while in the second case it can be rather big—a localized change in convexity of the boundary $\partial_1 X$ causes a distributed change in the shape of cascades.

[30]most of the band is in the rear and thus invisible.

The spirit of considerations centered on 0-surgery and 2-surgery of $\partial_1^+(X)$ is similar. It is illustrated in Figure 8 which portrays a small indent in a round 3-ball D^3 and its effect on the spine. (The indent generates a hole in the set $\partial_1^+(D^3)$ of the original round ball.)

Figure 11 shows the cascades that corresponds to diagram 6 in Figure 34. Again, the cascades before and after the dove tail cancelation are linked by elementary 2-collapses or expansions.

In short, we have seen that changing topology of the pair $\partial_1^+(X) \supset \partial_2^+(X)$ via surgery induces two-dimensional expansions and collapses of the corresponding gradient spines.

Next, let us examine how the spines are affected by deformations of the Morse data (f, v) that do not change the topology of the pair $\partial_1^+(X) \supset \partial_2^+(X)$. Evidently, under such deformations, the corresponding spine can change only via changing interactions of waterfalls among themselves and with the "ground" $\partial_1^+(X)$. We analyze these interactions by considering the v-projections of $\partial_2^+(X)$ into $\partial_1^+ X$, while keeping track of the f-height of the branches from $\partial_2^+ X$.

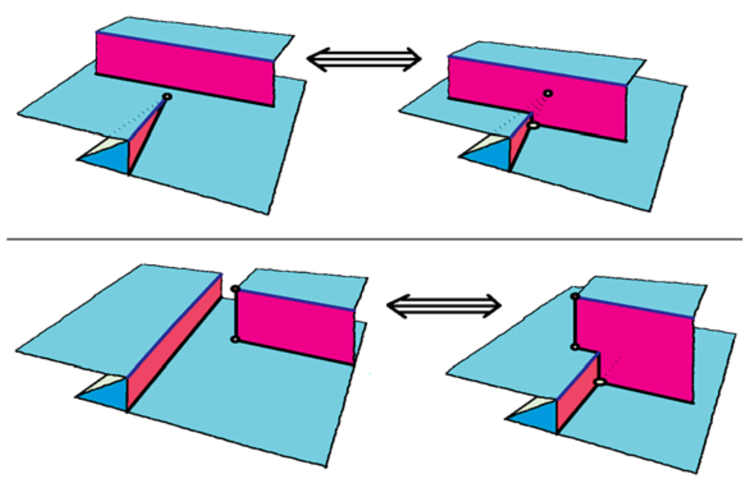

Figure 36. Generating a new Q-singularity by deforming the cascade, while keeping the topology of $(\partial_1^+ X, \partial_2^+ X)$ fixed.

Within generic families of deformations, these changes in the shapes of spines can be decomposed into sequences of three basic moves, the second and third of which resemble to the second and third Reidemeister moves, respectively. Let us describe them.

(1) Consider two waterfalls W_1 and W_2 that fall from two arcs C_1, C_2 $\subset \partial_2^+(X)$, locally C_1 being above C_2. Initially, W_1 and W_2 are disjoint in the vicinity of a given trajectory γ. Then, as the field v changes, C_2 can pierce W_1 transversally at a single point x (see Figure 36) (this change of the spine is impossible when $\partial_3 X = \emptyset$).

(2) Alternatively, C_2 can touch W_1 at a point x and then penetrate it at a pair of nearby points y and z as shown in Figure 31, (a). In such a case, the topology of the cascade is changing via the α-move, and its complexity changes by $+2$. Recall that the $(\oplus, \ominus)$-polarities of the two Q-singularities generated by an α-move are opposite.

(3) Finally, the β-move is generated where the three branches of $\partial_2^+ X$ form a "triangular" configuration that deforms into a new "triangular" configuration via Morse data that has a trajectory tangent to $\partial_1 X$ at three distinct points (see Figure 32). As a result of the β-move, the complexity of the gradient spine jumps by one; moreover, the polarized complexity also changes by one.

Note that these three moves resemble to the generic moves that describe deformations of plane curves in [1]. Before and after deformations, the curves are allowed to have only double-crossings and cusps (in our context, the cusps are "semi-stable": they are born and die in pairs), but no self-tangencies or triple intersections.

The proof of Theorem 9.1 is complete. □

It seems that it is hard to control the birth and death of Q-singularities that accompany deformations of Morse data. For example, canceling two cusps via a dove tail deformation reduces the number of

Q-singularities (equivalently, double-tangent trajectories) by one. The situation becomes more manageable in the category of Morse data with no cusps (see Corollary 9.3).

Next, consider the space $\mathcal{F}(X)$ of smooth nonsingular functions on a compact oriented manifold X with boundary. It coincides with the space of all submersions $f : X \to \mathbb{R}$. Let $\mathcal{V}(X)$ be the space of smooth nonsingular vector fields on X. Philips' remarkable Theorem B [31] claims that, for a fixed metric, the gradient map $\nabla : \mathcal{F}(X) \to \mathcal{V}(X)$ is a weak homotopy equivalence. When X is an oriented connected 3-fold with boundary, the tangent bundle of X is trivial, and Philips' Theorem reduces to the following known proposition:

Theorem 9.2. *Let X be a connected oriented Riemannian 3-fold with boundary. Fix a trivialization of the tangent bundle τX of X. Then the trivialization-induced normalized gradient map $\nabla/\| \nabla \| : \mathcal{F}(X) \to Map(X, S^2)$, where $Map(X, S^2)$ stands for the space of C^∞-maps from X to S^2, is a weak homotopy equivalence.*

Corollary 9.1. *Let X be in Theorem 9.2. Then the trivialization-dependent map $\nabla/\| \nabla \|$ induces a homotopy groups isomorphism*

$$\pi_n(\mathcal{F}(X)) \approx \pi_n(Map(X, S^2)).$$

In particular, as sets, $\pi_0(\mathcal{F}(X)) \overset{h}{\approx} H^2(X; \mathbb{Z})$ and $\pi_1(\mathcal{F}(X), f)$ can be identified with the set $H^1(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z})$.

Proof. In order to prove the corollary, we need to explain just the validity of the last two isomorphisms. With a trivialization β of τX being fixed, any nonsingular f gives rise to a map $\nabla f/\| \nabla f \| : X \to S^2$ whose homotopy class is an element of $\pi^2(X)$. Since X is 3-co-connected[31], by

[31]that is, $H^i(X; G) = 0$ for all $i \geq 3$ and any coefficient group G.

Hopf's Theorem (see Theorem 11.5, [16]), the natural map $h : \pi^2(X) \to H^2(X; \mathbb{Z})$ is an isomorphism. We denote by $h(f)$ the element $h(\nabla f / \| \nabla f \|) \in H^2(X; \mathbb{Z})$.[32] Thus, $h(f)$ detects the element $[f] \in \pi_0(\mathcal{F}(X))$.

Any loop in $\gamma \subset \mathcal{F}(X)$ can be viewed as function $F : X \times S^1 \to \mathbb{R}$ which is nonsingular when restricted to each fiber $X_\theta := X \times \theta$, $\theta \in S^1$. Hence, γ produces a map $G : X \times S^1 \to S^2$, and the homotopy class $[\gamma] \in \pi_1(\mathcal{F}(X), f)$ corresponds to an element $H(\gamma) \in \pi^2(X \times S^1)$ which restricts to $h(f) \in H^2(X; \mathbb{Z})$. Thus, $\pi_1(\mathcal{F}(X), f)$ can be identified with the elements of the set $\pi^2(X \times S^1)$ that map to $h(f)$ under the natural map $\pi^2(X \times S^1) \to H^2(X \times S^1; \mathbb{Z}) \to H^2(X; \mathbb{Z})$. Obstructions to linking any pair F_0, F_1 of such maps by a homotopy lie in $\oplus_j H^j(X \times S^1; \pi_j(S^2))$. Since X is 3-co-connected, $\pi_2(S^2) \approx \mathbb{Z} \approx \pi_3(S^2)$, and $F_0|_{X \times 0} = f = F_1|_{X \times 0}$, via the Künneth formula, these obstructions lie in $H^1(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z})$. With a little more work, one can show that any element of $H^1(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z})$ is realizable as an obstruction between $X \times S^1 \xrightarrow{p} X \xrightarrow{f} \mathbb{R}$ and some function F as above. □

Another invariant $e(f) \in H^2(X; \mathbb{Z})$ of nonsingular Morse functions $f : X \to \mathbb{R}$ on an orientable 3-fold X is available. Its definition is independent of the choice of a trivialization of τX. Consider an oriented 2-dimensional vector bundle η on X formed by the planes tangent to the constant level surfaces of f. The orientation of η is induced by the orientation of X and by ∇f. Note that $\eta \oplus \mathbb{R}$ is isomorphic to the tangent bundle of X and therefore is trivial. Let $e(f) \in H^2(X; \mathbb{Z})$ be the

[32]The choice of $h(f)$ is equivalent to the choice of a $Spin^c$-structure on X.

Euler class of η. The element $e(f)$ is invariant under homotopies of f through nonsingular functions.

Since the bundle $\eta \oplus \mathbb{R}$ is trivial, for each choice of the trivialization β, the isomorphism class of η is described by the homotopy class of the appropriate map $E(f) : X \to SO(3)/SO(2) \approx S^2$. In turn, the class of E is detected by the Euler class of the bundle η. The relation between $E(f)$ and $\nabla f/\|\nabla f\|$ is well known. It is described by the lemma below whose proof is left to the reader.

Lemma 9.1. *The homotopy class of the map* $E(f) : X \to S^2$ *is twice the homotopy class of the map* $h(f) : X \to S^2$. *Thus,* $e(f) = 2h(f)$.

Combining Theorem 9.1 with Corollary 9.1 we get one of our main results:

Theorem 9.3. *Let* (f_0, v_0) *and* (f_1, v_1) *be a pair of generic Morse data, where the fields* v_0, v_1 *are nonsingular, and such that* $h(f_0) = h(f_1)$.[33] *Then there exists a sequence of elementary* 2*-expansions,* 2*-collapses,* α, α^{-1}, β, β^{-1}*-moves which transforms the gradient spine* $K(v_0)$ *into the gradient spine* $K(v_1)$. *Therefore, if* $H^2(X; \mathbb{Z}) = 0$, *then any two gradient spines of* X *are linked by a sequence of elementary* 2*-expansions,* 2*-collapses, intermingled with* α *and* β*-moves and their inverses.*

It remains to sort out what happens to a gradient spine $K(v)$ when the value of the invariant $h(f) \in H^2(X; \mathbb{Z})$ jumps. In fact, due to Theorem 9.3, it suffices to analyze how the spine changes as a result of critical points of $\{f_t\}$, $0 < t < 1$, "traveling through X" along arcs that represent a generator of $H_1(X, \partial X; \mathbb{Z})$ (see Figure 37). Here we assume that f_0 and f_1 are nonsingular and that the traveling critical points are of the Morse type.

[33] Both fields define equivalent $Spin^c$-structures on X.

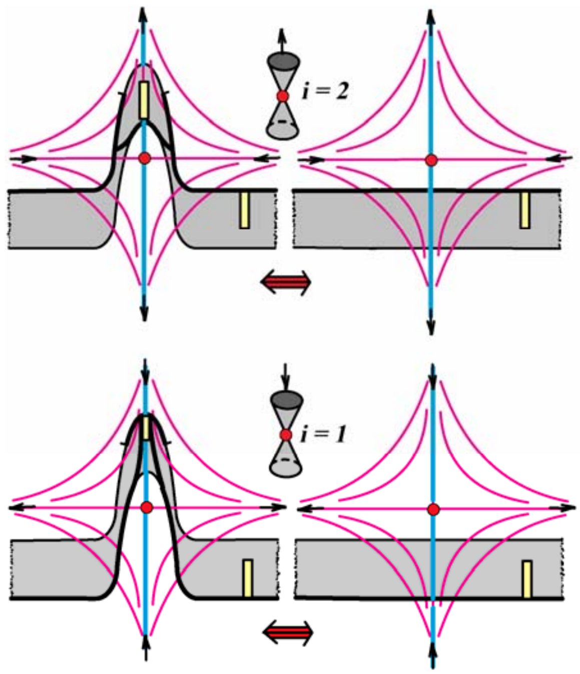

Figure 37. Changing $h(f)$ by the dual of the 1-cycle $(-1)^i[J]$ via an isotopy. The spine changes by mushroom flips. Note the change in the orientation (shown by small normal bars) of the newly formed disk, the mushroom head in the lower diagram.

The class $h(f) \in H^2(X;\, Z)$ is Poincaré-dual to the oriented 1-dimensional locus $J \subset X$ where v has fixed, up to proportionality, coordinates in the basis that trivializes τX. J is as a union of oriented loops and arcs in X with end in ∂X.

Suppose we have a homotopy $\{f_t\}_{t \in [0,1]}$, of Morse functions on X so that the singular set

$$\sum\{f_t\} = \{(x,\, t) \in X \times [0,\, 1] \mid x \text{ is a (Morse) critical point of } f_t\}$$

consists of a collection $\widetilde{J}$ of arcs in $X \times (0,1)$ with endpoint in $\partial X \times (0,1)$. Let J be the image of $\widetilde{J}$ under the projection onto X. By transversality, we assume that $J \subset X$ is a union of disjointly embedded arcs $\{J_\alpha\}$ with end points in ∂X. These arcs are oriented according to the direction in which the t parameter is increasing.

Lemma 9.2. *The cohomology class* $h(f_1) - h(f_0)$ *is dual to the relative* 1-*cycle* $\sum_\alpha (-1)^{i_\alpha}[J_\alpha]$, *where* i_α *is the Morse index of the critical point that traces the arc* J_α. *When all indices are even,* $h(f_1) - h(f_0) = g^*[S^2]$, *where* $g : X \to S^2$ *is the map defined by the Thom-Pontjagin construction on this union of arcs* $J\ (= g^{-1}(*))$.

Proof. Figure 37 reflects the spirit of the argument. It depicts the case of a critical point an index two tracing an oriented arc J in X. In fact, the figure shows an isotopy of the manifold X against a background of a "stationary" Morse function f and a trivialization β, both defined on a larger manifold $\hat{X} \supset X$ ($\hat{X}$ is obtained from X by two elementary expansions using 3-dimensional cells).

The isotopy of X is supported in a cylinder $C := D^2 \times \hat{J} \subset \hat{X}$. When we isotop X inside of $\hat{X}$, the original trivialization changes by a homotopy which is constant outside C. As long as $v \neq 0$, this deformation does not change the class of the map $\nabla f / \| \nabla f \|$ in $\pi^2(X)$.

Examining the locus where v is vertical and points up (see the upper diagram in Figure 37), we conclude that the Poincaré-dual of the variation of $h(f)$ can be represented by the oriented arc J—the bold arrow in the figure. The case of index one critical point is similar: the gradient ∇f will flip its direction (in comparison to the one shown in Figure 37) causing the change in the orientation of J. In the case of the Morse index i, the variation of the Poincaré-dual of $h(f)$ is given by the formula $(-1)^i[J]$. Note that when the critical point x of f is inside X, the map $\nabla f / \| \nabla f \|$ is only well-defined in $X \setminus x$.

Now we are in position to prove Lemma 9.2. Put $Y = X \times [0, 1]$. Consider a bundle τ tangent to the fibers of the obvious projection $Y \to X$ and its trivialization β. The gradient-like fields v_t define a section w of τ that vanishes on $\widetilde{J} = \bigcup_t \Sigma_t \times t$, and thus a map $\Phi : Y \setminus \widetilde{J} \to S^2$ is well defined. Denote my U a small regular neighborhood of $\widetilde{J}$ such that all the fibers U_x of the projection $U \subset Y \to X$ are homeomorphic to two-disks (the disks get truncated as they approach $\partial_1 X$) in which the Morse function f_t acquires its "almost canonical" form

$$a_1(t)x_1(t)^2 + a_2(t)x_2(t)^2 + a_3(t)x_3(t)^2,$$

with $|a_1(t)| < |a_2(t)| < |a_3(t)|$. Along $\widetilde{J}$, the Morse coordinates $(x_1(t), x_2(t), x_3(t))$ could disagree with the trivialization β. However, this disagreement happens along a bunch of arcs $\widetilde{J}_\alpha \subset X \times [0, 1]$ that have *disjoint* projections J_α in X. Because each arc is contractible, we can homotop β in the vicinity of each J_α so that the new trivialization will be adjusted to the Morse coordinates. (Recall, that a homotopy of β does not change the invariants $h(f_t)$.) Now, by general position, we can assume that (1) Φ is transversal to a base point $* \in S^2$, (2) $\Phi^{-1}(*) \cap (X \times 0) = (\nabla f_0 / |\nabla f_0|)^{-1}(*)$ and $\Phi^{-1}(*) \cap (X \times 1) = (\nabla f_1 / |\nabla f_1|)^{-1}(*)$, (3) $\Phi^{-1}(*) \cap U$ is given by $x_2(t) = 0 = x_3(t)$, $x_1(t) > 0$. Let Z be the surface $\Phi^{-1}(*) \cap (Y \setminus U)$ equipped with an orientation induced by the orientations of S^2 and Y. The boundary of the 2-chain Z is the 1-cycle which satisfies the equation

$$\partial Z = (\nabla f_1 / |\nabla f_1|)^{-1}(*) - (\nabla f_0 / |\nabla f_0|)^{-1}(*) + \sum_\alpha (-1)^{i_\alpha} \widetilde{J}'_\alpha,$$

where $\widetilde{J}'_\alpha \subset \partial U$ is parallel to $\widetilde{J}_\alpha$. Hence, $h(f_1) - h(f_0)$ is dual to the relative 1-cycle $\sum_\alpha (-1)^{i_\alpha} [J_\alpha]$. $\square$

Imagine cutting a disk D^2 out of an oriented surface K°, flipping the orientation of D^2, and gluing it back. This would be a clear violation of the orientation in K°. To protect the consistency of the orientation, let us erect a circular wall $W = S^1 \times [0, 1]$ along the cut and orient it so that the singularity locus $\partial D^2 = S^1 \times \{0\}$ contributes with the multiplicity ± 1

to the boundary of the oriented 2-chain $[K^\circ \cup W]$. This results in adding a disjoint oriented circle to the singularity set $s_\dagger(K)$. Although $K^\circ \cup W$ is collapsible on K°, within the category of *branched* spines, we cannot collapse the wall on its circular base: the reversed orientation of D^2 prevents the collapse. There are two ways of marking the new spine $K \cup W$ with *TN*-markers. Either (1) we mark the inner normal of ∂D^2 with *T*, and place marker *N* in *W* along $S^1 \times \{0\}$ so that it points towards $S^1 \times \{1\}$, or (2) we mark with *T* the outer normal of $D^2 \subset K$ and place *N* on wall $S^1 \times \{0\}$ so that it points towards $S^1 \times \{1\}$.

We call such spine changes *mushroom flips*. They are manifestations of jumps of the $Spin^c$-structure on *X*. The first type of flip occurs when a critical point of index 2 is traversing *X*, the second type corresponds to a critical point of index 1 (see Figure 37).

Theorem 9.4. *Let* (f_0, v_0) *and* (f_1, v_1) *be two generic pairs of Morse data on a compact oriented* 3*-fold X with boundary, the fields* v_0, v_1 *being nonsingular*. *Then there exists a sequence of* 2*-expansions*, 2*-collapses*, $\alpha, \alpha^{-1}, \beta, \beta^{-1}$*-moves, and mushroom flips which transform the gradient spine* $K(v_0)$ *into the gradient spine* $K(v_1)$.

Proof. Combining Theorem 9.3 with Lemma 9.2 reduces the problem to understanding the changes in the shape of gradient spine that are affected by critical points of an appropriate index traversing *X* along oriented arcs $\{J_\alpha\}$ representing a given generator in $H^1(X, \partial X; \mathbb{Z})$. The arcs representing γ can be chosen in general position with respect to a given gradient spine *K*. That is, they are transversal to K° and have an empty intersection with $s(K)$. Furthermore, if an oriented arc J_α hits the cascade $C \subset K$ transversally at a point *x*, then we can replace it with two oriented arcs J'_α and J''_α such that $J'_\alpha \cup J''_\alpha$ is homologous to J_a and $(J'_\alpha \cup J''_\alpha) \cap C = (J_a \cap C)\setminus x$. To construct the new arcs, use the down trajectory γ_x through *x* and a band *B* with the core γ_x, *B* being transversal to the waterfall that *W* contain *x*.

By somewhat similar construction, we can find a representative of γ so that each arc J_α hits $\partial_1^+ X$ at a single point x. For example, if we have an arc J_α with two ends, $x, y \in \partial_1^+ X$, then we pick a path δ connecting a point $z \in Int(J_\alpha)$ to a point $w \in \partial_1^- X$, form a narrow band B to with the core δ, and use two arcs in ∂B to replace J_α with two arcs J'_α and J''_α, each having the desired property. Therefore, we can assume that each J_α either is oriented in accordance with the vector $v(x)$, or opposite to it. In the first case, we send a critical point of an even index to trace J_α, in the second case, we send a critical point of an odd index.

Thus we need to describe only the case where an arc J_α hits $\partial_1^+ X \subset K$ transversally away from the cascade $C \subset K$. Each intersection of this type will be responsible for one mushroom flip as shown in Figure 37. □

Theorem 9.5. *Let X be an an oriented compact 3-fold. Any generic Morse data (f, v) can be deformed into new data (f', v'), so that the new $\partial_3 X(v') = \emptyset$, and $gc(v') = gc(v)$. In particular, there are no topological obstructions to the 3-convexity.*

Theorem 9.5 employes Theorem 9.6 below. It is similar in spirit to some results from [6], [7] concerned with folding maps of surfaces.

Theorem 9.6. *Let $S \subset X$ be a connected compact oriented and two-sided surface regularly embedded in the ambient 3-fold X, and let (f, v) be non-singular Morse data such that v is transversal to S along its boundary ∂S. Then there exists a deformation of (f, v) which is fixed in the vicinity of ∂S and such that the new generic data do not have cusps in S.*

Since the surface S has a preferred side in X, it can be divided with the help of v into two domains S^+ and S^- which share a common boundary L. In S^+, the field points into the preferred side of S and is tangent to S along locus L. Since v is transversal to S along ∂S, for a generic v, L is a collection of loops. The preferred orientation of S^+ induces a particular orientation on L. The 1-submanifold L is divided by the cusp locus C into portions L^+ and L^-. Along L^+, v points inside S^+. As before, the points from C acquire four flavors: $(+, \oplus)$, $(-, \ominus)$, $(-, \oplus)$, and $(+, \ominus)$. The first $\{+, -\}$ polarity reflects the fact that v points inside or outside of L^+. The second polarity $\{\oplus, \ominus\}$ tells us whether the field agrees or disagrees with the orientation of L. We denote by C^A the points of the first two flavors $(+, \oplus)$, $(-, \ominus)$ and by C^B of the last two flavors $(-, \oplus)$, $(+, \ominus)$.

We divide the proof of Theorem 9.6 in three lemmas.

Lemma 9.3. *Under the hypotheses of Theorem* 9.6, $\#(C^A) = \#(C^B)$. *In particular, for generic Morse data* (f, v),

$$\#(\partial_3^{+\oplus} X) - \#(\partial_3^{+\ominus} X) + \#(\partial_3^{-\ominus} X) - \#(\partial_3^{-\oplus} X) = 0. \tag{9.1}$$

Proof. We already remarked that L must be a disjoint union of simple loops. Each loop L_i from L either has no cusps, or the arcs from L^+ and L^- alternate along L_i. For an arc from L^+, we examine the four possible flavors attached to its end points a and b and see that one the two polarities of a and b must be different. Therefore, if $a \in C^A$, then $b \in C^B$. As a result, $\#(C^A) = \#(C^B)$. In the case of $S = \partial_1 X$, this leads to (9.1). □

Lemma 9.4. *Let* $a, b \in C$ *have flavors either* (1) $(+, \oplus)$ *and* $(+, \ominus)$ *or* (2) $(-, \oplus)$ *and* $(-, \ominus)$, *respectively. In the first case, assume that a and b can be connected by simple path* $\gamma \subset S^+$, *in the second case, assume that* $\gamma \subset S^-$. *Then a and b can be cancelled via a deformation of* (f, v) *as in Figure* 34, *diagrams* 1-2. *The deformation is an identity away from a regular neighborhood of* γ.

Proof. The two cusps are mirror images of each other in the sense that there are ambient coordinates x, y, z in the vicinity of $\gamma \subset X$ so that the Morse function is $f(x, y, z)$ has a form $z + g(x, y)$ and the surface S is given by $y = z^3 + (x - a)z$ at the cusp $(a, 0, 0)$ and by $y = z^3 - (x - b)z$ at the cusp $(b, 0, 0)$. In local coordinates, the path can be given by $y = z = 0$ with x ranging from a to b. The surface is transverse to ∇f along $Int(\gamma)$, and the space of all germs of two-sided oriented surfaces with this property has the homotopy type of S^0 (there is no topological obstruction to making the surface standard along γ). So, we can cancel the two cusps by embedding a standard model and the deformation as in Figure 35. □

Lemma 9.5. *If two consecutive cusps* a *and* b *along a loop* L *are as in Lemma* 9.4, *then* a, b *can be canceled so that the new Morse data has a tangency locus* $L' \subset S$ *which is the result of* 0*-surgery on* L. *The deformation of the Morse data has a support in an arbitrary small neighborhood of the arc* $[a, b] \subset L$.

Proof. There is a path γ along the surface that links a and b inside of S^+ in the case of $(+, \ominus)$ and $(+, \oplus)$ or S^- in the other case. This path can be found in an arbitrarily small neighborhood of the arc $[a, b] \subset L$. By Lemma 9.4, the cancellation is possible. □

Lemma 9.6. *Let* $S \subset X$ *be as in Theorem* 9.6. *Then in the vicinity of any point* $x \in L^+$ *of the tangency locus* $L \subset S$ *there is a deformation of the Morse data so that two new consecutive cusps of types* $(+, \oplus)$ *and* $(-, \oplus)$ *or of types* $(+, \ominus)$ *and* $(-, \ominus)$ *are introduced in* L *via the dove tail deformation. The deformation is an identity away from* x.

Proof. See Figure 34, diagrams 5-6, and Figure 11 depicting the dove tail surface. □

Proofs of Theorems 9.5 and 9.6. Take any two consecutive cusps a, b along L so that the connecting arc $[a, b]$ lies in L^+. Say a is of type $(+, \oplus)$. Then b is either of type $(+, \ominus)$ or $(-, \oplus)$. In the first case, according to Lemma 9.4, a and b can be cancelled. In the second case, by

Lemma 9.5, we can introduce two cusps $c, d \in [a, b]$ of type $(+, \ominus)$ and $(-, \ominus)$, respectively. Then, by Lemma 9.4, a and c can be cancelled, as well as d and b. The same argument works in the case when a is of any other type than $(+, \oplus)$. Continuing in this way, all cusps can be eventually eliminated, which completes the proof of Theorem 9.6.

In order to prove Theorem 9.5, we need to examine carefully the previous argument. Consider all the arcs (but not loops) $[a, b] \subset \partial_2^- X$ with the different first polarities of a and b (then, by an argument in Lemma 9.3, the second polarities of a, b agree). For every such arc $[a, b]$, we introduce a pair of cusps $c, d \in [a, b]$ as above. Because $[a, b] \subset \partial_2^- X$, the new waterfall of $[c, d] \subset \partial_2^+ X$ is "protected" by $\partial_1^- X$ and isolated from the rest of waterfalls; as a result, the original gradient complexity is not affected by the introduction of c, d (contrast this with Figure 11 where complexity increases by 1). Introducing c, d also has no affect on the degree $\deg(h) = \#(\partial_3^+ X) - \#(\partial_3^- X)$ of the map $h : \partial_2 X \to S^1$ from Lemma 2.1. Thus, we can assume that every arc from $\partial_2^- X$ has cusps of opposite second polarity.

Now, for each arc $[a, b] \subset \partial_2^- X$, we pick a path $\gamma \subset \partial_1^{\pm} X$ which connects two canceling cusps a, b (as in Lemma 9.5) and resides in the vicinity of $[a, b]$. The new portion of the waterfall that is generated after the cusps' cancelation is also localized in the vicinity of $[a, b]$. Hence it is "protected" by $\partial_1^- X$ and separated from the old waterfalls that existed before the cancellation (see Figure 35, the upper diagram). Therefore, canceling a, b via such γ, again, does not change the gradient complexity of the original Morse data. Since the first polarities of a and b agree, each cancelation does change the degree of the map h by one. So, we will need at least $\deg(h)$ cancellations to get to the Morse data with $\partial_3 X = \varnothing$. □

Corollary 9.2. *In terms of the polarized cusps, the degree of the map* $h : \partial_2 X \to S^1$ *can be expressed as follows*:

$$\deg(h : \partial_2 X \to S^1) = [\#(\partial_3^{+\oplus} X) - \#(\partial_3^{-\oplus} X)]$$

$$= [\#(\partial_3^{+\ominus} X) - \#(\partial_3^{-\ominus} X)] = \chi(X) - 2\chi(\partial_1^+ X). \quad (9.2)$$

Hence, for the Morse data with fixed values of $\chi(\partial_1^+ X)$,[34] *the number* $\#(\partial_3^{+\oplus} X) - \#(\partial_3^{-\oplus} X) = \#(\partial_3^{+\ominus} X) - \#(\partial_3^{-\ominus} X)$ *is a topological invariant.*

Proof. By Lemma 9.3, $\#(\partial_3^{+\oplus} X) - \#(\partial_3^{+\ominus} X) + \#(\partial_3^{-\ominus} X) - \#(\partial_3^{-\oplus} X) = 0$. Combined with Lemma 2.1, this leads to the formula for the degree of $h : \partial_2 X \to S^1$ claimed in the corollary. □

Now, consider only nonsingular Morse data (f, v) such that v is transversal to the *submanifold* $\partial_2 X \subset \partial_1 X$. Denote by $\mathcal{W}(X)$ their space. In particular, for elements $(f, v) \in \mathcal{W}(X)$, $\partial_3 X = \varnothing$.

Similar spaces of smooth maps with folds only from a manifold M^n to a manifold N^n have been studied in great generality in [6], [7]. In our context, we are lacking a nice target space N^2. Its role is played by the space $X/{\sim_v}$ of v-trajectories, a space which has a structure of a cellular 2-complex and is singular in general.

Recall that, according to Theorem 9.5,

$$gc(X) = \min_{\{(f,\, v) \in \mathcal{W}(X)\}} gc(f, v). \quad (9.3)$$

Let $\mathcal{W}_*(X) \subset \mathcal{W}(X)$ be an open and dense subspace of Morse data (f, v) for which no v-trajectories, tangent to $\partial_1 X$ at three distinct points, exist. Note that no β-move is possible within $\mathcal{W}_*(X)$ (see Figure 32). The codimension one walls $\mathcal{W}(X) \setminus \mathcal{W}_*(X)$ can be cooriented by the following rule: a path $\gamma \subset \mathcal{W}(X)$, which represents a β-move, defines a positive coorientation when (as a result of the β-move) the difference between the

[34]For example, for Morse data with $\partial_1^+ X$ being a disk.

numbers of $\oplus$ and $\ominus$ double-tangent trajectories[35] jumps by +1. This coorientation is similar in spirit but different from the one used by V. Arnold in his studies of the spaces of immersions of plane curves [1].

We have shown that any pair (f, v) can be deformed into a pair with no cusps. For Morse data with $\partial_3 X = \emptyset$, both $\partial_2^+ X$ and $\partial_2^- X$ are collections of simple oriented loops in $\partial_1 X$, the orientation being induced by the orientation of $\partial_1^+ X$.

Within the space $\mathcal{W}(X)$, no surgery on $\partial_2 X$, induced by deformations of Morse data (f, v), is possible (see Figures 33 and 34). Indeed, the transversality of v to $\partial_2 X$ is imposed by the nature of $\mathcal{W}(X)$ and prevents loops from $\partial_2 X$ from touching each other, or being born/annahilated. Thus, each component of $\mathcal{W}(X)$ has its own oriented and polarized loop pattern $\theta(v) = \partial_2^+ X \coprod \partial_2^- X \subset \partial_1 X$.

Questions. For a given X, what is the minimal number of positive/negative loops for nonsingular Morse data with $\partial_3 X = \emptyset$? Evidently, the equation $\chi(\partial_1^+ X) = -\chi(X)$ imposes constraints on the number of loops in $\partial_2 X$. For a given X, which oriented loop patterns $\partial_2^\pm X$ in $\partial_1 X$ are realizable?

Within the space $\mathcal{W}_*(X)$ no β-moves are permitted. Therefore, there is a well-defined map from $\pi_0(\mathcal{W}_*(X))$ to skew-symmetric integral-valued bilinear forms Ψ. In a sense, the forms are induced by the intersections of 1-cycles forming $\partial_2^+ X$ in the v-orbit space $X/\sim_v$. Since $X/\sim_v$ has singularities of codimension one, in general, this intersection has no homological interpretation. However, within the constraints of a given chamber of $\mathcal{W}_*(X)$, it is well-defined. Consider a free $\mathbb{Z}$-module M

[35] equivalently, $\#[Q(K)^{\oplus}] - \#[Q(K)^{\ominus}]$.

generated by the oriented loops γ_i from $\partial_2^+(X)$. Define $\Psi(\gamma_i, \gamma_j)$ to be the sum of ± 1 which are contributed by the Q-singularities $x_{ij} \in \partial_1^+ X$ that correspond to the double-tangent trajectories linking γ_i to γ_j. The sign contributed by x_{ij} is the $\oplus/\ominus$ polarity that has been associated with x_{ij}. Evidently, the form Ψ is preserved under the α-moves—the only admissible transformations within a given chamber of $\mathcal{W}_*(X)$ (see Figures 31, 32).

If $\partial_3 X = \emptyset$, then, for each component $\partial_1 X_j$ of the boundary, the degree $\#(\partial_3^+ X_j) - \#(\partial_3^- X_j)$ of the map $h_j : \partial_2 X_j \to S^1$ is zero. By Lemma 4.3, we get $\chi(\partial_1^+ X) = \chi(\partial_1^- X)$. This property is shared by all Morse data from $\mathcal{W}(X)$ (cf. [6]).

These considerations lead to

Theorem 9.7. *The oriented and polarized loop patterns* $\theta(v) \subset \partial_1 X$ *are locally constant on the space* $\mathcal{W}(X)$ *of Morse data with the property* $\partial_3 X = \emptyset$.

The skew-symmetric form Ψ *and the linking number*

$$lk_v(\partial[K], \partial[K]) = \#(Q(K)^{\oplus}) - \#(Q(K)^{\ominus}),$$

$K = K(f, v)$, *are locally constant on the subspace* $\mathcal{W}_*(X) \subset \mathcal{W}(X)$ *of Morse data* (f, v) *with no triple-tangent trajectories.*

Corollary 9.3. *If* (f, v) *and* (f', v') *belong to different chambers of* $\mathcal{W}_*(X)$, *then any generic path* γ *which connects in* $\mathcal{W}(X)$ *the points* (f, v) *and* (f', v') *must have at least* $|\, gc_{\ominus}^{\oplus}(f, v) - gc_{\ominus}^{\oplus}(f', v')\,|$ *intersections with the walls* $\mathcal{W}(X) \setminus \mathcal{W}_*(X)$ *of various chambers of* $\mathcal{W}_*(X)$, *that is, the deformation family must have at least* $|\, gc_{\ominus}^{\oplus}(f, v) - gc_{\ominus}^{\oplus}(f', v')\,|$ *members with triple-tangent trajectories.*

Most likely, these and many other results of the paper admit multidimensional generalizations.

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