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PARAMETRIC INSTABILITY OF GENERALIZED PHILLIPS' MODEL

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Abstract

The parametric instability of the generalized Phillips' model, for which the velocity shear is a periodic function of time and the top and bottom surfaces are either rigid or free, has been studied in the neighborhood of the classical threshold of instability. It is shown that for the linear problem, the ignorance of the influences of the free surface parameter cannot change the essential character of the instability.

1. Introduction

The generalized Phillips' model is a fluid with the surfaces of top and bottom are either rigid or free. It has the advantage of the simplifying the actual fluid motions while retaining the essential dynamics of the instability. Many researchers discussed its baroclinic instability (see, for example, Li and Mu [6], Olascoaga and Ripa [8] and Li [5]), when the

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shear of the basic current is independent of time. However, the time dependence is important since it can destabilize (stabilize) a flow, which consequently alters the transport of heat and momentum (see, for example, Davis [2]). The importance of time dependence has been emphasized in the context of internal waves in Broutman et al. [1]. Though the instability of time-dependent shear flows has received relatively little attention in comparison to that of time-independent ones, many researchers have done some contributions to it (see, for example, Davis et al. [3], Farrell and Ioannou [4], and Pedlosky and Thomson [10]).

In this paper, the parametric instability of the generalized Phillips' model on the beta-plane has been studied includes the effects of time-varying baroclinic shear in the neighborhood of the classical threshold of instability. And we introduce a free surface parameter α to discuss the influences of the free surface approximation on the stability of atmosphere and oceanic motions. The results reveal that the influences of the free surface parameter α may be ignored in the linear problem.

2. Model

We consider the instability of the zonal flow in the generalized quasi-geostrophic two-layer model on beta-plane. The governing equations for the evolution of the disturbances are (see Li and Mu [6], and Pedlosky and Thomson [10]):

$$\left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x}\right) q_i + \frac{dQ_i}{dy}\frac{\partial \varphi_i}{\partial x} + J(\varphi_i, q_i) = -\mu q_i, \quad i = 1, 2, \tag{2.1}$$

$$q_i = \nabla^2 \varphi_i - d_i^{-1} \sum_{j=1}^{2} T_{ij} \varphi_j, \tag{2.2}$$

here t is time variable, the subscript i refers to the upper $(i = 1)$ and lower $(i = 2)$ layers, U_i, q_i, φ_i are horizontal velocity, perturbation potential vorticity and perturbation streamfunction in the ith layer, respectively. J is the Jacobian of the two sequential functions with respect to x and y. μ is dissipation coefficient, ∇^2 is the two-dimensional Laplace operator, d_i is the height of the ith layer, and

$$T = (T_{ij}) = \begin{pmatrix} f_0^2(g_0^{-1} + g_1^{-1}) & -f_0^2 g_1^{-1} \\ -f_0^2 g_1^{-1} & f_0^2(g_1^{-1} + g_2^{-1}) \end{pmatrix},$$

where f_0 is Coriolis parameter (constant), g_i is the buoyancy jump across the interface between the ith and $(i+1)$th layer, and if the top (or bottom) surface is rigid, then $g_0^{-1} = 0$ $(g_2^{-1} = 0)$; and when the top (or bottom) surface is free, then $g_0^{-1} > 0$ $(g_2^{-1} > 0)$. For convenience, without loss of generality, we consider the case that $g_0^{-1} = g_2^{-1}$ and the height of each layer is the same. We define $\alpha = g_0^{-1}/g_1^{-1}$ as the free surface parameter, so when the surface is rigid, then $\alpha = 0$ and when the surface is free, then $\alpha > 0$. And assume $F = d_i^{-1} f_0^2 g_1^{-1}$, so that

$$\begin{cases} q_1 = \nabla^2\varphi_1 - F[(1+\alpha)\varphi_1 - \varphi_2], \\ q_2 = \nabla^2\varphi_2 + F[\varphi_1 - (1+\alpha)\varphi_2]. \end{cases} \tag{2.3}$$

The potential vorticity gradient of the mean flow is

$$\begin{cases} \dfrac{dQ_1}{dy} = \beta + F[(1+\alpha)U_1 - U_2], \\ \dfrac{dQ_2}{dy} = \beta - F[U_1 - (1+\alpha)U_2]. \end{cases}$$

The horizontal domain under consideration is a periodic channel in x direction and $D_p = [0, X] \times [0, 1]$. For the large-scale atmosphere, $\alpha = O(10^{-1})$, here we take $0 \le \alpha \le 1$.

It is helpful to reformulate the problem in the terms of the barotropic and baroclinic models of the perturbation fields. With the definitions

$$\varphi_1 = \varphi_t + \varphi_c, \quad \varphi_2 = \varphi_t - \varphi_c, \tag{2.4}$$

the following equations are obtained from (2.3):

$$q_t = \nabla^2\varphi_t - F\alpha\varphi_t, \quad q_c = \nabla^2\varphi_t - F(\alpha + 2)\varphi_c. \tag{2.5}$$

From (2.1), (2.4) and (2.5), the barotropic and baroclinic models are obtained respectively:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + U_m \dfrac{\partial}{\partial x} + \mu\right) q_t + U_s \dfrac{\partial}{\partial x} \nabla^2 \varphi_c + (\beta + F\alpha U_m) \dfrac{\partial \varphi_t}{\partial x} \\ \qquad\qquad\qquad + J(\varphi_t, q_t) + J(\varphi_c, q_c) = 0, \\ \left(\dfrac{\partial}{\partial t} + U_m \dfrac{\partial}{\partial x} + \mu\right) q_c + U_s \dfrac{\partial}{\partial x} (\nabla^2 + 2F) \varphi_t + (\beta + F\alpha U_m) \dfrac{\partial \varphi_c}{\partial x} \\ \qquad\qquad\qquad + J(\varphi_t, q_c) + J(\varphi_c, q_t) = 0, \end{cases} \tag{2.6}$$

where

$$U_m = \frac{U_1 + U_2}{2}, \quad U_s = \frac{U_1 - U_2}{2},$$

and U_s is the shear of the basic current, which is the function of time.

3. Parametric Instability

In order to study the nature of the linear problem, we ignore the Jacobian and consider the equation as follows:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + U_m \dfrac{\partial}{\partial x} + \mu\right) q_t + U_s \dfrac{\partial}{\partial x} \nabla^2 \varphi_c + (\beta + F\alpha U_m) \dfrac{\partial \varphi_t}{\partial x} = 0, \\ \left(\dfrac{\partial}{\partial t} + U_m \dfrac{\partial}{\partial x} + \mu\right) q_c + U_s \dfrac{\partial}{\partial x} (\nabla^2 + 2F) \varphi_t + (\beta + F\alpha U_m) \dfrac{\partial \varphi_c}{\partial x} = 0. \end{cases} \tag{3.1}$$

We consider solutions of the form:

$$\varphi_{t,c} = A_{t,c}(t) \exp(ikx) \sin m_p y + *, \tag{3.2}$$

where $A_{t,c}(t)$ represent the wave amplitudes, m_p is any integral multiple of π and $*$ denotes the complex conjugate of the preceding expression.

Through direct computation, two coupled ordinary differential equations in time for the wave amplitudes are obtained:

$$\begin{cases} \dfrac{\partial A_t}{\partial t} + ik\left(U_m + \dfrac{\mu}{ik} - \dfrac{\beta}{K^2 + F\alpha}\right) A_t = -\dfrac{ikK^2 U_s}{K^2 + F\alpha} A_c, \\ \dfrac{\partial A_c}{\partial t} + ik\left(U_m + \dfrac{\mu}{ik} - \dfrac{\beta}{K^2 + F(\alpha + 2)}\right) A_c = -\dfrac{ik(K^2 - 2F) U_s}{K^2 + F(\alpha + 2)} A_t, \end{cases} \tag{3.3}$$

where $K^2 = k^2 + m_p^2$.

Consider the situation in which U_m is independent of time and only the shear varies with time. We rewrite the problem's variables as

$$\begin{cases} A_t = B_t e^{-ikt\left(U_m + \frac{\mu}{ik} - \frac{c_t + c_c}{2}\right)}, \\ A_c = B_c e^{-ikt\left(U_m + \frac{\mu}{ik} - \frac{c_t + c_c}{2}\right)}, \\ c_t = \dfrac{\beta}{K^2 + F\alpha}, \\ c_c = \dfrac{\beta}{K^2 + F(\alpha + 2)}, \end{cases}$$

substitute to (3.3) yields

$$\begin{cases} \dfrac{\partial B_t}{\partial t} - ik\dfrac{c_t - c_c}{2} B_t = -\dfrac{ikK^2 U_s}{K^2 + F\alpha} B_c, \\ \dfrac{\partial B_c}{\partial t} + ik\dfrac{c_t - c_c}{2} B_c = -\dfrac{ik(K^2 - 2F)U_s}{K^2 + F(\alpha + 2)} B_t. \end{cases} \tag{3.4}$$

Since we subsequently consider the finite-amplitude behavior of the waves, we will, in this study, restrict our attention to the vicinity of the marginal curve. It is convenient to introduce new variables:

$$a^2 \doteq \frac{K^2}{F}, \quad b \doteq \frac{\beta}{FU_{s0}}, \quad U_s \doteq U_{s0}(1 + \delta(\vartheta)), \quad \vartheta \doteq kU_{s0}t,$$

here U_{s0} is the initial critical value and $\delta(\vartheta)$ represents the increment above it. The last relation implies that we have scaled time with the advective time scale. So the equations for the amplitudes B_t and B_c are obtained:

$$\begin{cases} \dfrac{dB_t}{d\vartheta} - \dfrac{ib}{(a^2 + \alpha)(a^2 + 2 + \alpha)} B_t = -\dfrac{ia^2}{a^2 + 2}(1 + \delta)B_c, \\ \dfrac{dB_c}{d\vartheta} + \dfrac{ib}{(a^2 + \alpha)(a^2 + 2 + \alpha)} B_c = -\dfrac{i(a^2 - 2)}{a^2 + \alpha + 2}(1 + \delta)B_t. \end{cases} \tag{3.5}$$

If the linear problem has the steady shear, i.e., $\delta = 0$, then the last equations become the ordinary differential equations, so a critical value of shear is given by:

$$b^2 = a^2(2 - a^2)(a^2 + \alpha)(a^2 + \alpha + 2). \tag{3.6}$$

Note that for the stability of the classical type $a^2 < 2$, so in the following discussion the limit condition is always true.

Now, we will examine the role of the increment of the shear in the neighborhood of the classical marginal curve for instability. For small δ, i.e., near the marginal curve, a perturbation expansion is useful. We let ϱ be a small parameter and expand δ as follows:

$$\delta = \varrho G + \varrho H \cos wT, \tag{3.7}$$

here ϱG represents the small, additional steady increment of the shear above critical and ϱH represents a larger value of the amplitude of the oscillating part of the shear. We also introduce a slow time variable $T = \varrho^{1/2}\vartheta$ and allow the amplitudes to be functions both ϑ and T. An expansion of the form:

$$B_{t,c} = B_{t,c}^{(0)} + \varrho^{1/2}B_{t,c}^{(1)} + \varrho B_{t,c}^{(2)} + \cdots. \tag{3.8}$$

When (3.7) and (3.8) are substituted into (3.5) and like orders in ϱ are equated, yields the following results:

at lowest order ϱ^0,

$$\begin{cases} \dfrac{\partial B_t^{(0)}}{\partial \vartheta} - \dfrac{ib}{(a^2 + \alpha)(a^2 + 2 + \alpha)} B_t^{(0)} = -\dfrac{ia^2}{a^2 + 2} B_c^{(0)}, \\[2ex] \dfrac{\partial B_c^{(0)}}{\partial \vartheta} + \dfrac{ib}{(a^2 + \alpha)(a^2 + 2 + \alpha)} B_c^{(0)} = -\dfrac{i(a^2 - 2)}{a^2 + \alpha + 2} B_t^{(0)}. \end{cases} \tag{3.9}$$

Suppose

$$\gamma_1^2 \doteq \frac{a^2}{a^2 + \alpha}, \quad \gamma_2^2 \doteq \frac{2 - a^2}{a^2 + \alpha + 2}, \tag{3.10}$$

(3.9) can be written

$$\begin{cases} \dfrac{\partial B_t^{(0)}}{\partial \vartheta} - i\gamma_1\gamma_2 B_t^{(0)} + i\gamma_1^2 B_c^{(0)} = 0, \\[2ex] \dfrac{\partial B_c^{(0)}}{\partial \vartheta} + i\gamma_1\gamma_2 B_c^{(0)} - i\gamma_2^2 B_t^{(0)} = 0, \end{cases} \tag{3.11}$$

$(3.11a)\cdot\gamma_2 - (3.11b)\cdot\gamma_1$ yields

$$\frac{\partial}{\partial\vartheta}(\gamma_2 B_t^{(0)} - \gamma_1 B_c^{(0)}) = 0, \tag{3.12}$$

so $B_{t,c}^{(0)}$ can be considered as functions only of T, and

$$B_t^{(0)}(T) = \frac{\gamma_1}{\gamma_2} B_c^{(0)}(T). \tag{3.13}$$

at the next-order $\varrho^{1/2}$

$$\begin{cases} \dfrac{\partial B_t^{(1)}}{\partial\vartheta} + \dfrac{\partial B_t^{(0)}}{\partial T} - i\gamma_1\gamma_2 B_t^{(1)} = -i\gamma_1^2 B_c^{(1)}, \\ \dfrac{\partial B_c^{(1)}}{\partial\vartheta} + \dfrac{\partial B_c^{(0)}}{\partial T} + i\gamma_1\gamma_2 B_c^{(1)} = i\gamma_2^2 B_t^{(1)}, \end{cases} \tag{3.14}$$

we easily obtain

$$\frac{\partial}{\partial\vartheta}(\gamma_2 B_t^{(1)} - \gamma_1 B_c^{(1)}) = 0. \tag{3.15}$$

The solution follows directly from (3.15)

$$B_t^{(1)} = \frac{\gamma_1}{\gamma_2} B_c^{(1)} + C(T), \tag{3.16}$$

where $C(T)$ is integral constant.

If (3.16) is inserted in (3.14), we get

$$\frac{\partial B_c^{(1)}}{\partial\vartheta} + \frac{\partial B_c^{(0)}}{\partial T} = i\gamma_2^2 C(T). \tag{3.17}$$

Without loss of generality, let $C(T) = -\dfrac{i}{\gamma_2^2}\dfrac{\partial B_c^{(0)}}{\partial T}$, i.e., $\dfrac{\partial B_c^{(1)}}{\partial\vartheta} = 0$, we have

$$\begin{cases} B_c^{(1)} = 0, \\ B_t^{(1)} = \dfrac{\gamma_1}{\gamma_2} B_c^{(1)} + C(T) = -\dfrac{i}{\gamma_2^2}\dfrac{\partial B_c^{(0)}}{\partial T}. \end{cases} \tag{3.18}$$

at ϱ

$$\begin{cases} \dfrac{\partial B_t^{(2)}}{\partial \vartheta} - i\gamma_1\gamma_2 B_t^{(2)} + i\gamma_1^2 B_c^{(2)} = -\dfrac{\partial B_t^{(1)}}{\partial T} - i\gamma_1^2 (G + H\cos wT) B_c^{(0)}, \\ \dfrac{\partial B_c^{(2)}}{\partial \vartheta} + i\gamma_1\gamma_2 B_c^{(2)} - i\gamma_2^2 B_t^{(2)} = -\dfrac{\partial B_c^{(1)}}{\partial T} + i\gamma_2^2 (G + H\cos wT) B_t^{(0)}, \end{cases} \tag{3.19}$$

$(3.19a)\cdot\gamma_2 - (3.19b)\cdot\gamma_1$ yields

$$\frac{\partial}{\partial \vartheta}(\gamma_2 B_t^{(2)} - \gamma_1 B_c^{(2)})$$

$$= -\gamma_2 \frac{\partial B_t^{(1)}}{\partial T} + \gamma_1 \frac{\partial B_c^{(1)}}{\partial T} - i\gamma_1^2\gamma_2 (G + H\cos wT) B_c^{(0)} - i\gamma_1\gamma_2^2 (G + H\cos wT) B_t^{(0)}.$$

Using (3.13) and (3.18), the evolution equation of amplitude can be obtained

$$\frac{\partial^2 B_c^{(0)}}{\partial T^2} - 2\gamma_1^2\gamma_2^2 (G + H\cos wT) B_c^{(0)} = 0, \tag{3.20}$$

where $\gamma_1^2\gamma_2^2 = \dfrac{a^2(2 - a^2)}{(a^2 + \alpha)(a^2 + \alpha + 2)}$.

If we let $\alpha = 0$, then the equation is just the classical Mathieu equation (see, for example, Morse and Feshbach [7]).

For slightly subcritical shears, i.e., when $G < 0$. In the limit $H \to 0$, a natural frequency of oscillation of the system is given by

$$\sigma = \sqrt{-2\gamma_1^2\gamma_2^2 G}, \tag{3.21}$$

and the critical frequencies for so-called parametric instability occur when

$$w = \frac{2}{n}\sigma, \quad n = 1, 2, \ldots. \tag{3.22}$$

The perturbations with the largest growth rates correspond the $n = 1$ mode corresponding to a frequency which, for small H, is twice the natural frequency σ.

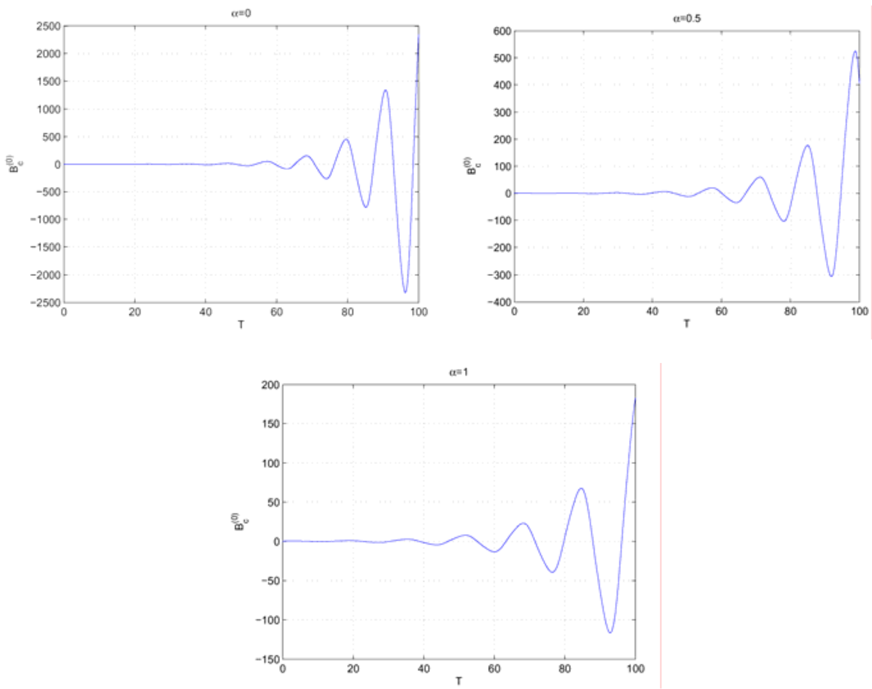

Figure 1. The amplitude behavior in the linear system (3.20) for different α, where $a = (2.1)^{1/4}$, $G = -1$, $H = 0.7$, $\alpha = 0, 0.5, 1$, and the corresponding values of w are 1.1303, 0.91085, 0.76556, respectively.

In the case that the shear of the basic current slightly less than the threshold value for instability, we take $a = (2.1)^{1/4}$, $G = -1$, $H = 0.7$. Thus, the time-averaged shear is stable. In order to study the influences of the free surface parameter α, we take $\alpha = 0, 0.5, 1$, respectively. We easily obtain $w = 1.1303$, 0.91085 and 0.76556 for corresponding α, with the aid of (3.22). Figure 1 shows the amplitude behavior in the linear system for different α, here the initial values are $[B_c(0), dB_c(0)/dT]$ $= [0.1, 0.1]$.

As we have seen, the disturbance amplitude will exponentially grow for any surface parameter α due to the parametric instability, in the neighborhood of the classical threshold of instability. So for the linear problem, the essential character of the instability will not be changed if the influences of the free surface parameter α are ignored.

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